Consider the standard LP relaxation for the standard variant of the k-median problem:

$$\min \sum_{u,v} d(u,v) x_{uv}$$
$$\sum_{v} x_{uv} = 1$$
$$\sum_{u} y_v \leq k$$
$$x_{uv} \leq y_v$$
$$x_{uv} \geq 0, y_v \geq 0$$

Problem 1 Let x, y be an optimal LP solution. Prove that $x_{uu} = y_u$ for every u.

Problem 2 Assume that (1) $y_u \geq 1/2$ for all u and (2) every u has a unique closest point u' other than u.

- Show that $|X| \leq 2k$.
- Prove that $x_{uu'} = 1 - y_u$ and $x_{uv} = 0$ for all $v \notin \{u, u'\}$.

Problem 3 Let C^* be an optimal set of centers and OPT be the cost of the optimal clustering. For every u, define Δ_u as follows: Δ_u is the smallest Δ such that $|B(u, \Delta/2)| \cdot \Delta/2 \geq OPT$. Prove that $d(u, C^*) \leq \Delta_u$ for all u. Conclude that we can add constraint $x_{uv} = 0$ if $d(u,v) > \Delta_u$ to the LP.