1 Simplex Algorithm

1.1 Iteration of Simplex Algorithm (a “pivot”)

1. start with basis Matrix $A_{B(1)},...,A_{B(m)} \rightarrow$ basic feasible solution $x$.

2. compute reduced costs $\bar{c}_j = c_j - c^T_B A^{-1}_B A_j$ for each nonbasic variable $x_j$.
   
   (a) if all $\bar{c}_j \geq 0$ then we are optimal.
   
   (b) choose some $j$ with $\bar{c}_j < 0$

3. compute $u = A^{-1}_B A_j = -d^T_B$. If $u \leq 0$ then the optimum is $-\infty$ and we stop.

4. Choose index $l$ such that $u_l > 0$ and
   
   $\frac{x_B(l)}{u_l} = \theta^* = min\{\frac{x_B(i)}{u_i} | i \in [m] \text{ and } u_i > 0\}$

5. Form new basis by replacing $A_{B(l)}$ with $A_j$.

1.2 Faster Implementation

1st iteration: basic feasible solution $x$, basis matrix $A_B$, compute $A^{-1}_B$.
2nd iteration: basic feasible solution $\bar{x}$, basis matrix $A_{\bar{B}}$, compute $A^{-1}_{\bar{B}}$ 
   \rightarrow derive $A^{-1}_B$ from $A^{-1}_{\bar{B}}$.

We know, that $A_B$ and $A_{\bar{B}}$ are very similar.
Idea: Are $A^{-1}_B$ and $A^{-1}_{\bar{B}}$ also similar?
We also know: \( A_B^{-1}A_B = I, \ A_B^{-1}A_B(i) = e_i, \ A_B^{-1}A_j = u \)

\[
A_B^{-1}A_B = \begin{bmatrix}
1 & 0 & \ldots & u_1 & \ldots & 0 & 0 \\
0 & \ddots & \ldots & u_2 & \ldots & 0 & 0 \\
\vdots & \vdots & 1 & \vdots & \vdots & \vdots & \vdots \\
0 & \ldots & 0 & u_l & 0 & \ldots & 0 \\
\vdots & \vdots & 1 & \vdots & \vdots & \vdots & \vdots \\
0 & \vdots & \ddots & 0 \\
0 & \ldots & 0 & u_m & \ldots & 0 & 1
\end{bmatrix}
\]

We want to find a matrix \( Q \), such that \( Q A_B^{-1}A_B = I \).

### 1.3 Elementary Row Operations

Multiply \( i \)-th row by some \( \alpha \neq 0 \)

\[ \Leftrightarrow \text{ multiplying from left with} \]

\[
Q_1 = \begin{bmatrix}
1 & 0 & \ldots & \ldots & \ldots & \ldots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & 1 & 0 & \vdots & \ddots & \vdots \\
\vdots & 0 & \alpha & 0 & 0 & \ddots & \vdots \\
\vdots & 0 & 1 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \ldots & \ldots & \ldots & 0 & 1
\end{bmatrix}
\]

This matrix is like the unit matrix, but with \( \alpha \) at position \((l, l)\). Obviously \( Q_1 \) is invertible.

Now add \( \beta \) times the \( j \)-th row to the \( i \)-th row for \( i \neq j \) to eliminate the non-diagonal components of \( u \):
\[ Q_2 = \begin{bmatrix}
1 & 0 & \ldots & \ldots & \ldots & 0 \\
0 & \ddots & \ddots & \beta & \vdots \\
\vdots & \ddots & 1 & 0 & \vdots \\
\vdots & 0 & 1 & 0 & 0 \\
\vdots & 0 & 1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \ldots & \ldots & \ldots & 0 & 1
\end{bmatrix} \]

This matrix is like the unit matrix, but with $\beta$ at position $(i,j)$.

With these elementary row operations turn $A_B^{-1}A_{\bar{B}}$ into $I$.

- For each $i \neq l$:
  - Add the $l$-th row $-\frac{u_l}{u_i}$ times to the $i$-th row.
- Multiply $l$-th row by $\frac{1}{u_l}$

In other words: find $Q_1, \ldots, Q_m$ such that:

\[
Q_m \ldots Q_2 Q_1 A_B^{-1}A_{\bar{B}} = \begin{bmatrix} 1 & \ldots & \ldots & \ldots & 0 \\
\end{bmatrix} \Rightarrow QA_B^{-1} = A_{\bar{B}}^{-1}
\]

### 1.4 Simplex: full tableau implementation

\[
\begin{array}{c|cc}
-c_B^T x_B & \bar{c}_1 & \ldots & \bar{c}_n \\
\hline
x_{B(1)} & A_B^{-1} A_1 & \ldots & A_B^{-1} A_n \\
x_{B(m)} & & & \\
\end{array}
\]

Note that $-c_B^T x_B$ is the negated objective value of $x$, $\bar{c}_1, \ldots, \bar{c}_n$ the reduced costs and the vectors $A_B^{-1} A_i = u_n$.

