1 One iteration of the Simplex algorithm a.k.a. a ”pivot”

1.1 Pseudocode

1. Start with basis matrix $A_{B(1)}, \ldots, A_{B(m)} \rightarrow$ basic feasible solution $x$.

2. Compute reduced costs $\bar{c}_j = c_j - c^T_j A_B^{-1} A_j$ for each nonbasic variable $x_j$.
   
   (a) if all $\bar{c}_j \geq 0$ then we are optimal. STOP.
   
   (b) choose some $j$ with $\bar{c}_j < 0$

3. Compute $u = A_B^{-1} A_j = -d_j^T$. If $u \leq 0$ then optimum $=-\infty$ and we stop.

4. Choose index $l$ such that $u_l > 0$ and

   \[
   \frac{x_{B(l)}}{u_l} = \theta^* = \min \left\{ \frac{x_{B(i)}}{u_i} \mid i \in [m] \text{ and } u_i > 0 \right\}
   \]  

   (1)

5. Form new basis by replacing $A_{B(l)}$ with $A_j$.

1.2 Introduction

The computational most expensive operation used in a pivot is the generation of $A_B^{-1}$ which has to be recomputed whenever the basis $B$ changes. Computing the inverse of a matrix costs $O(n^3)$ by applying brute-force, e.g. Gauss-Elimination.

We observe that only one column of $B$ will change per pivot. Assume that $A_B^{-1}$ is at our disposal from the last pivot and we want to compute $A_{\bar{B}}^{-1}$ where $\bar{B}$ denotes a new basis.

The idea we get is that $A_B^{-1}$ might look similar to $A_{\bar{B}}^{-1}$ if $A_B$ and $A_{\bar{B}}^{-1}$ differ only by one column. This lecture introduces a method exploiting elementary row operations within a tableau-scheme to compute $A_{\bar{B}}^{-1}$ from $A_B^{-1}$ in $O(n^2)$. 
1.2.1 Mathematical point of view

\[ A_B^{-1}A_B = I \]
\[ A_B^{-1}A_B(i) = e_i \]
\[ A_B^{-1}A_j = u \]

\[ A_B^{-1}A_B = \begin{bmatrix}
1 & 0 & 0 & \cdots & u_1 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & u_2 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & u_3 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & u_{l-1} & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & u_l & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1
\end{bmatrix} \]

Our goal is to find a matrix Q such that \( QA_B^{-1}A_B = I \). In other words we search for elementary row operations transforming \( A_B^{-1} \) into \( A_B^{-1} \) via \( Q \) (\( QA_B^{-1} = A_B^{-1} \)).

1.3 Elementary row operations

We will use elementary row operations to setup the matrix Q. A short recap, there are three different elementary row operations:

1. Switching: A row within the matrix can be switched with another row,
2. Multiplication: Components of a row-vector may be multiplicated with a non-zero scalar \( \lambda \in \mathbb{R} \),
3. Addition: Multiples of a row \( j \) can be added to another row \( i \) where \( i \neq j \).

Starting from equation 2 our first goal is to bring the entry \((l, j) = u_l\) to 1. Multiplying \( i \)-th row by some \( \alpha \neq 0 \) is equivalent to multiply with \( Q_1 \) from left. For simplicity we assume \( l = j \) here.

\[ Q_1 = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \alpha & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1
\end{pmatrix} \]
Since $Q_1$ is a square matrix with full rank, it is invertible. Another argumentation is that $\det(Q_1) \neq 0$.

To eliminate the non-diagonal entries of $u$ a multiple of the $l$-th row can be subtracted from the $i$-th row. Written in terms of a matrix operation multiplied from left:

$$
Q_2 = \begin{pmatrix}
1 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & 1 & 0 & \cdots & \beta & 0 \\
\vdots & \vdots & \ddots & \cdots & \vdots & \vdots \\
0 & 0 & 0 & 1 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
$$

\[(4)\]

In summary, elementary row operations will help us to turn $A_B^{-1}A_B$ into $I$.

- For each $i \neq l$ add $l$-th row $-\frac{u_i}{u_l}$ times to the $i$-th row.
- Multiply $l$-th row by $\frac{1}{u_l}$

In matrix notation we need a series of matrices $Q_i$ such that:

$$
\underbrace{Q_m \cdots Q_2 Q_1}_Q A_B^{-1} A_B = I \implies QA_B^{-1} = A_B^{-1}
$$

\[(5)\]

Implementation of the elementary row operations to accelerate the Simplex method is often done in a tableau-scheme.

