Löwenheim - Skolem (1920):

If a first-order sentence is valid over an infinite domain, then it is valid over <u>any</u> infinite domain

$U = \mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$ ⇝ you cannot define the natural numbers in FOL

It is possible with FOL (LRA)   LRA ⇝ Domain is $\mathbb{Q}$

$Nat(0)$

$y = x + 1 \wedge Nat(x) \rightarrow Nat(y)$

$0 < x < 1 \rightarrow \neg Nat(x)$

$x < 0 \rightarrow \neg Nat(x)$

$y + 1 = x \wedge x > 1 \wedge \neg Nat(y) \rightarrow \neg Nat(x)$

$\Big\} \; \mathbb{N} \subseteq Nat^{\mathcal{A}}$

$\mathbb{N} = Nat^{\mathcal{A}}$

$P(1+1)$

$\neg P(2)$

$x \neq 1+1 \lor \underline{P(x)}$

$y \neq 2 \lor \underline{\neg P(y)}$

$\xrightarrow{\text{Resolve}} \quad 2 \neq x \lor x \neq 1+1 \rightsquigarrow$

$\underbrace{2 = x \land x = 1+1}_{\text{has a solution}} \to \bot$

$\downarrow$

problem is unsat

$x > 2 \to P(x)$

$x < 3 \to \neg P(x)$

$\xrightarrow{\text{Resolve}} \quad \underbrace{x > 2 \land x < 3}_{\text{has solution}} \to \bot$

has solution $\to$ problem is unsat

$x > 3 \to P(x)$

$x < 2 \to \neg P(x)$

$\xrightarrow{\text{Resolve}} \quad \underbrace{x > 3 \land x < 2}_{\text{has no solution}} \to \bot$

has no solution $\to$ clause is tautology

$$P(f(x+g(1))) \lor \neg P(g(y-3z))$$ | Not abstracted

$$\downarrow$$

$$y_1 \not\approx 1 \lor g(y_1) \not\approx y_2 \lor y_3 \not\approx x + y_2 \lor P(f(y_3))$$
$$\lor\ y_4 \not\approx y - 3z \lor \neg P(g(y_4))$$ | abstracted

$$\downarrow$$

$$y \approx 1 \land y_3 \approx x + y_2 \land y_4 \approx y - 3z \;\|\;$$ | constrained clause form

$$g(y_1) \not\approx y_2 \lor P(f(y_3)) \lor \neg P(g(y_4))$$

$N' = \{\Delta_1 \| C_1, \ldots, \Delta_n \| C_n\} \subseteq N$

If N is unsat, then there exists a subset N' of clauses that can be resolved together to get $\Delta \| \bot$ where Δ has a satisfiable solution

iff

$N^* = \{\Delta_1' \| C_1', \ldots, \Delta_n' \| C_n'\} \subseteq gnd(N, \mathbb{Q})$

there exists a subset N^* of $gnd(N, \mathbb{Q})$ that can be resolved together to get $\Delta \| \bot$ where Δ simplify to true

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$M = P(a,b),\ a \leq b,\ \neg P(a,a),\ a \leq 0,\ b \geq 2,$
$\neg P(b,b),\ P(b,a)$

$M = P(a),\ a < b,\ \neg P(b)$

$\beta_1 = \{ a \mapsto \cancel{0},\ b \mapsto 1 \}$

$\beta_2 = \{ a \mapsto 1,\ b \mapsto 2 \}$

$(1)\ x = 0 \mid\mid P(x)$

$(2)\ x = 0 \mid\mid \neg P(x)$

$(3)\ 1 \leq x \leq 2 \mid\mid P(x)$

$\beta = \{ a, b, \ldots \}$

$\Rightarrow$ Propagate* $P(a)^{(3)\{x \mapsto a\}},\ 1 \leq a \leq 2,$

$P(b)^{(3)\{x \mapsto b\}},\ 1 \leq b \leq 2, \ldots,$

$\overline{M_1 = P(a)^{(3)\{x \mapsto 1\}},\ 1 \leq a \leq 2}$

$\beta = \{ a \}$

$M_2 = P(a)^{(1)\{x \mapsto a\}},\ a = 0$

$\neg P(a) \vee a < 1 \vee a > 2$