5.2.1 Theorem (Superposition Soundness)

All inference rules of the superposition calculus are sound, i.e., for every rule $N \cup \{ C_1, \ldots, C_n \} \Rightarrow N \cup \{ C_1, \ldots, C_n \} \cup \{ D \}$ it holds that $\{ C_1, \ldots, C_n \} \models D$.

5.2.2 Definition (Abstract Redundancy)

A clause $C$ is redundant with respect to a clause set $N$ if for all ground instances $C\sigma$ there are clauses $\{ C_1, \ldots, C_n \} \subseteq N$ with ground instances $C_1\tau_1, \ldots, C_n\tau_n$ such that $C_i\tau_i \prec C\sigma$ for all $i$ and $C_1\tau_1, \ldots, C_n\tau_n \models C\sigma$.

Given a set $N$ of clauses $\text{red}(N)$ is the set of clauses redundant with respect to $N$. 
5.2.3 Definition (Saturation)

A clause set $N$ is saturated up to redundancy if for every derivation $N \setminus \text{red}(N) \Rightarrow_{\text{SUPE}} N \cup \{C\}$ it holds $C \in (N \cup \text{red}(N))$. 

saturated up to redundancy
5.2.4 Definition (Partial Model Construction)

Given a clause set $N$ and an ordering $\succ$ a (partial) model $N_\mathcal{I}$ can be constructed inductively over all ground clause instances of $N$ as follows:

$$N_C := \bigcup_{D \in \text{grd}(\Sigma, N) \prec C} E_D$$

$$N_\mathcal{I} := \bigcup_{C \in \text{grd}(\Sigma, N)} N_C$$

where $N_D$, $N_\mathcal{I}$, $E_D$ are also considered as rewrite systems with respect to $\succ$. If $E_D \neq \emptyset$ then $D$ is called *productive*. 
\( E_D \) := \[
\begin{cases}
\{ s \approx t \} & \text{if } D = D' \lor s \approx t, \\
(i) \ s \approx t \text{ is strictly maximal in } D \\
(ii) \ s \succ t \\
(iii) \ D \text{ is false in } N_D \\
(iv) \ D' \text{ is false in } N_D \cup \{ s \rightarrow t \} \\
(v) \ s \text{ is irreducible by } N_D \\
(vi) \text{ no negative literal is selected in } D' \\
\emptyset & \text{otherwise}
\end{cases}
\]
5.2.5 Lemma (Maximal Terms in Productive Clauses)
If $E_C = \{ s \rightarrow t \}$ and $E_D = \{ l \rightarrow r \}$, then $s \succ l$ if and only if $C \succ D$.

5.2.6 Corollary (Partial Models are Convergent Rewrite Systems)
The rewrite systems $N_C$ and $N_I$ are convergent.
5.2.7 Lemma (Ordering Consequences in Productive Clauses)
If $D \preceq C$ and $E_C = \{s \rightarrow t\}$, then $s \succ r$ for every term $r$ occurring in a negative literal in $D$ and $s \succeq l$ for every term $l$ occurring in a positive literal in $D$.

5.2.8 Corollary (Model Monotonicity True Clauses)
If $D$ is true in $N_D$, then $D$ is true in $N_I$ and $N_C$ for all $C \succ D$. 
5.2.9 Corollary (Model Monotonicity False Clauses)

If $D = D' \lor s \approx t$ is productive, then $D'$ is false and $D$ is true in $N_I$ and $N_C$ for all $C \succ D$.