1.4 Simplex: A full tableau implementation

Idea: Every iteration of the Simplex algorithm maintains a tableau (matrix) which stores important properties of the current Simplex iteration.
1.5 Pivot step

1. If $\bar{c} \geq 0$ then STOP (costs already computed by previous iteration).
   Otherwise choose $j$ such that $\bar{c}_j < 0$.

2. Consider $u = A_B^{-1} A_j$ (is already in tableau from previous step).
   If $u \leq 0$ then STOP (Move to new vertex or equivalently switch basis).

3. For each $i$ with $u_i > 0$ compute $\frac{x_{B(i)}}{u_i}$.
   Let $l$ be the index of a row minimizing this ratio.

4. Column $A_j$ enters the basis and column $A_{B(l)}$ leaves the basis (We have to update the tableau now by performing elementary row operations).

5. Performing elementary row operations such that:
   Turn $u_l$ into $u_i$ by transforming $u_l$ to 1 and all other entries $i \neq l$ of the $j$-th column to 0.

1.5.1 Example

\[
\begin{array}{cccccc}
\text{min} & -10 & x_1 & -12 & x_2 & -12 & x_3 \\
\text{subject to} & 1 & x_1 & +2 & x_2 & +2 & x_3 & +1 & x_4 & = & 20 \\
& 2 & x_1 & +1 & x_2 & +2 & x_3 & +1 & x_5 & = & 20 \\
& 2 & x_1 & +2 & x_2 & +1 & x_3 & +1 & x_6 & = & 20 \\
& & x_1, & x_2, & x_3, & x_4, & x_5, & x_6 \geq & 0
\end{array}
\]

Table 1: LP for the tableau method example.

\[
\begin{array}{cccccc}
0 & -10 & -12 & -12 & 0 & 0 & 0 \\
20 & 1 & 2 & 2 & 1 & 0 & 0 \\
20 & 2 & 1 & 2 & 0 & 1 & 0 \\
20 & 2 & 2 & 1 & 0 & 0 & 1
\end{array}
\]

Table 2: Initialize tableau.

Initial solution $x = (0, 0, 0, 20, 20, 20)$, $B(1) = 4$, $B(2) = 5$, $B(3) = 6$. $A_B = I = A_B^{-1}$.

$c_B = 0$. Perform 3rd pivot step (see equation 1 or enumeration 1.5).

\[
\frac{x_{B(1)}}{u_1} = \frac{x_4}{u_1} = 20 \\
\frac{x_{B(2)}}{u_2} = \frac{x_5}{u_2} = 10 \\
\frac{x_{B(3)}}{u_3} = \frac{x_6}{u_2} = 10
\]

\[6\]
Minimal index $l = 2 \implies A_1$ enters the basis and $A_5$ leaves the basis. New basis: $B(1) = 4,$ $B(2) = 1$, $B(3) = 6$.

### Table 3: Tableau after 1st iteration.

<table>
<thead>
<tr>
<th>100</th>
<th>0</th>
<th>-7</th>
<th>-2</th>
<th>0</th>
<th>5</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0</td>
<td>0.5</td>
<td>1</td>
<td>1</td>
<td>-0.5</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>0</td>
<td>0.5</td>
<td>0</td>
<td>0</td>
<td>0.5</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>1</td>
</tr>
</tbody>
</table>

1.6 Lemma 44

The elementary row operations lead to tableau: where $\bar{B}$ is obtained from adding $j$ to $B$

$$
\begin{array}{c|c}
-c_B^T A_B^{-1} b & c^T - c_B^T A_B^{-1} A \\
A_B^{-1} b & A_B^{-1} A
\end{array}
$$

Table 4: Tableau after performing elementary row operations.

and removing $B(l)$ from $B$.

1.6.1 Proof

Consider entries of $A_B^{-1}$ and $A_B^{-1} A$:

Elementary row operations are equivalent to left-multiplying with a matrix $Q$ such that $Q A_B^{-1} B = A_B^{-1} A$.

0-th row: We started with $[0 | c^T] - g^T [b | A]$. with $g^T = c_B^T A_B^{-1}$.

After the $i$-th iteration: $[0 | c^T] - p^T [b | A] \implies c_j - p^T A_j = 0$ where $j = B(l)$. Consider columns of old basis $\to$ identity matrix. $l$-th row was zero at column $j \to$ new reduced cost stays 0. Let $i \neq l : \bar{c}_{B(i)} = 0$ and entry stays 0 after update. $c_B^T - p^T A_B = 0 \implies p^T = c_B^T A_B^{-1} \implies 0 \square$