Motivation

1 Algorithm: WhatDoIDo\((n, m)\)
   Input : Two positive integers \(n, m\).
   Output: The number contained in \(n\).
2 while \((m > 0)\) do
3      \(m = m - 1\);  
4      \(n = n + 1\);
5 end
6 return \(n\);
In First-Order Logic Modulo LIA

1. \( \forall n, m. \ (m > 0, R(2, n, m) \rightarrow R(3, n, m)) \)
2. \( \forall n, m. \ (m = 0, R(2, n, m) \rightarrow R(6, n, m)) \)
3. \( \forall n, m, m'. \ (m' = m - 1, R(3, n, m) \rightarrow R(4, n, m')) \)
4. \( \forall n, m, n'. \ (n' = n + 1, R(4, n, m) \rightarrow R(5, n', m)) \)
5. \( \forall n, m. \ (R(5, n, m) \rightarrow R(2, n, m)) \)

\( \forall n, m. \ (R(2, n, m) \rightarrow R(6, n + m, 0)) \)
2-Counter Machines (Minsky 1967)

The memory of the machine are two integer counters $k_1$, $k_2$, where the integers are not limited in size, resulting in the name. The counters may be initialized at the beginning with arbitrary positive values.

A program consists of a finite number of programming lines, each coming with a unique and consecutive line number and containing exactly one instruction. The available instructions are:

- **inc($k_i$)**: increment counter $k_i$ and goto the next line,
- **td($k_i$, $n$)**: if $k_i > 0$ then decrement $k_i$ and goto the next line, otherwise goto line $n$ and leave counters unchanged,
- **goto $n$**: goto line $n$,
- **halt**: halt the computation.
Example: WhatDoIDo

2 \ td(k_2, 6)  
4 \ inc(k_1)  
5 \ goto 2  
6 \ halt
8.7.1 Theorem (2-Counter Machine Halting Problem)
The halting problem for 2-counter machines is undecidable (Minsky 1967).

Proof.
(Idea) By a reduction to the halting problem for Turing machines.

8.7.2 Proposition (FOL(LIA) Undecidability with a Single Ternary Predicate)
Unsatisfiability of a FOL(LIA) clause set with a single ternary predicate is undecidable.
FOL(LIA) Decidable for Binary or Monadic Predicates?

No: translate 2-counter machine halting problem to FOL(LIA) with a single monadic predicate.

Idea: translate state \((i, n, m)\) where the program is at line \(i\) with respective counter values \(n, m\) by the integer \(2^n \cdot 3^m \cdot p_i\) where \(p_i\) is the \(i^{th}\) prime number following 3
Example: WhatDoIDo

1. \( \text{td}(k_2, 4) \)
2. \( \text{inc}(k_1) \)
3. goto 1
4. halt

\[
5y = x, 3y' = y, x' = 7y', S(x) \rightarrow S(x')
\]
\[
5y = x, 3y' + 1 = y, x' = 13y', S(x) \rightarrow S(x')
\]
\[
5y = x, 3y' + 2 = y, x' = 13y', S(x) \rightarrow S(x')
\]
\[
7y = x, x' = 2y, x'' = 11x', S(x) \rightarrow S(x'')
\]
\[
11y = x, x' = 5y, S(x) \rightarrow S(x')
\]
\[
13y = x, S(x) \rightarrow
\]
8.7.3 Proposition (FOL(LIA) Undecidability with a Single Monadic Predicate)

Unsatisfiability of a FOL(LIA) clause set with a single monadic predicate is undecidable (Downey 1972).