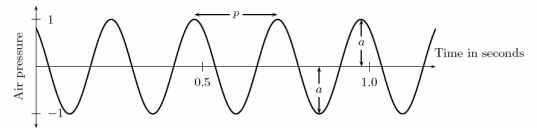


Signals and Fourier Transform



Signals

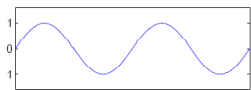
Sinusoidal



Signals

Sinusoidal $f(t) = A \sin(2\pi(\omega t - \varphi))$ for $t \in [0, 2]$

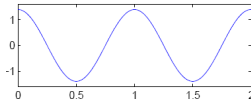
$A = 1, \omega = 1, \varphi = 0$



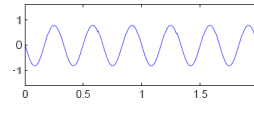
$A = 1, \omega = 3, \varphi = 0$



$A = 1.4, \omega = 1, \varphi = 0.25$

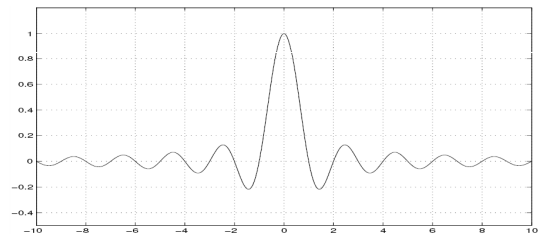


$A = 0.8, \omega = 3, \varphi = 0.5$



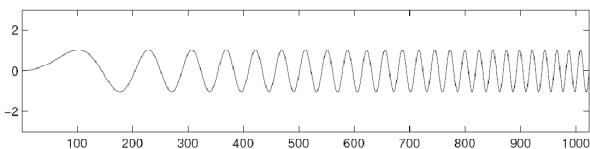
Signals

Sinc-function $\text{sinc}(t) := \begin{cases} \frac{\sin \pi t}{\pi t} & \text{if } t \neq 0 \\ 1 & \text{if } t = 0 \end{cases}$



