$E = \{ g(a) \approx b,\ a \approx b \}$ $\qquad \Sigma = \{ S, \{a, b, g\}, \emptyset \}$

$E \models g(b) \approx b$ $\qquad$ TRS R s.t. $g(b){\downarrow_R}\ b{\downarrow_R}$

First try: $R_0 = \{ g(a) \mapsto b,\ a \mapsto b \}$

then $g(b){\downarrow_{R_0}} = \underbrace{g(b),\ b{\downarrow_{R_0}} = b}_{\neq}$

$R_1 = R_0 \cup \{ \boxed{g(b) \to b} \}$

$g(b){\downarrow_{R_1}} = b = b{\downarrow_{R_1}}$

$$E_c = \{ g(a) \approx b,\ a \approx b \} \qquad \Sigma = \{ S, \{a, b, g\}, \phi \}$$

$$E_0 \models g(b) \approx b$$

$$E_0 \Rightarrow^{\text{Symmetry}}_{E_0} E_c \cup \{ b \approx a \} = E_1$$

$$\Rightarrow^{\text{Transitivity}}_{E_1} E_1 \cup \{ g(a) \approx a \} = E_2$$

$$\Rightarrow^{\text{Congruence}}_{E} E_2 \cup \{ g(a) \approx g(b) \} = E_3$$

$$\Rightarrow^{\text{Symmetry}}_{E} E_3 \cup \{ g(b) \approx g(a) \} = E_4$$

$$\Rightarrow^{\text{Transitivity}}_{E} E_4 \cup \{ g(b) \approx b \}$$

$$E = \{ g(a) \approx b,\ a \approx b \}$$

$$[g(a)] = \{ a, b, g(a), g(b), g(g(a)), g(g(b)) \ldots \}$$

$$= [g(b)] = [a] \ldots$$

$$S^I = \{ [a] \}$$

local confluence

$$
\begin{tikzcd}
 & t \arrow[dl] \arrow[dr] & \\
t_1 \arrow[dr, "*"] & & t_2 \arrow[dl, "*"'] \\
 & s &
\end{tikzcd}
$$

disjoint redexes

$$
\begin{tikzcd}
 & f(a,b) \arrow[dl, "(1)"'] \arrow[dr, "(2)"] & \\
f(c,b) \arrow[dr, "(2)"'] & & f(a,d) \arrow[dl] \\
 & f(c,d) &
\end{tikzcd}
$$

$a \mapsto c \quad (1)$

$b \mapsto d \quad (2)$

overlap

$$\begin{tikzcd}[column sep=small] & f(a,b) \arrow[dl, "(1)"'] \arrow[dr, "(2)"] & \\ f(a,c) & & a \end{tikzcd}$$

$\underline{b} \mapsto c$ (1)

$f(x,\underline{b}) \mapsto x$ (2)

$\Rightarrow$ is not confluent

$$\begin{tikzcd}[column sep=small] & f(a,b) \arrow[dl, "(1)"'] \arrow[dr, "(2)"] & \\ f(c,b) \arrow[rr, bend right] & & a \end{tikzcd}$$

$a \mapsto c$ (1)

$f(x,b) \mapsto a$ (1)

$\Rightarrow$ is confluent

Critical pair $\quad l_1 \to r_1 \qquad \langle r_1\sigma, (l_1\sigma)[r_2\sigma]_p \rangle$
$\qquad\qquad\qquad\;\; l_2 \to r_2$

$$R = \{ \underbrace{f(f(x,x),y)}_{l_1} \to \underbrace{g(x)}_{r_1},\ \underbrace{f(z,y')}_{l_2} \to \underbrace{z}_{r_2} \}$$

$p = \varepsilon$

1) $f(f(x,x),y)$ and $f(z,y')$ overlap

$$\sigma_1 = \{ z \mapsto f(x,x),\ y' \mapsto y \}$$

$$cp = \langle g(x)\sigma_1, z\sigma_1 \rangle = \langle g(x), f(x,x) \rangle$$

2) $f(x,x)$ and $f(z,y')$ overlap, $\sigma_2 = \{ z \mapsto x,\ y' \mapsto x \}$

$$cp = \langle g(x)\sigma_2, f(z\sigma, y) \rangle = \langle g(x), f(x,x) \rangle$$