An Improved Distributed Algorithm for Maximal Independent Set (MIS)

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Model:

distributed LOCAL model -
all nodes learn in k-rounds the subgraph
induced by their k-hop neighborhood (+ random bits)

Maximal Independent Set:

For a graph \( G = (V, E) \) an independent set \( S \) is a maximal
(the subgraph induced on
the set is empty)

independent set if \( \forall v \in V: \) either \( v \in S \) or \( N(v) \cap S = \emptyset \).

Result (main): MIS in \( O(\log \Delta) + 2^{O(\sqrt{\log \log n})} \) (randomized)

improving Borenboim et al. \( O(\log^2 \Delta) + 2^{O(\sqrt{\log \log n})} \)

best lower bound: \( \Omega(\frac{\log \Delta}{\log \log \Delta}) \) and \( \Omega(\sqrt{\frac{\log n}{\log \log n}}) \) Kuhn et al.

High level overview of the Alg:

Step 1: run \( O(\log \Delta + \log \frac{1}{\epsilon}) \) rounds of "local algorithm"
st. for each node \( v \), wp \( \geq 1 - \epsilon \) \( v \) is in the MIS
or one of its neighbors is.

Step 2: w.h.p the remaining graph is shattered into small components,
for which we can solve MIS deterministically. \( \Rightarrow \) the \( 2^{O(\sqrt{\log \log n})} \) term.
The "Local" algorithm.

In each round $t$:

$\forall v \in V$: $P_0(v)$ - the desired level of $v$

$V$ gets marked w.p. $P_t(v)$, if none of $v$'s neighbors is marked then $v$ joins the MIS and we remove $N^+(v)$ from $G$.

$$P_0(v) = \frac{1}{2}$$

$$d_e(v) = \sum_{u \in N(v)} P_e(u)$$ - effective degree

$$P_{t+1}(v) = \begin{cases} P_t(v) / 2, & \text{if } d_e(v) > 2 \\ \min \left\{ 2 P_t(v), \frac{1}{2} \right\}, & \text{if } d_e(v) \leq 2 \end{cases}$$

**Thm 1**: Fix $v \in V$ by round $X = p \left( \log a + \log \frac{1}{\epsilon} \right)$$v$ made its decision w.p. $\geq 1 - s$.

$\forall v \in \text{MIS or } N(v) \text{ MIS } \not\in d$

Golden rounds of $v$:

**Type 1**: $d_e(v) < 2$ and $P_e(v) = \frac{1}{2}$

$[v$ has a good chance to join the MIS$]$ 

**Type 2**: $d_e(v) \geq 1$ and at least $\frac{d_e(v)}{10}$ of its neighbors ($d_e < 2$)

$[v$ has a good chance that one of its neighbors joins the MIS$]$
Lemma (easy proof): in golden rounds $r$ has constant pr. of being removed from $G$.

$\Rightarrow$ to prove that suffices to show that a constant fraction of the rounds are golden.

Proof: $g_1$ = # golden rounds of type 1.

$g_2$ = # golden rounds of type 2.

Assume $g_1 \leq C \cdot X$ (otherwise we are done).

Define $h = \# \text{rounds } d_t(w) \leq 2 = \# \text{rounds } P_t(v) \text{ dec} \geq \# \text{rounds } P_t(v) \text{ inc}.

$\Rightarrow P_t(v) \text{ inc or dec. } \leq 2h \text{ rounds}

$\Rightarrow \geq X-2h \text{ rounds } P_t(v) = \frac{1}{2}

$\Rightarrow \geq X-3h \text{ rounds } P_t(v) = \frac{1}{2} \text{ and } d_t(w) < 2 \text{ (golden type 1)}

$g_1 \geq X-3h \text{ and } \text{so } h > \frac{1}{3} X \text{ (const. fraction of } X)\text{.}

Lemma: If $d_t(v) > 1$ and t not golden type 2j then $d_{t+1}(v) < \frac{2}{3} d_t(v)$

Proof: $d_{t+1}(v) \leq 2 \cdot \frac{1}{10} d_t(w) + \frac{1}{2} \cdot \frac{5}{10} d_t(w) < \frac{2}{3} d_t(w)$
focus on rounds for which \( d_t(v) \geq 2 \).

Here \( 2 \leq d_t(v) \leq 4 \).

Inner boundary round

A - rounds from for which \( d_t(v) \geq 2 \)

I - rounds in A which are inner-boundary.

observe \( |I| \leq g_2 \)

the rounds in \( \overline{A \setminus I} \) can be golden or non-golden.

\[ a = \# \text{ golden rounds in } \overline{A \setminus I} \quad (d_t(v) \text{ incr. by at least factor 2}) \]

\[ b = \# \text{ non-golden rounds in } \overline{A \setminus I} \quad (d_t(v) \text{ dec. by at least factor } 2/3) \]

\[ |A| = |I| + |\overline{A \setminus I}| + a + b \]

Lemma:

\[ \alpha \quad \text{or} \quad \beta \leq 2|I| + 2a + \log_{\frac{3}{2}} \Delta \]

proof: assume otherwise and obtain that \( d_t(v) < 2 \).

use:

1) \( (\frac{2}{3})^2 \cdot 2 < 1 \)
2) \( \Delta c_t(v) \leq \frac{\Delta}{2} \quad \forall t \)
3) \( \frac{\Delta}{2} \cdot (\frac{2}{3}) \log_{\frac{3}{2}} \Delta < 2 \)
Since \(|A| = h = |T| + a + b \leq g_2 + a + b\)

we obtain from Lemma \(*\) that:

\[ a \Rightarrow h - g_2 - b \Rightarrow h - g_2 = (2|T| + 2a + \log \frac{3}{2} \Delta) \]

\[ \Rightarrow h - 3g_2 - 2a = \log \frac{3}{2} \Delta \]

\[(6g_2/\Delta) \Rightarrow 3a + 3g_2 \Rightarrow h - \log \frac{3}{2} \Delta \]

and so \(6g_2 \Rightarrow h - \log \frac{3}{2} \Delta \)

\[ h - \log \frac{3}{2} \Delta \]

\[ g_2 \Rightarrow \frac{\log \frac{3}{2} \Delta}{6} \]

\[ g_2 \Rightarrow \frac{\log \frac{3}{2} \Delta}{6} \] is a constant fraction of \(X\).

\[ \square \]

**Shattering facts:**

1) in each round \(v\) decides whether to join the MIS based on random coins of \(N^+(v) \Rightarrow \) there is a lot of independence only

2) By Panconesi and Srinivasan, deterministic MIS

\[ |MIS| \leq O(\log m) \]

\[ |MIS| \leq O(\log m) \] on \(m\) nodes.