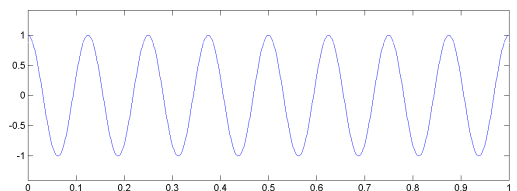
Signals

Chirp signal



Sampling

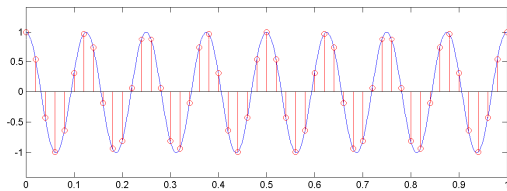
Original CT signal



Sampling

Original CT signal

DT signal sampled with 50 Hz

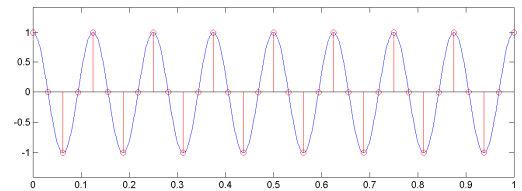


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Sampling

Original CT signal

DT signal sampled with 32 Hz

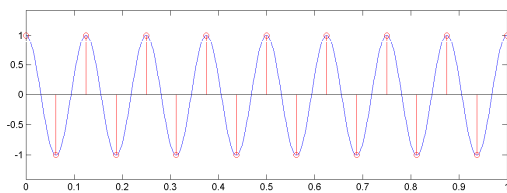


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Sampling

Original CT signal

DT signal sampled with 16 Hz

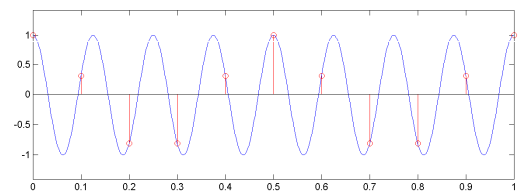


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Sampling

Original CT signal

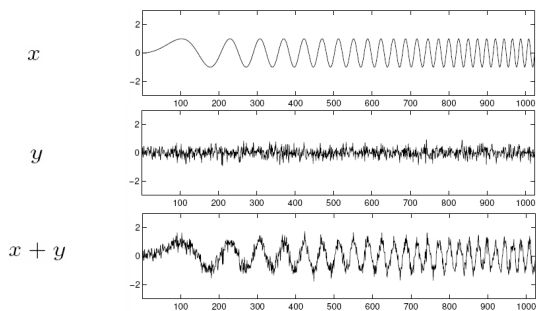
DT signal sampled with 10 Hz



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Signals

Superposition



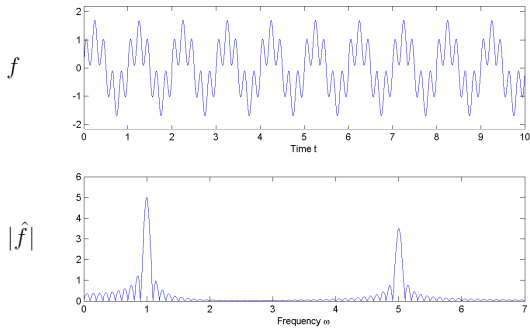
11

Fourier Transform

| Signal space | $L^2(\mathbb{R})$ | $L^2([0, 1])$ | $\ell^2(\mathbb{Z})$ |
|-------------------------------|--|--|---|
| Inner product | $\langle f g \rangle = \int_{t \in \mathbb{R}} f(t)\overline{g(t)}dt$ | $\langle f g \rangle = \int_{t \in [0, 1]} f(t)\overline{g(t)}dt$ | $\langle f g \rangle = \sum_{n \in \mathbb{Z}} x(n)\overline{y(n)}$ |
| Norm | $\ f\ _2 = \langle f f \rangle^{\frac{1}{2}}$ | $\ f\ _2 = \langle f f \rangle^{\frac{1}{2}}$ | $\ x\ _2 = \langle x x \rangle^{\frac{1}{2}}$ |
| Definition | $L^2(\mathbb{R}) := \{f: \mathbb{R} \rightarrow \mathbb{C} \mid \ f\ _2 < \infty\}$ | $L^2([0, 1]) := \{f: [0, 1] \rightarrow \mathbb{C} \mid \ f\ _2 < \infty\}$ | $L^2(\mathbb{Z}) := \{f: \mathbb{Z} \rightarrow \mathbb{C} \mid \ x\ _2 < \infty\}$ |
| Elementary frequency function | $\mathbb{R} \rightarrow \mathbb{C}$ $t \mapsto e^{2\pi i \omega t}$ | $[0, 1] \rightarrow \mathbb{C}$ $t \mapsto e^{2\pi i k t}$ | $\mathbb{Z} \rightarrow \mathbb{C}$ $n \mapsto e^{2\pi i \omega n}$ |
| Frequency parameter | $\omega \in \mathbb{R}$ | $k \in \mathbb{Z}$ | $\omega \in [0, 1]$ |
| Fourier representation | $f(t) = \int_{\omega \in \mathbb{R}} c_{\omega} e^{2\pi i \omega t} d\omega$ | $f(t) = \sum_{k \in \mathbb{Z}} c_k e^{2\pi i k t}$ | $x(n) = \int_{\omega \in [0, 1]} c_{\omega} e^{2\pi i \omega n} d\omega$ |
| Fourier transform | $\hat{f}: \mathbb{R} \rightarrow \mathbb{C}$ $\hat{f}(\omega) = \int_{t \in \mathbb{R}} f(t) e^{-2\pi i \omega t} dt$ $c_{\omega} = \hat{f}(\omega)$ | $\hat{f}: \mathbb{Z} \rightarrow \mathbb{C}$ $\hat{f}(k) = \int_{t \in [0, 1]} f(t) e^{-2\pi i k t} dt$ $c_k = \hat{f}(k)$ | $\hat{x}: [0, 1] \rightarrow \mathbb{C}$ $\hat{x}(\omega) = \sum_{n \in \mathbb{Z}} x(n) e^{-2\pi i \omega n}$ $c_{\omega} = \hat{x}(\omega)$ |

