1 Recap

1.1 Simplex algorithm

Let $P$ be a polyhedron that defines feasible space and $c$ be a vector of objective function. The simplex algorithm takes the following steps:

1. Start at some vertice of $P$
2. Move to the next vertice, that would improve objective function
3. Repeat step 2 until objective function cannot be improved anymore
4. Obtained vertice is the optimal solution

There are several ways to describe vertices mathematically:

- **Vertex.** There exists an objective function such that point $v$ is the unique optimal solution
- **Extreme point.** The point of polyhedron $P$ cannot be expressed as a convex combination of two other points in $P$.
- **Basic feasible solution.** There exist $n$ linearly independent constraints that are tight in this point.

2 Optimal solution at vertices

2.1 When polyhedron has vertices

Applying the simplex algorithm we want it to finish in finite time. Therefore polyhedron $P$ should not have infinite amount of vertexes. We claim that there exists an optimal solution, which is a vertice, apart from the following exceptions:
• $P$ is unbounded
• there are no feasible solutions
• $P$ does not have a vertex

We define a property of polyhedron, that detects whether a polyhedron has no vertexes.

**Definition** A polyhedron $P$ has no vertexes if it contains a line.

**Definition** $P$ contains a line if there exists a point $x \in P$ and vector $d$ denoting direction, such that every point of the form $x + \lambda d \in P$.

**Theorem** The polyhedron $P = \{x \in \mathbb{R}^n | Ax \geq b\} \neq \emptyset$. The following statements are equal:

a) $P$ has at least one extreme point

b) $P$ does not contain a line

c) there are no $n$ linearly independent constraints

**Proof:** b) $\Rightarrow$ a) Let’s construct a polyhedron. Then we start in some point and move in direction of boundary of $P$ considering we cannot have infinite amount of steps. We know that we are on the edge if some constraints are tight.

Let point $x \in P$, and define set of indexes $I = \{i | a_i^T x = b_i\}$ and set of rows, for which constraints are tight $S = \{a_i | i \in I\}$. If amount of rows in $S$ is equal to $n$, then $x$ is basic feasible solution, hence (according to the theorem from Lecture 5) $x$ is an extreme point.

Assume it’s not the case. Then $a_i$ lie in the subspace of $\mathbb{R}^n \Rightarrow \exists d \neq 0$ such that $a_i^T d = 0 \forall i \in I$. $d$ is a direction in which we want to move. Now we are interested in points of a form $x + \lambda d$.

As we move in direction of boundary, after some finite $\lambda$ some constraint has to become tight (was not tight before). Hence $\exists \lambda^* > 0$ and $j \notin I$ such that $a_j^T (x + \lambda^* d) = b_i$. If we add $a_j$ to $S$, rank of $S$ increases. Then we need the following claim to be true.

**Claim:** $a_j$ is not a linear combination os vectors in $S$.

**Proof:** By contradiction assume there exists a way to express $a_j$ as a linear combination $a_j = \sum_{i \in I} \lambda_i a_i$. We know that $a_j^T x \neq b_j$, because $j \notin I$. But also:

$$a_j^T (x + \lambda^* d) = b_j \Rightarrow a_j^T \neq b_j, a_j^T \lambda^* d \neq 0$$

but $a_j^T = 0 \forall i \in I$ by definition. So we know that $a_j^T \neq 0$ and $\sum_{i \in I} \lambda_i a_i^T d = 0 \neq a_j^T d = 0$. Contradiction.

Now get back to the proof of the theorem. The rank of $S$ increases $\text{rank}(S) < \text{rank}(S \cup \{a_j\})$. 

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We take a point \( x + \lambda d \) and move again. After we move \( n \) iterations \( \text{rank}(S') = n \) we obtain the point \( x' \) with corresponding \( S' \) with \( \text{rank}(S') = n \), hence we have \( n \) linearly independent constraints, hence \( x' \) is a vertex, hence also an extreme point.

\[ a) \Rightarrow c) \text{Suppose } x \text{ is an extreme point. That means that } x \text{ is also basic feasible solution. Hence, there are } n \text{ linearly independent constraints active at } x. \]

\[ c) \Rightarrow b) \text{Suppose we have } n \text{ linearly independent constraints } a_1 \ldots a_n. \text{Suppose by contradiction } p \text{ contains a line, means it contains all points of a form } \{ x + \lambda d | \lambda \in \mathbb{R}, d \neq 0 \}. \text{This also means that every point on this line satisfies all constraints:} \]

\begin{align*}
    a_i^T (x + \lambda d) & \geq b_i \forall i, \lambda \in \mathbb{R} \\
    a_i^T x + \lambda a_i^T d & \geq b_i \forall i, \lambda \in \mathbb{R}
\end{align*}

We want this to hold for all \( \lambda \), so that it is not possible to pick arbitrary small negative \( \lambda \), that would break the condition. Hence we need \( \lambda a_i^T d = 0 \forall i \). But if \( a_i^T d = 0 \forall i, d \neq 0 \) then \( a_i \) are not linearly independent. Contradiction.

### 2.2 Optimal solution in extreme point

**Theorem.** Let \( P \subset \mathbb{R}^n \) be polyhedron with at least one extreme point. Consider the LP \( \max \{ c^T x | x \in P \} \) and assume a (finite) optimal solution exists. Then there exists optimal solution which is an extreme point.