We call the vector $(-c_B^T x_B, \bar{c}_1, \ldots, \bar{c}_n)^T$ the 0-th row of our tableau and the vector $(-c_B^T x_B, x_{B(1)}, \ldots, x_{B(m)})^T$ the 0-th column.

### 1.5 Pivot step

1. If $\bar{c} \geq 0$ then STOP.
   Otherwise choose $j$ such that $\bar{c}_j \leq 0$. 
2. Consider \( u = A^{-1}B j \).
   If \( u \leq 0 \) then STOP.

3. For each \( i \) with \( u_i > 0 \) compute \( \frac{x_{B(i)}}{u_i} \).
   Let \( l \) be the index of a row that minimizes this ratio.

4. Column \( A_j \) enters basis,
   column \( A_{B(l)} \) leaves basis.

5. Perform elementary row operation such that:
   (a) \( u_l \) becomes 1
   (b) All other entries in the \( j \)-th column become 0, including entries in 0-th row.

1.5.1 Example

\[
\begin{align*}
\text{minimize} & \quad -10x_1 - 12x_2 - 12x_3 \\
\text{subject to} & \quad x_1 + 2x_2 + 2x_3 + x_4 = 20 \\
& \quad 2x_1 + x_2 + 2x_3 + x_5 = 20 \\
& \quad 2x_1 + 2x_2 + x_3 + x_6 = 20 \\
\end{align*}
\]

Initial solution \( x = (0, 0, 0, 20, 20, 20) \)
\( A_B = I = A_B^{-1}, c_B = 0 \)

The tableau for this LP looks like this:
\[
\begin{array}{cccccc|c}
0 & -10 & -12 & -12 & 0 & 0 & 0 \\
20 & 1 & 2 & 2 & 1 & 0 & 0 \\
20 & 2 & 1 & 2 & 0 & 1 & 0 \\
20 & 2 & 2 & 1 & 0 & 0 & 1 \\
\end{array}
\]

* This is the \( u_l \) that has to become 1. The other entries in this column (\( = u \)) have to become 0.

\[
\frac{x_{B(1)}}{u_1} = \frac{x_4}{u_1} = 20
\]
\[
\frac{x_{B(2)}}{u_2} = \frac{x_5}{u_2} = 10
\]
\[
\frac{x_{B(3)}}{u_3} = \frac{x_6}{u_3} = 10
\]

Index \( l = 2 \implies A_1 \) enters the basis, \( A_5 \) leaves the basis
\( B(1) = 4, B(2) = 1, B(3) = 6 \)
\[
\begin{array}{c|ccccc}
100 & 0 & -7 & -2 & 0 & 5 & 0 \\
10 & 0 & 0.5 & 1 & 1 & -0.5 & 0 \\
10 & 1 & 0.5 & 1 & 0 & 0.5 & 0 \\
0 & 0 & 1 & -1 & 0 & -1 & 1 \\
\end{array}
\]

1.6 Lemma 44

The elementary row operations lead to tableau

\[
\begin{pmatrix}
-c_T A_B^{-1} b \\
A_B^{-1} b
\end{pmatrix} = 
\begin{pmatrix}
c^T - c_T A_B^{-1} A \\
A_B^{-1} A
\end{pmatrix}
\]

where \( \bar{B} \) is obtained from adding \( j \) to \( B \) and removing \( B(l) \) from \( B \).

1.6.1 Proof

Entries \( A_B^{-1} b \) and \( A_B^{-1} A \):

Elementary row operations are equivalent to left-multiplying with matrix \( Q \) such that \( QA_B^{-1} = A_B^{-1} \).

0-th row:

we started with \([0|c^T] - g^T [b|A]\) with \( g^T = c_T A_B^{-1} \).

After iteration, 0-th row equals \([0|c^T] - p^T [b|A] \).

\( \Rightarrow c_j - p^T A_j = 0 \) where \( j = B(l) \)

Let \( i \neq l \), \( \bar{c}(i) = 0 \) and entry stays 0 after update.

\( c_B^T - p^T A_B = 0 \) \( \Rightarrow p^T = c_B^T A_B^{-1} \)

\( \Rightarrow 0 \)-th row equals \([0|c^T] - c_B^T A_B^{-1} [b|A] \) \( \Box \)