1 Recap

The simplex algorithm – which we studied in the first half of this class – is a prominent algorithm proposed for linear programming and has shown to be very fast in practice, however, there is yet no variant of the simplex algorithm which is shown to run in polynomial time in general. On the other hand, the Ellipsoid algorithm – which was first applied to linear programming problems by Leonid Khachiyan in 1979 – is the first polynomial time algorithm for linear programming. Besides being efficient in theory, the ellipsoid algorithm is not very fast in practice (other than simplex), however, it plays an important role in theory.

We start today’s lecture by recapping the concept of a separation oracle and by stating the Ellipsoid Theorem.

1.1 Separation Oracle and the Ellipsoid Theorem

Definition 1. Separation oracle

Given a polytope $P \subseteq \mathbb{R}^n$ and a point $x^* \in \mathbb{R}^n$, a separation oracle either decides that $x^* \in P$ or finds an inequality $a^T x \leq b$ which is satisfied by all $x \in P$ but $a^T x^* > b$.

Theorem 1. Ellipsoid Theorem

Given a polytope $P \subseteq \mathbb{R}^n$. We can find a point $x \in P$ by calling the separation oracle $\text{poly}(n)$-times.

(Notice that if we have an efficient algorithm which finds a feasible point in $P$, then there also exists an efficient algorithm that computes $\max\{c^T x \mid x \in P\}$. For the remainder of this lecture we assume this to hold. A proof will be given in the tutorial.)

2 Background

2.1 Linear Algebra Background

To get started with our survey on the ellipsoid method we first need to establish some linear algebra background.
Definition 2. Let $A$ be a symmetric matrix, i.e $A^T = A$. $\lambda$ is said to be an eigenvalue of $A$ if $\exists x \neq 0$ s.t. $Ax = \lambda x$.

Theorem 2. Suppose $\lambda_1, \ldots, \lambda_n$ are eigenvalues of matrix $A$, then:

1. $\det(A) = \prod_i \lambda_i$
2. $\text{tr}(A) = \sum_i \lambda_i$

Definition 3. A matrix $A$ is said to be positive semi-definite if $\forall x. x^T A x \geq 0$.

Theorem 3. Let $A$ be an arbitrary matrix. The following are equivalent:

1. $A$ is positive semi-definite.
2. All eigenvalues of $A$ are non-negative.
3. $\exists B \text{ s.t. } A = B^T B$. ($\sim B = A^{1/2}$)

Definition 4. Ellipsoid
Let $A$ be a positive semi-definite matrix. The ellipsoid $E(A, a)$ is defined as follows:

$$E(A, a) = \{ x | (x - a)^T A^{-1} (x - a) \leq 1 \},$$

where $a$ is the center of the ellipsoid.

3 The Ellipsoid Algorithm

We start the discussion on the ellipsoid algorithm by giving a high level overview of how it works.

Given a polytope $P \subseteq \mathbb{R}^n$. We start with an initial ellipsoid $E_0$ that contains $P$. In each iteration $k$ we check if the center of the ellipsoid $E_k$ is contained in $P$. If so, we are done. If not, we use the separation oracle to find a separating hyperplane $H$ that separates the center of the ellipsoid $E_k$ from $P$. Having found such a separating hyperplane we construct a new ellipsoid $E_{k+1}$ of minimal volume which contains the set $E_k \cap H$ and start the next iteration.

The abstract algorithm looks as follows:

1. $E_0$ = initial ellipsoid containing $P$
2. While center $a_k$ of $E_k$ is not in $P$ do:
• Find a separating hyperplane $c^T x$ s.t. $P \subseteq \{ x \mid c^T x \leq c^T a_k \}$
• Construct minimum-volume ellipsoid $E_{k+1}$ which contains $E_k \cap \{ x \mid c^T x \leq c^T a_k \}$
• $k = k + 1$

To show that the algorithm above terminates we need to proof the following theorem.

**Theorem 4. Volume Reduction**

\[
\frac{\text{vol}(E_{k+1})}{\text{vol}(E_k)} \leq e^{-\frac{1}{2(n+1)}} < 1
\]

Intuitively the theorem says that the next ellipsoid $E_{k+1}$ is guaranteed to be smaller than its predecessor $E_k$.

**Corollary.** The number of iterations of the ellipsoid algorithm is at most $O(n^2 \cdot \log(\frac{R}{r}))$.

We start with a proof of the corollary but first we need some assumptions.

1. We assume that the volume reduction theorem holds.
2. The polyhedron $P$ has non-zero volume, i.e. $\exists r > 0, c \in \mathbb{R}^n$ s.t. $\text{Ball}(c, r) \subseteq P$.

**Proof.** The volume of the n-dimensional sphere is given by:

\[
\text{vol}(E_0) = \frac{\pi \frac{R}{n}^{n}}{\Gamma(\frac{n}{2}+1)}
\]

Using assumption (2) from above we get the following invariant:

\[
\forall i. P \subseteq E_i. \text{ Hence, } \text{vol}(E_i) \geq \text{vol}(\text{Ball}(c, r)) \geq \frac{\pi \frac{R}{n}^{n}}{\Gamma(\frac{n}{2}+1)}
\]

Using assumption (1) we get:

\[
\left( \frac{R}{r} \right)^n = \frac{\text{vol}(E_k)}{\text{vol}(E_0)} = \frac{\text{vol}(E_k)}{\text{vol}(E_{k-1})} \ast \frac{\text{vol}(E_{k-1})}{\text{vol}(E_{k-2})} \ast \cdots \ast \frac{\text{vol}(E_1)}{\text{vol}(E_0)} \leq e^{-\frac{k}{2(n+1)}}
\]

This gives us:

\[
e^{-\frac{k}{2(n+1)}} \leq \left( \frac{R}{r} \right)^n
\]

Taking the logarithm on both sides we get:
\[
\frac{k}{2(n+1)} \leq n \ast lg((\frac{R}{r}))
\]

Hence,

\[
k \leq 2n \ast (n + 1) \ast lg((\frac{R}{r}))
\]

Which proofs the claim.

Figure 1: A single iteration of the ellipsoid method.