**Proof.** Let \( P = \{ x \in \mathbb{R}^n | Ax \geq b \} \), \( v \) is a finite optimum \( v = \max \{ c^T x | x \in P \} \). Define a new polyhedron \( Q \), that contains only those points of \( P \), that are optimal \( Q = \{ x \in \mathbb{R}^n | Ax \geq b \land c^T x = v \} \). \( P \) has an extreme point, thus it does not contain a line. \( Q \) is a subset of \( P \), thus \( Q \) also does not contain a line. Hence \( Q \) has an extreme point \( x^* \).

**Claim:** \( x^* \) is also an extreme point of \( P \).

**Proof:** By contradiction assume that \( x^* \) is not an extreme point of \( P \). Then there exist two points of \( P \), that give a linear combination of \( x^* \): \( \exists y, z \in P, \lambda \in [0, 1] \) such that \( x^* = \lambda y + (1 - \lambda)z \). As \( x^* \in Q \) we know \( v = c^T x^* = \lambda c^T y + (1 - \lambda)c^T z \). As \( v \) is optimal solution then \( c^T y \leq v, c^T z \leq v \).

If \( y \in P \setminus Q \) then \( c^T y < v \Rightarrow v = \lambda c^T y + (1 - \lambda)c^T z < v \). Then \( y \) gives strictly better solution for LP, hence \( y \in Q \land z \in Q \). Hence \( x^* \) is not an extreme point of \( Q \). Contradiction.

We know that \( x^* \) is an extreme point of \( P \) and \( x^* \in Q \). Thus \( c^T x^* = v \) and \( x^* \) is an optimal extreme point solution for LP.
3 Full rank assumption

**Theorem.** Let $P = \{ x | Ax = b, x \geq 0 \}$ where $A \in \mathbb{R}^{m \times n}$ but $\text{rank}(A) = k < m$. Assume $P \neq \emptyset$ and w.l.o.g. that rows $a_1^T, \ldots, a_k^T$ are linearly independent. Define $Q = \{ x | a_1^T x = b_1, \ldots, a_k^T x = b_k, x \geq 0 \}$. Then $Q = P$.

**Proof.** 1) Every point that satisfies $P$ also satisfies $Q \Rightarrow P \subseteq Q$.

2) Prove $Q \subseteq P$. Every row $a_i^T$ of $A$ can be expressed as $a_i^T = \sum_{j=1}^{k} a_i^T \lambda_{ij}$ for some $\lambda \in \mathbb{R}$. Because $P \neq \emptyset$ we can say let $x \in P$, $b_i = a_i^T x = \sum_{j=1}^{k} a_i^T \lambda_{ij} \forall i$. Let $y \in Q \forall i a_i^T y = \sum_{j=1}^{k} \lambda_{ij} a_i^T y = \sum_{j=1}^{k} \lambda_{ij} b_j = b_i$. Hence we know that $y$ satisfied all constraints in $Q$. Then $y \in P$ and $Q \subseteq P$.

From 1) and 2) we conclude that $Q = P$.

From now on we consider that all $A$ have full row rank.

Let $A \in \mathbb{R}^{m \times n}, m \leq n$. If $x$ is a feasible solution, then first $m$ constraints are tight, vector $x$ is $m-$dimensional. $n - m$ constraints of a form $x_j \geq 0$ have to be also tight at $x$ to satisfy linear independence. How to choose these?

4 Extreme points of LP in standard form

**Theorem.** Given LP $Ax = b, x \geq 0$ and assume that rows of $A$ are linearly independent. A vector $x \in \mathbb{R}^n$ is basic solution if and only if $Ax = b$ and there are indexes $B \subseteq \{1, \ldots, n\}, |B| = m$ such that:

- a) the columns $A_j, j \in B$ are linearly independent
- b) if $j \notin B$ then $x_j = 0$

**Proof.** 1) Direction $\Leftarrow$. Let $x^* \in \mathbb{R}^n$ such that $Ax = b$ and let $B$ be a set of indexes satisfying a) and b). Consider a system of equations:

$$Ax = b$$
$$x_j = 0 \forall j \notin B$$

We know that $b = Ax^* = \sum_{i=1}^{n} A_i x_i^*$. Let $x$ be an arbitrary solution for the system. Then $b = Ax = \sum_{i=1}^{n} A_i x = \sum_{j \in B} A_j x_j$.

By assumption we know that $A_j$ are linearly independent, hence the system has only one solution, that is $x^*$. Thus there are $n$ linearly independent **tight** constraints at $x^*$. Then by definition $x^*$ is a basic solution.

2) Direction $\Rightarrow$. Let $x$ be a basic solution. Let’s define a set of indexes $B_1$ such that $x_j \neq 0$. Consider the system of equations, that are tight at $x$. By assumption $x$ is basic solution,
hence the system has a unique solution. Thus columns $A_j, j \in B$ are linearly independent.

If not then there will be $\lambda_j \forall j \in B_1$ such that $\sum_{j \in B_1} A_j \lambda_j = 0$ where $\lambda_j$ are not all 0. Then it would be $\sum_{j \in B_1} A_j (x_j + \lambda_j) = b$ and solution $x$ is not unique in this case. Contradiction.

Since row rank is the same as column rank $|B_1| \leq m$. And we know $\text{rank}(A) = m$, thus there exist $m$ linearly independent columns. We can find $m - |B_1|$ columns $B_2$ with $B_1 \cap B_2 = 0$ such that columns represented by $B_1 \cup B_2$ are linearly independent $\Rightarrow$ a) is satisfied.

If there exist $j \notin B$ and $j \notin B_1$ then $x_j = 0$ by definition of $B_1$. b) is satisfied.