Geometric Registration for Deformable Shapes

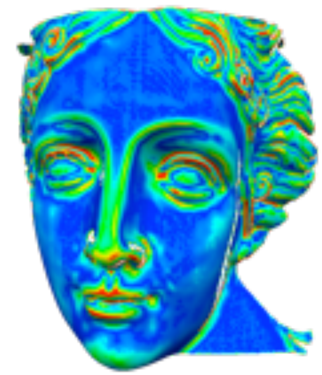
1.2 Differential Geometry & Deformation

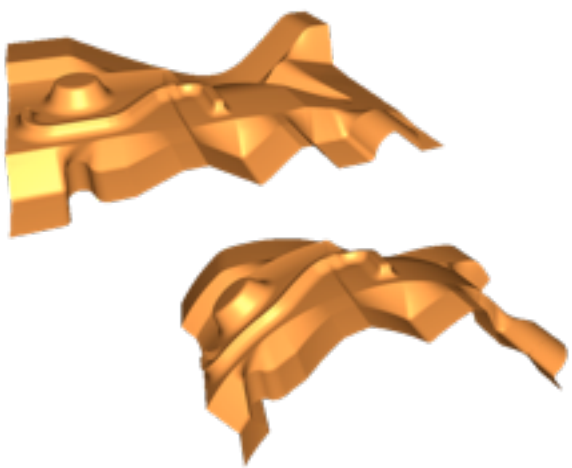
31st Annual Conference of the European Association for Computer Graphics
eurographics 2010

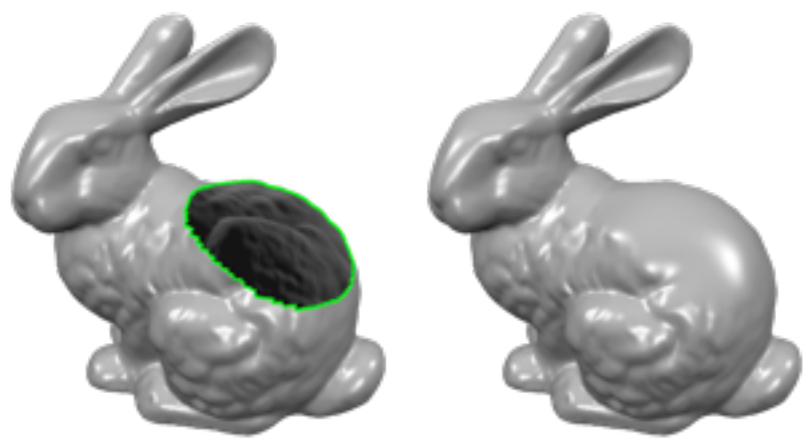
Motivation

We need differential geometry to

- compute surface curvature
- evaluate deformation energies
- fill holes







Differential Geometry

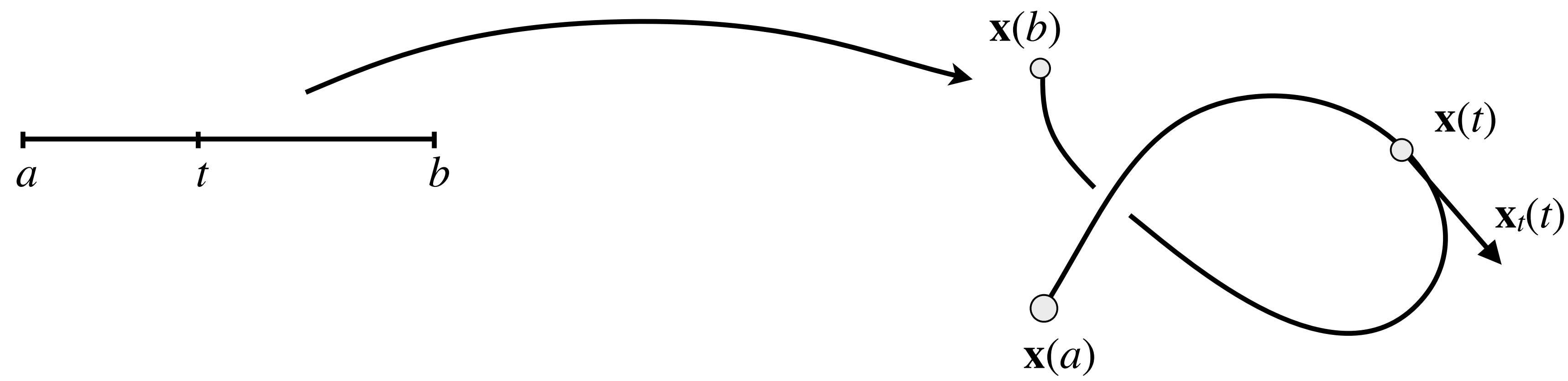
Manfredo P. do Carmo: *Differential Geometry of Curves and Surfaces*, Prentice Hall, 1976

