D5: Databases and Information Systems
Data Mining and Matrices, SS 2017
Homework #2: Linear algebra & SVD
Tutorial: **24 May 2015** at 10:15

You can discuss these problems with other students, but everybody must do and present their own answers. You can use computers etc. to perform the algebraic operations, but you must show the intermediate steps (and "computer said so" is never a valid answer). You are of course free to use material from the Internet, but again, you must present the intermediate steps and you must also be able to explain why the steps are valid and why you chose them. You can mark an answer even if it is not complete or correct, as long as you have made significant progress towards solving it. Note, however, that the TA does the final decision on whether your solution is complete (or correct) enough for a mark.

NB! The tutorial will take place from 16:15 onwards on room 021, building E1.4.

Problem 1 (Frobenius norm and SVD). A function $f\colon \mathbb{R}^{n\times m} \to \mathbb{R}$ is *unitarily invariant* if

$$f(\boldsymbol{A}) = f(\boldsymbol{U}\boldsymbol{A}\boldsymbol{V}) \tag{1.1}$$

for any $\boldsymbol{A} \in \mathbb{R}^{n\times m}$ and for any orthogonal $\boldsymbol{U} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{V} \in \mathbb{R}^{m\times m}$.

Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and let $\boldsymbol{A}_k = \boldsymbol{U}_k\boldsymbol{\Sigma}_k\boldsymbol{V}_k^T$ be the rank-k truncated SVD of it, for some $k < n$. Use the fact that the Frobenius norm is unitarily invariant to show that if $(\sigma_i)_{i=1}^n$ are the singular values of $\boldsymbol{A}$, then

$$\|\boldsymbol{A} - \boldsymbol{A}_k\|_F^2 = \sum_{i=k+1}^{n} \sigma_i^2 \; . \tag{1.2}$$

Problem 2 (Orthogonal matrices as isometries). An *isometry* of Euclidean space $\mathbb{R}^n$ is a mapping that preserves the distances. That is, $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is an isometry if and only if $\|\boldsymbol{x} - \boldsymbol{y}\|_2 = \|f(\boldsymbol{x}) - f(\boldsymbol{y})\|_2$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$. Show that if $\boldsymbol{Q} \in \mathbb{R}^{n\times n}$ is an orthogonal matrix, it defines an isometric mapping.

Problem 3 (Unitarily invariant Frobenius). In Problem 1 we claimed that the Frobenius norm is unitarily invariant. Prove this claim.

Problem 4 (KLT). Let $\boldsymbol{A} \in \mathbb{R}^{n\times m}$ and let $\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$ be the SVD of it. In the lecture it was said that, assuming $\boldsymbol{A}$ is normalized, its k-dimensional Karhunen–Lòeve transform can be computed as

$$\tilde{\boldsymbol{A}}_k = \boldsymbol{A}\boldsymbol{V}_k \; , \tag{4.1}$$

where $\boldsymbol{V}_k$ contains the first k columns of $\boldsymbol{V}_k$. Show that we can equivalently compute it as

$$\tilde{\boldsymbol{A}}_k = \boldsymbol{U}_k\boldsymbol{\Sigma}_k \; , \tag{4.2}$$

where $\boldsymbol{U}_k$ contains the first k columns of $\boldsymbol{U}$ and $\boldsymbol{\Sigma}_k$ is the principal k-by-k submatrix of $\boldsymbol{\Sigma}$.

Problem 5 (Singular values and rank). Let $\boldsymbol{H}_n = (h_{ij})$ be an n-by-n matrix such that $h_{ij} = 1/(i+j-1)$ (where $i, j \in \{1, 2, \ldots, n\}$). Use your favorite linear algebra software (e.g. R or Matlab) to compute the SVD of the 12-by-12 matrix $\boldsymbol{H}_{12}$. Does the matrix have full rank? Why?

Problem 6 (Singular values and condition numbers). Consider again the family of matrices $\boldsymbol{H}_n$ of the previous question. Compute (and report) the condition numbers for $\boldsymbol{H}_4$, $\boldsymbol{H}_5$, and $\boldsymbol{H}_6$. Do you think (some of) these condition numbers are high? Why?