5.2.1 Theorem (Superposition Soundness)

All inference rules of the superposition calculus are *sound*, i.e., for every rule $N \uplus \{C_1, \ldots, C_n\} \Rightarrow N \cup \{C_1, \ldots, C_n\} \cup \{D\}$ it holds that $\{C_1, \ldots, C_n\} \models D$.

5.2.2 Definition (Abstract Redundancy)

A clause C is *redundant* with respect to a clause set N if for all ground instances $C\sigma$ there are clauses $\{C_1, \ldots, C_n\} \subseteq N$ with ground instances $C_1\tau_1, \ldots, C_n\tau_n$ such that $C_i\tau_i \prec C\sigma$ for all i and $C_1\tau_1, \ldots, C_n\tau_n \models C\sigma$.
Given a set N of clauses red(N) is the set of clauses redundant with respect to N.

5.2.3 Definition (Saturation)

A clause set N is *saturated up to redundancy* if for every derivation $N \setminus \mathrm{red}(N) \Rightarrow_{\mathrm{SUPE}} N \cup \{C\}$ it holds $C \in (N \cup \mathrm{red}(N))$.

5.2.4 Definition (Partial Model Construction)

Given a clause set N and an ordering $\succ$ a (partial) model $N_{\mathcal{I}}$ can be constructed inductively over all ground clause instances of N as follows:

$$
\begin{aligned}
N_C &:= \bigcup_{D \prec C}^{D \in \mathrm{grd}(\Sigma, N)} E_D \\
N_{\mathcal{I}} &:= \bigcup_{C \in \mathrm{grd}(\Sigma, N)} N_C
\end{aligned}
$$

where N_D, $N_{\mathcal{I}}$, E_D are also considered as rewrite systems with respect to $\succ$. If $E_D \neq \emptyset$ then D is called *productive*.

$$
E_D \quad := \quad \begin{cases} \{s \approx t\} & \text{if } D = D' \vee s \approx t, \\ & (i)\ s \approx t \text{ is strictly maximal in } D \\ & (ii)\ s \succ t \\ & (iii)\ D \text{ is false in } N_D \\ & (iv)\ D' \text{ is false in } N_D \cup \{s \to t\} \\ & (v)\ s \text{ is irreducible by } N_D \\ & (vi) \text{ no negative literal is selected in } D' \\ \emptyset & \text{otherwise} \end{cases}
$$

5.2.5 Lemma (Maximal Terms in Productive Clauses)

If $E_C = \{s \rightarrow t\}$ and $E_D = \{l \rightarrow r\}$, then $s \succ l$ if and only if $C \succ D$.

5.2.6 Corollary (Partial Models are Convergent Rewrite Systems)

The rewrite systems N_C and $N_{\mathcal{I}}$ are convergent.

5.2.7 Lemma (Ordering Consequences in Productive Clauses)

If $D \preceq C$ and $E_C = \{s \to t\}$, then $s \succ r$ for every term r occurring in a negative literal in D and $s \succeq l$ for every term l occurring in a positive literal in D.

5.2.8 Corollary (Model Monotonicity True Clauses)

If D is true in N_D, then D is true in $N_{\mathcal{I}}$ and N_C for all $C \succ D$.

5.2.9 Corollary (Model Monotonicity False Clauses)

If $D = D' \vee s \approx t$ is productive, then D' is false and D is true in $N_{\mathcal{I}}$ and N_C for all $C \succ D$.

5.2.10 Lemma (Lifting Single Clause Inferences)

Let C be a clause and let σ be a substitution such that $C\sigma$ is ground. Then every equality resolution or equality factoring inference from $C\sigma$ is a ground instance of an inference from C.

5.2.11 Lemma (Lifting Two Clause Inferences)

Let $D = D' \vee u \approx v$ and $C = C' \vee [\neg]s \approx t$ be two clauses (without common variables) and let σ be a substitution such that $D\sigma$ and $C\sigma$ are ground. If there is a superposition inference between $D\sigma$ and $C\sigma$ where $u\sigma$ and some subterm of $s\sigma$ are overlapped and $u\sigma$ does not occur in $s\sigma$ at or below a variable position of s then the inference is a ground instance of a superposition inference from D and C.

5.2.12 Theorem (Model Construction)

Let N be a set of clauses that is saturated up to redundancy and does not contain the empty clause. Then for every ground clause $C\sigma \in \text{grd}(\Sigma, N)$ it holds that:

1. $E_{C\sigma} = \emptyset$ if and only if $C\sigma$ is true in $N_{C\sigma}$.
2. If $C\sigma$ is redundant with respect to $\text{grd}(\Sigma, N)$ then it is true in $N_{C\sigma}$.
3. $C\sigma$ is true in $N_{\mathcal{I}}$ and in N_D for every $D \in \text{grd}(\Sigma, N)$ with $D \succ C\sigma$.

5.2.13 Lemma (Lifting Models)

Let N be a set of clauses with variables and let $\mathcal{A}$ be a term-generated Σ-algebra. Then $\mathcal{A}$ is a model of $\mathrm{grd}(\Sigma, N)$ if and only if it is a model of N.

5.2.14 Theorem (Refutational Completeness: Static View)

Let N be a set of clauses that is saturated up to redundancy. Then N has a model if and only if N does not contain the empty clause.

5.2.15 Definition (Superposition Run)

A *run* of the superposition calculus is a derivation $N_0 \Rightarrow_{\mathrm{SR}} N_1 \Rightarrow_{\mathrm{SR}} N_2 \Rightarrow_{\mathrm{SR}} \ldots$, so that

1. $N_i \models N_{i+1}$, and
2. all clauses in $N_i \setminus N_{i+1}$ are redundant with respect to N_{i+1}.

For a run, $N_\infty = \bigcup_{i \geq 0} N_i$ and $N_* = \bigcup_{i \geq 0} \bigcap_{j \geq i} N_j$. The set N_* of all *persistent* clauses is called the *limit* of the run.

5.2.16 Lemma (Redundancy is Monotone)

If $N \subseteq N'$, then $\text{red}(N) \subseteq \text{red}(N')$.

5.2.17 Lemma (Redundant Clauses Do not Contribute)

If $N' \subseteq \text{red}(N)$, then $\text{red}(N) \subseteq \text{red}(N \setminus N')$.

5.2.18 Lemma (Redundancy is Monotone in Runs)

Let $N_0 \Rightarrow N_1 \Rightarrow_{\text{SR}} N_2 \Rightarrow_{\text{SR}} \ldots$ be a run. Then $\text{red}(N_i) \subseteq \text{red}(N_\infty)$ and $\text{red}(N_i) \subseteq \text{red}(N_*)$ for every i.

5.2.19 Corollary (Redundancy is Monotone Modulo Persistent Clauses)

$N_i \subseteq N_* \cup \text{red}(N_*)$ for every i.

5.2.20 Definition (Fair Run)

A run is called *fair*, if $(N_* \setminus \text{red}(N_*)) \Rightarrow_{\text{SUPE}} (N_* \setminus \text{red}(N_*)) \cup \{C\}$ then $C \in (N_i \cup \text{red}(N_i))$ for some i.

5.2.21 Lemma (Saturation of Fair Runs)

If a run is fair, then its limit is saturated up to redundancy.

5.2.22 Theorem (Refutational Completeness: Dynamic View)

Let $N_0 \Rightarrow_{\mathrm{SR}} N_1 \Rightarrow_{\mathrm{SR}} N_2 \Rightarrow_{\mathrm{SR}} \ldots$ be a fair run, let N_* be its limit. Then N_0 has a model if and only if $\bot \notin N_*$.