Leonard Euler (1707 - 1783)

Carl Friedrich Gauss (1777 - 1855)

Parametric Curves

$$\mathbf{x} : [a, b] \subset \mathbb{R} \to \mathbb{R}^3$$



$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \qquad \mathbf{x}_t(t) := \frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \begin{pmatrix} \mathrm{d}x(t)\,/\mathrm{d}t \\ \mathrm{d}y(t)\,/\mathrm{d}t \\ \mathrm{d}z(t)\,/\mathrm{d}t \end{pmatrix}$$

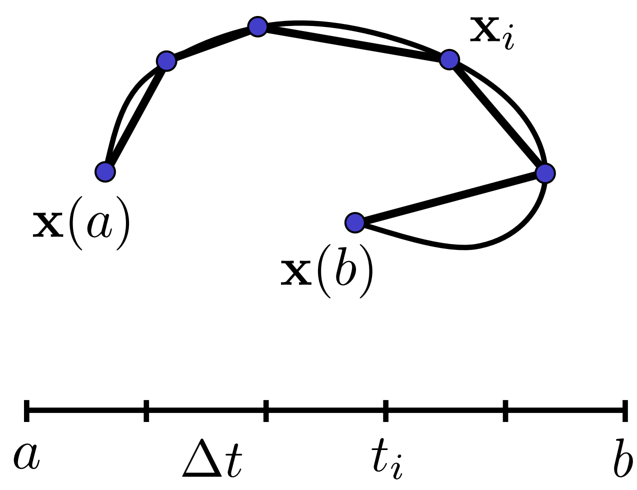
Length of a Curve

Polyline *chord length*

$$S = \sum_i \|\Delta \mathbf{x}_i\| = \sum_i \left\| \frac{\Delta \mathbf{x}_i}{\Delta t} \right\| \Delta t \,, \quad \Delta \mathbf{x}_i := \|\mathbf{x}_{i+1} - \mathbf{x}_i\|$$

Curve *arc length* ($\Delta t \rightarrow 0$)

$$s = s(t) = \int_a^t \|\mathbf{x}_t\| \, \mathrm{d}t$$



Curvature

Mapping of parameter domain:

$$t \;\mapsto\; s(t) = \int_a^t \|\mathbf{x}_t\| \,\mathrm{d}t$$

Special properties of resulting curve

$$\|\mathbf{x}_s(s)\| = 1\,, \quad \mathbf{x}_s(s) \cdot \mathbf{x}_{ss}(s) = 0$$

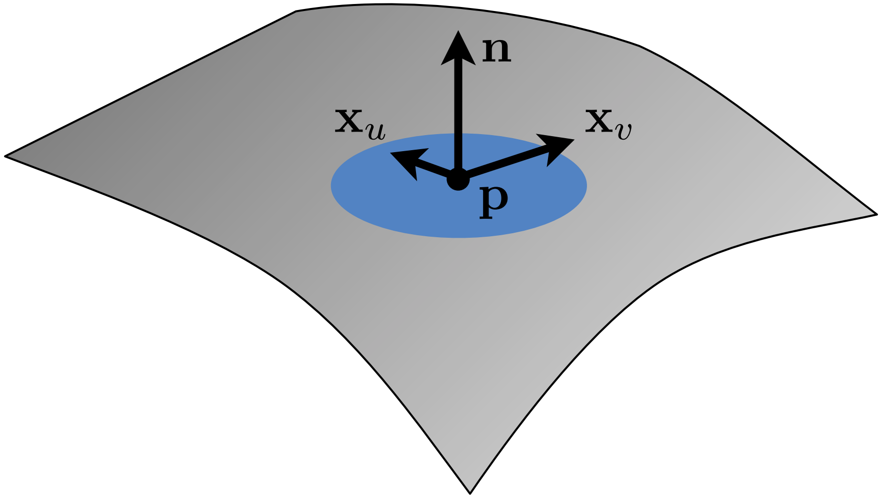
Curvature (deviation from straight line)

