## Rewrite Systems on Logics: Calculi

<table>
<thead>
<tr>
<th></th>
<th>Validity</th>
<th>Satisfiability</th>
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<tbody>
<tr>
<td><strong>Sound</strong></td>
<td>If the calculus derives a proof of validity for the formula, it is valid.</td>
<td>If the calculus derives satisfiability of the formula, it has a model.</td>
</tr>
<tr>
<td><strong>Complete</strong></td>
<td>If the formula is valid, a proof of validity is derivable by the calculus.</td>
<td>If the formula has a model, the calculus derives satisfiability.</td>
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<tr>
<td><strong>Strongly Complete</strong></td>
<td>For any validity proof of the formula, there is a derivation in the calculus producing this proof.</td>
<td>For any model of the formula, there is a derivation in the calculus producing this model.</td>
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Propositional Logic: Syntax

2.1.1 Definition (Propositional Formula)

The set $\text{PROP}(\Sigma)$ of *propositional formulas* over a signature $\Sigma$, is inductively defined by:

<table>
<thead>
<tr>
<th>$\text{PROP}(\Sigma)$</th>
<th>Comment</th>
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<tbody>
<tr>
<td>$\bot$</td>
<td>connective $\bot$ denotes “false”</td>
</tr>
<tr>
<td>$\top$</td>
<td>connective $\top$ denotes “true”</td>
</tr>
<tr>
<td>$P$</td>
<td>for any propositional variable $P \in \Sigma$</td>
</tr>
<tr>
<td>$\neg \phi$</td>
<td>connective $\neg$ denotes “negation”</td>
</tr>
<tr>
<td>$(\phi \land \psi)$</td>
<td>connective $\land$ denotes “conjunction”</td>
</tr>
<tr>
<td>$(\phi \lor \psi)$</td>
<td>connective $\lor$ denotes “disjunction”</td>
</tr>
<tr>
<td>$(\phi \rightarrow \psi)$</td>
<td>connective $\rightarrow$ denotes “implication”</td>
</tr>
<tr>
<td>$(\phi \leftrightarrow \psi)$</td>
<td>connective $\leftrightarrow$ denotes “equivalence”</td>
</tr>
</tbody>
</table>

where $\phi, \psi \in \text{PROP}(\Sigma)$. 
Propositional Logic: Semantics

2.2.1 Definition ((Partial) Valuation)

A \( \Sigma \)-valuation is a map

\[ \mathcal{A} : \Sigma \to \{0, 1\}. \]

where \( \{0, 1\} \) is the set of truth values. A partial \( \Sigma \)-valuation is a map \( \mathcal{A}' : \Sigma' \to \{0, 1\} \) where \( \Sigma' \subseteq \Sigma \).
2.2.2 Definition (Semantics)

A $\Sigma$-valuation $\mathcal{A}$ is inductively extended from propositional variables to propositional formulas $\phi, \psi \in \text{PROP}(\Sigma)$ by

\[
\begin{align*}
\mathcal{A}(\bot) & := 0 \\
\mathcal{A}(\top) & := 1 \\
\mathcal{A}(\neg \phi) & := 1 - \mathcal{A}(\phi) \\
\mathcal{A}(\phi \land \psi) & := \min(\{\mathcal{A}(\phi), \mathcal{A}(\psi)\}) \\
\mathcal{A}(\phi \lor \psi) & := \max(\{\mathcal{A}(\phi), \mathcal{A}(\psi)\}) \\
\mathcal{A}(\phi \rightarrow \psi) & := \max(\{1 - \mathcal{A}(\phi), \mathcal{A}(\psi)\}) \\
\mathcal{A}(\phi \leftrightarrow \psi) & := \text{if } \mathcal{A}(\phi) = \mathcal{A}(\psi) \text{ then } 1 \text{ else } 0
\end{align*}
\]
If $\mathcal{A}(\phi) = 1$ for some $\Sigma$-valuation $\mathcal{A}$ of a formula $\phi$ then $\phi$ is \textit{satisfiable} and we write $\mathcal{A} \models \phi$. In this case $\mathcal{A}$ is a \textit{model} of $\phi$.

If $\mathcal{A}(\phi) = 1$ for all $\Sigma$-valuations $\mathcal{A}$ of a formula $\phi$ then $\phi$ is \textit{valid} and we write $\models \phi$.

If there is no $\Sigma$-valuation $\mathcal{A}$ for a formula $\phi$ where $\mathcal{A}(\phi) = 1$ we say $\phi$ is \textit{unsatisfiable}.

A formula $\phi$ \textit{entails} $\psi$, written $\phi \models \psi$, if for all $\Sigma$-valuations $\mathcal{A}$ whenever $\mathcal{A} \models \phi$ then $\mathcal{A} \models \psi$. 
2.1.2 Definition (Atom, Literal, Clause)

A propositional variable $P$ is called an *atom*. It is also called a *(positive) literal* and its negation $\neg P$ is called a *(negative) literal*. The functions $\text{comp}$ and $\text{atom}$ map a literal to its complement, or atom, respectively: if $\text{comp}(\neg P) = P$ and $\text{comp}(P) = \neg P$, $\text{atom}(\neg P) = P$ and $\text{atom}(P) = P$ for all $P \in \Sigma$. Literals are denoted by letters $L, K$. Two literals $P$ and $\neg P$ are called *complementary*.