5.2.10 Lemma (Lifting Single Clause Inferences)

Let $C$ be a clause and let $\sigma$ be a substitution such that $C\sigma$ is ground. Then every equality resolution or equality factoring inference from $C\sigma$ is a ground instance of an inference from $C$. 
5.2.11 Lemma (Lifting Two Clause Inferences)

Let \( D = D' \lor u \approx v \) and \( C = C' \lor [\neg]s \approx t \) be two clauses (without common variables) and let \( \sigma \) be a substitution such that \( D_\sigma \) and \( C_\sigma \) are ground. If there is a superposition inference between \( D_\sigma \) and \( C_\sigma \) where \( u_\sigma \) and some subterm of \( s_\sigma \) are overlapped and \( u_\sigma \) does not occur in \( s_\sigma \) at or below a variable position of \( s \) then the inference is a ground instance of a superposition inference from \( D \) and \( C \).
5.2.12 Theorem (Model Construction)

Let $N$ be a set of clauses that is saturated up to redundancy and does not contain the empty clause. Then for every ground clause $C_{\sigma} \in \text{grd}(\Sigma, N)$ it holds that:

1. $E_{C_{\sigma}} = \emptyset$ if and only if $C_{\sigma}$ is true in $N_{C_{\sigma}}$.
2. If $C_{\sigma}$ is redundant with respect to $\text{grd}(\Sigma, N)$ then it is true in $N_{C_{\sigma}}$.
3. $C_{\sigma}$ is true in $N_I$ and in $N_D$ for every $D \in \text{grd}(\Sigma, N)$ with $D \succ C_{\sigma}$.
5.2.13 Lemma (Lifting Models)
Let $N$ be a set of clauses with variables and let $\mathcal{A}$ be a term-generated $\Sigma$-algebra. Then $\mathcal{A}$ is a model of $\text{grd}(\Sigma, N)$ if and only if it is a model of $N$.

5.2.14 Theorem (Refutational Completeness: Static View)
Let $N$ be a set of clauses that is saturated up to redundancy. Then $N$ has a model if and only if $N$ does not contain the empty clause.
5.2.15 Definition (Superposition Run)

A *run* of the superposition calculus is a derivation

\[ N_0 \Rightarrow_{SR} N_1 \Rightarrow_{SR} N_2 \Rightarrow_{SR} \ldots, \]

so that

1. \( N_i \models N_{i+1} \), and
2. all clauses in \( N_i \setminus N_{i+1} \) are redundant with respect to \( N_{i+1} \).

For a run, \( N_\infty = \bigcup_{i \geq 0} N_i \) and \( N_* = \bigcup_{i \geq 0} \bigcap_{j \geq i} N_j \). The set \( N_* \) of all *persistent* clauses is called the *limit* of the run.
5.2.16 Lemma (Redundancy is Monotone)
If $N \subseteq N'$, then $\text{red}(N) \subseteq \text{red}(N')$.

5.2.17 Lemma (Redundant Clauses Do not Contribute)
If $N' \subseteq \text{red}(N)$, then $\text{red}(N) \subseteq \text{red}(N \setminus N')$. 
5.2.18 Lemma (Redundancy is Monotone in Runs)
Let \( N_0 \Rightarrow N_1 \Rightarrow_{SR} N_2 \Rightarrow_{SR} \ldots \) be a run. Then \( \text{red}(N_i) \subseteq \text{red}(N_\infty) \) and \( \text{red}(N_i) \subseteq \text{red}(N_*) \) for every \( i \).

5.2.19 Corollary (Redundancy is Monotone Modulo Persistent Clauses)
\[ N_i \subseteq N_* \cup \text{red}(N_*) \] for every \( i \).

5.2.20 Definition (Fair Run)
A run is called \textit{fair}, if \( (N_* \setminus \text{red}(N_*)) \Rightarrow_{\text{SUPE}} (N_* \setminus \text{red}(N_*)) \cup \{C\} \) then \( C \in (N_i \cup \text{red}(N_i)) \) for some \( i \).
5.2.21 Lemma (Saturation of Fair Runs)
If a run is fair, then its limit is saturated up to redundancy.

5.2.22 Theorem (Refutational Completeness: Dynamic View)
Let $N_0 \Rightarrow_{SR} N_1 \Rightarrow_{SR} N_2 \Rightarrow_{SR} \ldots$ be a fair run, let $N_*$ be its limit. Then $N_0$ has a model if and only if $\bot \notin N_*$. 