$$\kappa \;=\; \|\mathbf{x}_{ss}\|$$

Parametric Surfaces

Continuous surface

$$\mathbf{x}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$$



Normal vector

$$\mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}$$

Assume *regular* parameterization

$$\mathbf{x}_u \times \mathbf{x}_v \neq \mathbf{0}$$

Angles on Surface

Curve $[u(t), v(t)]$ in *uv*-plane defines curve on the surface x(*u*,*v*)

$$\mathbf{c}(t) = \mathbf{x}(u(t), v(t))$$

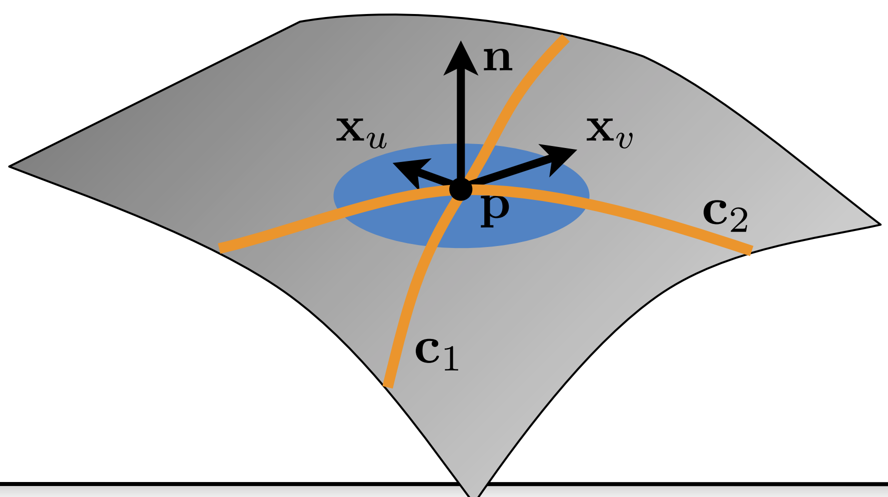
Two curves c_1 and c_2 intersecting at p

- Angle of intersection?
- Two tangents $\mathbf{t}_1$ and $\mathbf{t}_2$

$$\mathbf{t}_i = \alpha_i \mathbf{x}_u + \beta_i \mathbf{x}_v$$

- Compute inner product

$$\mathbf{t}_1^T \mathbf{t}_2 = \cos\theta \, \|\mathbf{t}_1\| \, \|\mathbf{t}_2\|$$



Angles on Surface

Curve [*u*(*t*), *v*(*t*)] in *uv*-plane defines curve on the surface x(*u*,*v*)

$$\mathbf{c}(t) = \mathbf{x}(u(t)\,, v(t))$$

Two curves c_1 and c_2 intersecting at p

$$\mathbf{t}_1^T \mathbf{t}_2 = (\alpha_1 \mathbf{x}_u + \beta_1 \mathbf{x}_v)^T (\alpha_2 \mathbf{x}_u + \beta_2 \mathbf{x}_v)$$

$$= \alpha_1 \alpha_2 \mathbf{x}_u^T \mathbf{x}_u + (\alpha_1 \beta_2 + \alpha_2 \beta_1)\, \mathbf{x}_u^T \mathbf{x}_v + \beta_1 \beta_2 \mathbf{x}_v^T \mathbf{x}_v$$

$$= (\alpha_1, \beta_1) \begin{pmatrix} \mathbf{x}_u^T \mathbf{x}_u & \mathbf{x}_u^T \mathbf{x}_v \\ \mathbf{x}_u^T \mathbf{x}_v & \mathbf{x}_v^T \mathbf{x}_v \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}$$