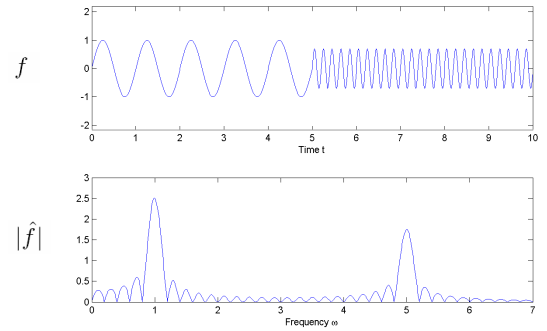
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Fourier Transform



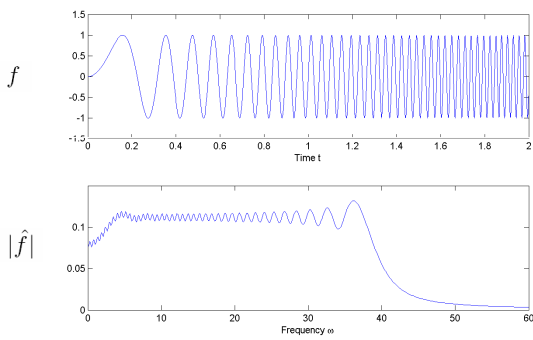
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Fourier Transform



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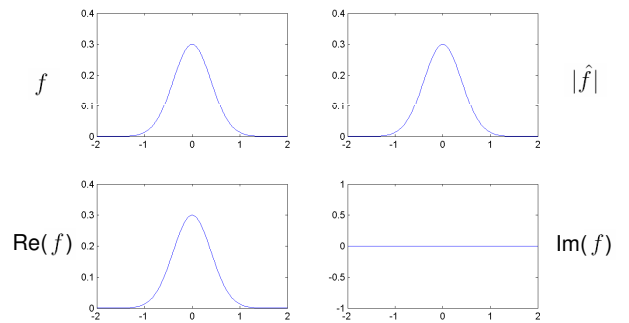
Fourier Transform



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Fourier Transform

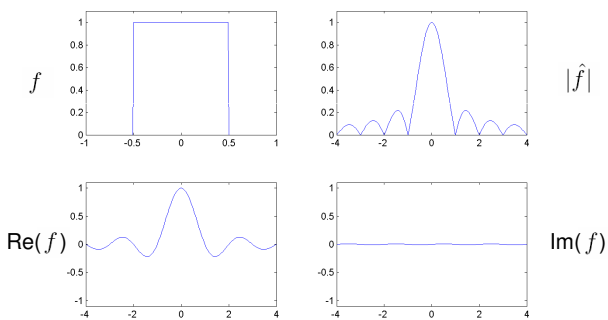
Gaussian function $f(t) = (2\pi)^{-\frac{1}{2}}\pi^{-\frac{1}{4}}e^{-\pi t^2}$