A disjunction of literals $L_1 \lor \ldots \lor L_n$ is called a *clause*. A clause is identified with the multiset of its literals.
### 2.1.3 Definition (Position)

A *position* is a word over $\mathbb{N}$. The set of positions of a formula $\phi$ is inductively defined by

$$
\begin{align*}
\text{pos}(\phi) &:= \{\epsilon\} \text{ if } \phi \in \{\top, \bot\} \text{ or } \phi \in \Sigma \\
\text{pos}(\neg \phi) &:= \{\epsilon\} \cup \{1p \mid p \in \text{pos}(\phi)\} \\
\text{pos}(\phi \circ \psi) &:= \{\epsilon\} \cup \{1p \mid p \in \text{pos}(\phi)\} \cup \{2p \mid p \in \text{pos}(\psi)\}
\end{align*}
$$

where $\circ \in \{\land, \lor, \to, \leftrightarrow\}$.
The prefix order $\leq$ on positions is defined by $p \leq q$ if there is some $p'$ such that $pp' = q$. Note that the prefix order is partial, e.g., the positions 12 and 21 are not comparable, they are “parallel”, see below.

The relation $<$ is the strict part of $\leq$, i.e., $p < q$ if $p \leq q$ but not $q \leq p$.

The relation $\parallel$ denotes incomparable, also called parallel positions, i.e., $p \parallel q$ if neither $p \leq q$, nor $q \leq p$.

A position $p$ is above $q$ if $p \leq q$, $p$ is strictly above $q$ if $p < q$, and $p$ and $q$ are parallel if $p \parallel q$. 
The size of a formula $\phi$ is given by the cardinality of $\text{pos}(\phi)$:

$$|\phi| := |\text{pos}(\phi)|.$$ 

The subformula of $\phi$ at position $p \in \text{pos}(\phi)$ is inductively defined by $\phi|_{\epsilon} := \phi$, $\neg \phi|_p := \phi|_p$, and $(\phi_1 \circ \phi_2)|_i := \phi_i|_p$ where $i \in \{1, 2\}$, $\circ \in \{\land, \lor, \to, \leftrightarrow\}$.

Finally, the replacement of a subformula at position $p \in \text{pos}(\phi)$ by a formula $\psi$ is inductively defined by $\phi[\psi]_{\epsilon} := \psi$, $(\neg \phi)[\psi]_1 := \neg \phi[\psi]_1$, and $(\phi_1 \circ \phi_2)[\psi]_1 := (\phi_1[\psi]_1 \circ \phi_2)$, $(\phi_1 \circ \phi_2)[\psi]_2 := (\phi_1 \circ \phi_2[\psi]_2)$, where $\circ \in \{\land, \lor, \to, \leftrightarrow\}$.
2.1.5 Definition (Polarity)

The *polarity* of the subformula $\phi|_p$ of $\phi$ at position $p \in \text{pos}(\phi)$ is inductively defined by

\[
\begin{align*}
\text{pol}(\phi, \epsilon) & := 1 \\
\text{pol}(\neg \phi, 1p) & := - \text{pol}(\phi, p) \\
\text{pol}(\phi_1 \circ \phi_2, ip) & := \text{pol}(\phi_i, p) \quad \text{if } \circ \in \{\land, \lor\}, \ i \in \{1, 2\} \\
\text{pol}(\phi_1 \rightarrow \phi_2, 1p) & := - \text{pol}(\phi_1, p) \\
\text{pol}(\phi_1 \rightarrow \phi_2, 2p) & := \text{pol}(\phi_2, p) \\
\text{pol}(\phi_1 \leftrightarrow \phi_2, ip) & := 0 \quad \text{if } i \in \{1, 2\}
\end{align*}
\]
Valuations can be nicely represented by sets or sequences of literals that do not contain complementary literals nor duplicates.

If \( \mathcal{A} \) is a (partial) valuation of domain \( \Sigma \) then it can be represented by the set 
\[
\{ P \mid P \in \Sigma \text{ and } \mathcal{A}(P) = 1 \} \cup \{ \neg P \mid P \in \Sigma \text{ and } \mathcal{A}(P) = 0 \}.
\]

Another, equivalent representation are *Herbrand* interpretations that are sets of positive literals, where all atoms not contained in an Herbrand interpretation are false. If \( \mathcal{A} \) is a total valuation of domain \( \Sigma \) then it corresponds to the Herbrand interpretation 
\[
\{ P \mid P \in \Sigma \text{ and } \mathcal{A}(P) = 1 \}.
\]
2.2.4 Theorem (Deduction Theorem)

\[ \phi \models \psi \text{ iff } \models \phi \rightarrow \psi \]
2.2.6 Lemma (Formula Replacement)

Let $\phi$ be a propositional formula containing a subformula $\psi$ at position $p$, i.e., $\phi|_p = \psi$. Furthermore, assume $\models \psi \leftrightarrow \chi$. Then $\models \phi \leftrightarrow \phi[\chi]_p$. 