Karl Bringmann and Sebastian Krinninger **Summer 2016**

Exercises for Complexity Theory of Polynomial-Time Problems

https://www.mpi-inf.mpg.de/departments/algorithms-complexity/teaching/summer16/poly-complexity/

Exercise sheet 3 Due: **Monday, June 13, 2016**

Total points: 40

Either email an electronic version of your assignment submission to gjindal@mpi-inf.mpg.de or give it to Gorav in his office at Room 425, Building E1.3. If Gorav is not in his office then you can just slide your submission under the door of his office.

You are allowed to collaborate on the exercise sheets, but you have to write down a solution on your own, using your own words. Please indicate the names of your collaborators for each exercise you solve. Further, cite all external sources that you use (books, websites, research papers, etc.).

You need to collect at least 50% of all points on exercise sheets.

Exercise 1 (*7 points*) Recall the following formal definition of subcubic reductions: Let A and B be computational problems with a common size measure n on inputs. We say that there is a subcubic reduction from A to B if there is an algorithm $\mathcal{A}$ with oracle access to B satisfying three properties:

- For every instance x of A, $\mathcal{A}(x)$ solves the problem A on x.
- For some $\gamma > 0$, $\mathcal{A}$ runs in $O(n^{3-\gamma})$ time on instances of size n.
- For every $\varepsilon > 0$ there is a $\delta > 0$ such that for every instance x of A of size n we have $\sum_i n_i^{3-\varepsilon} \leq n^{3-\delta}$, where n_i is the size of the ith oracle call to B in $\mathcal{A}(x)$.

We use the notation $A \leq B$ to denote the existence of a subcubic reduction from A to B. Prove that subcubic reductions are transitive. In other words, prove that if $A \leq B$ and $B \leq C$ then $A \leq C$.

Exercise 2 (*8 points*) The **Metricity Problem** is defined as follows: Given an $n \times n$ matrix A with entries in $\{0, \dots, \lfloor n^c \rfloor\}$ for some constant $c > 0$, decide whether $\forall i, j, k \in [n] : A_{ij} \leq A_{ik} + A_{kj}$. Prove that **Metricity Problem** is equivalent to APSP under subcubic reductions.

Hint: *Solve it using Min-Plus Product and reduce Negative Triangle to it.*

Exercise 3 (*9 points*) Recall the following problem defined on the previous exercise sheet:

Hitting Set Problem: Given two lists of n subsets over a universe U of size d, determine if there is a set in the first list that intersects every set in the second list, i.e. a "hitting set".

The **HSH** (Hitting set Hypothesis) states that the **Hitting Set Problem** cannot be solved in time $O(n^{2-\epsilon} \cdot \mathrm{poly}(d))$. Prove that **HSH** implies **OVH**.

Hint: *In the lecture, we showed a reduction from All-Pairs-Negative-Triangle to Negative-Triangle. The same kind of reduction can work here.*

Exercise 4 (*16 points*) Recall the following problem from the lecture. Let A be a matrix with entries in $\{\lfloor n^c \rfloor, -\lfloor n^c \rfloor + 1, \ldots, \lfloor n^c \rfloor\}$ for some constant $c > 0$. In the Maximum Submatrix problem the task is to find the maximum sum of all entries of any submatrix of A. A submatrix here means a choice of some consecutive rows and some consecutive columns.

a) (7 points) Describe an $O(n^3)$ time algorithm for the maximum submatrix problem (solve it directly without any reductions).

b) (9 points) A submatrix of A is called **centered** if it contains the center (entry $A_{\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor}$) of matrix A. Now the problem is to find a **Maximum Centered Submatrix**. Show that if APSP has subcubic algorithm then so does this problem, i.e, show that there is subcubic reduction from this problem to APSP.

Exercise 5 (*10 Bonus points*) Let $G = (V, E)$ be a directed weighted graph with edge weights in $\{-\lfloor n^c \rfloor, \ldots, \lfloor n^c \rfloor\}$. The **Betweenness Centrality** of a given node $v \in V$ is the number of pairs s, t such that v lies on a shortest path from s to t:

$$BC(v) = \left| \{(s,t) \mid s, t \in V \setminus \{v\}, s \neq t \colon d(s,t) = d(s,v) + d(v,t)\} \right|$$

Show that computing $BC(v)$ is equivalent to APSP under subcubic reductions.

Hint: *Modify the reductions between Radius and APSP shown in the lecture.*