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Fourier Transform

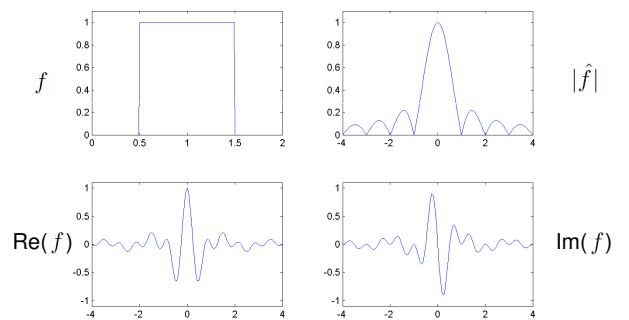
Box function



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Fourier Transform

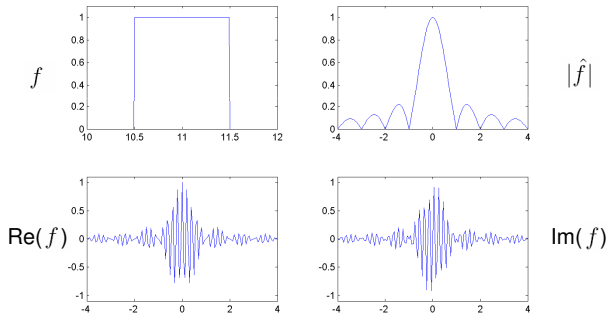
Box function (translated)



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Fourier Transform

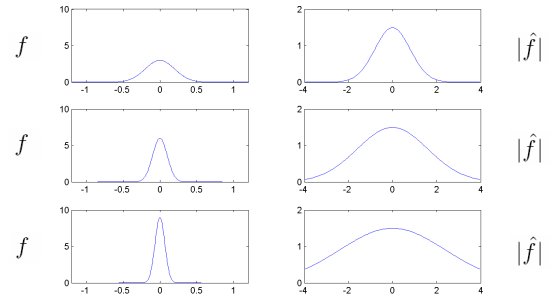
Box function (translated)



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Fourier Transform

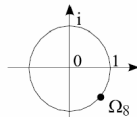
Dirac sequence



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Discrete Fourier Transform (DFT)

$$\Omega_N := e^{-2\pi i/N}$$



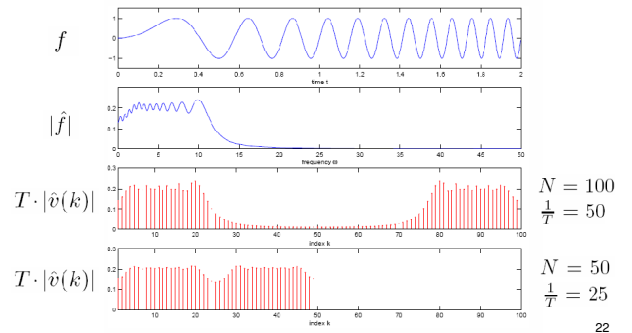
$$\text{DFT}_N := \frac{1}{\sqrt{N}} (\Omega_N^{kj})_{0 \leq k, j < N}$$

$$= \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ 1 & \Omega_N & \dots & \Omega_N^{(N-2)} & \Omega_N^{(N-1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \Omega_N^{(N-2)} & \dots & \Omega_N^{(N-2)(N-2)} & \Omega_N^{(N-2)(N-1)} \\ 1 & \Omega_N^{(N-1)} & \dots & \Omega_N^{(N-1)(N-2)} & \Omega_N^{(N-1)(N-1)} \end{pmatrix}$$

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Discrete Fourier Transform (DFT)

$$v(k) = f(Tk), \quad k \in [0 : N - 1], \quad \hat{v} = \text{DFT}_N(v)$$



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Fast Fourier Transform (FFT)

$$N = 2M$$

$$\text{DFT}_N \cdot \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{N-1} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \text{id}_M & \Delta_M \\ \text{id}_M & -\Delta_M \end{pmatrix} \begin{pmatrix} \text{DFT}_M & 0 \\ 0 & \text{DFT}_M \end{pmatrix} \begin{pmatrix} v_0 \\ v_2 \\ \vdots \\ v_{N-2} \\ v_1 \\ v_3 \\ \vdots \\ v_{N-1} \end{pmatrix}$$

$$\text{id}_M = \text{diag}(1, 1, \dots, 1)$$

$$\Delta_M = \text{diag}(1, \Omega_N, \dots, \Omega_N^{M-1})$$

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