Optimization **Summer 2010**

Lecture 2 — April 14

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2.1 Basic notation

Let us review some basic definitions and notation of matrices and vectors.

Definition 2.1. *A matrix* $\mathbf{A} \in \mathcal{R}^{n\times m}$ *is defined by numbers* $a_{ij} \in \mathcal{R}$ *where* $1 \le i \le n$ *and* $1 \le j \le m$*:*

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

Definition 2.2. *The transpose of a matrix* $\mathbf{A} \in \mathcal{R}^{n\times m}$ *is the matrix* $\mathbf{B} \in \mathcal{R}^{m\times n}$ *such that* $a_{ij} = b_{ji}$ *for all* $1 \le i \le n$ *and* $1 \le j \le m$*. We denote the transpose of* $\mathbf{A}$ *by* $\mathbf{A}'$*.*

Definition 2.3. *The product of two matrices* $\mathbf{A} \in \mathcal{R}^{n\times k}$ *and* $\mathbf{B} \in \mathcal{R}^{k\times m}$ *is the matrix* $\mathbf{C} \in \mathcal{R}^{n\times m}$ *such that* $c_{ij} = \sum_{p=1}^{k} a_{ip}\, b_{pj}$ *for all* $1 \le i \le n$ *and* $1 \le j \le m$*.*

Always remember that to be able to multiply matrix **A** and **B**, the number of columns of **A** must equal the number of rows of **B**.

Definition 2.4. *A vector* $\mathbf{x} \in \mathcal{R}^n$ *is just a* $n \times 1$ *matrix and we write*

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \text{or} \quad x = (x_1, \ldots, x_n).$$

By default vectors will be *column vectors* as above. An n dimensional *row vector* will be a matrix of dimension $1 \times n$, that is, the transpose of a column vector.

Definition 2.5. *The dot product of two vectors* $\mathbf{x}, \mathbf{y} \in \mathcal{R}^n$ *is* $\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i\, y_i$*.*

When viewing vectors $\mathbf{x}, \mathbf{y} \in \mathcal{R}^n$ as matrices we sometimes write $\mathbf{x}'\mathbf{y}$ instead of $\mathbf{x} \cdot \mathbf{y}$.

2.2 Polyhedra

Definition 2.6. *A polyhedron is the set* $\{x \in \mathcal{R}^n : \mathbf{A}\mathbf{x} \ge \mathbf{b}\}$ *defined by a matrix* $\mathbf{A} \in \mathcal{R}^{m\times n}$ *and a vector* $\mathbf{b} \in R^m$*.*

We'll say that the polyhedron $P \subseteq \mathcal{R}^n$ is *bounded* if $P \subseteq [-k, k]^n$ for some $k > 0$, and *unbounded* otherwise.

Definition 2.7. *Let* $\mathbf{a}$ *be a non-zero vector in* $\mathcal{R}^n$ *and* $\mathbf{b} \in \mathcal{R}$. *Then* $\{\mathbf{x} \in \mathcal{R}^n : \mathbf{a}'\mathbf{x} = \mathbf{b}\}$ *is called a hyperplane and* $\{\mathbf{x} \in \mathcal{R}^n : \mathbf{a}'\mathbf{x} \leq \mathbf{b}\}$ *is called a halfspace.*

Notice that in $\mathcal{R}^2$ a hyperplane is a line and a halfspace is a halfplane.

Halfspaces are the building blocks of polyhedra: A polyhedron is nothing but the intersection of a bunch of halfspaces.

Definition 2.8. *A set* $S \subseteq \mathcal{R}^n$ *is convex if for all* $\mathbf{x}, \mathbf{y} \in S$ *and* $\lambda \in [0, 1]$ *the vector* $\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}$ *belong to* S *as well.*

Theorem 2.9. *Any polyhedron is a convex set.*

Proof: It is not hard to show that convex sets are closed under intersections and that halfspaces are convex. It follows then that polyhedra are always convex. □

2.3 Extreme points, vertices, and basic feasible solutions

Definition 2.10. *Let* P *be a polyhedron. A vector* $\mathbf{x} \in P$ *is an extreme point of* P *if* $\nexists\, \mathbf{y}, \mathbf{z} \in P$ *such that* $\mathbf{y}, \mathbf{z} \neq \mathbf{x}$ *and* $\mathbf{x}$ *is a convex combination of* $\mathbf{y}$ *and* $\mathbf{z}$.

Definition 2.11. *Let* P *be a polyhedron. A vector* $\mathbf{x} \in P$ *is a vertex of* P *if* $\exists\, \mathbf{c}$ *such that* $\mathbf{c} \cdot \mathbf{x} < \mathbf{c} \cdot \mathbf{y}$ *for all* $\mathbf{y} \in P - \mathbf{x}$.

Given a polyhedron $P = \{\mathbf{x} \in \mathcal{R}^n : \mathbf{a_i} \cdot \mathbf{x} \geq b_i \text{ for } i = 1, \ldots, m\}$ and a vector $\mathbf{x} \in \mathcal{R}^n$, we will say that the ith constraint is active if $\mathbf{a_i} \cdot \mathbf{x} = b_i$.

Definition 2.12. *Let* P *be a polyhedron. Then* $\mathbf{x} \in P$ *is a basic feasible solution if the set of active constraints at* $\mathbf{x}$ *has full rank.*

Theorem 2.13. *Let* P *be a non-empty polyhedron and* $\mathbf{x} \in P$. *The following statements about* x *are equivalent:*

1. $\mathbf{x}$ *is a vertex.*
2. $\mathbf{x}$ *is an extreme point.*
3. $\mathbf{x}$ *is a basic feasible solution.*

2.4 Polyhedra in standard form

It will be convenient for our algorithm for deal with polyhedra in standard form. Such polyhedra are defined by a matrix $\mathbf{A} \in \mathcal{R}^{m \times n}$ having m linearly independent rows and a vector $\mathbf{b} \in R^m$.

$$P = \{\mathbf{x} \in \mathcal{R}^n : \mathbf{A}\mathbf{x} = \mathbf{b} \text{ and } \mathbf{x} \geq \mathbf{0}\} \text{ is a polyhedron in standard form.}$$

Definition 2.14. *Given a polyhedron* $P \subseteq \mathcal{R}^n$ *in standard form, the set* $B \subseteq [1, n]$ *is a basis if* $|B| = m$ *and* $\mathbf{A_B}$ *(the columns of* $\mathbf{A}$ *indexed by* B*) have full rank. The vector* $\mathbf{x}$ *defined as* $\mathbf{x_B} = \mathbf{A_B^{-1}}\mathbf{b}$ *and* $\mathbf{x}_{[\mathbf{1},\mathbf{n}]\setminus\mathbf{B}} = \mathbf{0}$ *is the basic solution induced by the basis* B.

Notice that if the basic solution $\mathbf{x}$ also happens to be feasible it agrees with our previous definition of basic feasible solution.