First Fundamental Form

First fundamental form

$$\mathbf{I} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} := \begin{pmatrix} \mathbf{x}_u^T \mathbf{x}_u & \mathbf{x}_u^T \mathbf{x}_v \\ \mathbf{x}_u^T \mathbf{x}_v & \mathbf{x}_v^T \mathbf{x}_v \end{pmatrix}$$

Defines inner product on tangent space

$$\left\langle \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}, \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} \right\rangle := \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}^T \mathbf{I} \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}$$

First Fundamental Form

First fundamental form $\mathbf{I}$ allows to measure (*with respect to surface metric*)

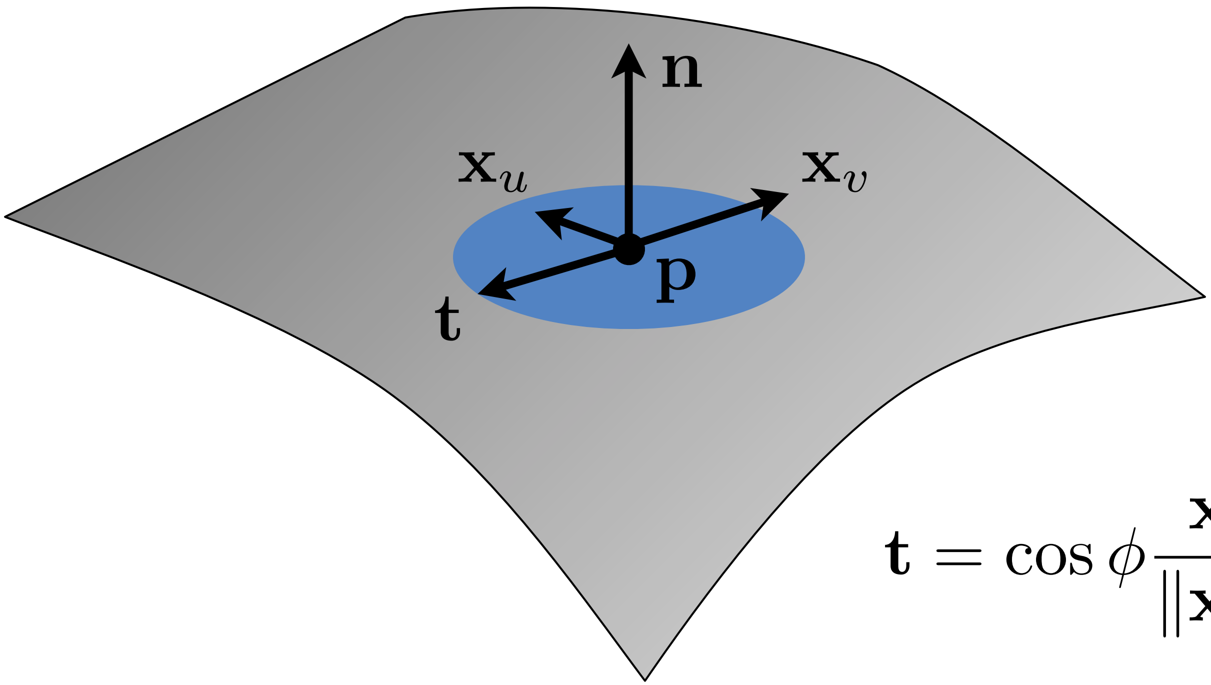
- Angles $\mathbf{t}_1^T \mathbf{t}_2 = \langle (\alpha_1, \beta_1) \ , \ (\alpha_1, \beta_1) \rangle$

- Length
$$\begin{aligned} \mathrm{d}s^2 &= \langle (\mathrm{d}u, \mathrm{d}v), (\mathrm{d}u, \mathrm{d}v) \rangle \\ &= E\mathrm{d}u^2 + 2F\mathrm{d}u\mathrm{d}v + G\mathrm{d}v^2 \end{aligned}$$

- Area
$$\begin{aligned} \mathrm{d}A &= \|\mathbf{x}_u \times \mathbf{x}_v\| \, \mathrm{d}u \, \mathrm{d}v \\ &= \sqrt{\mathbf{x}_u^T \mathbf{x}_u \cdot \mathbf{x}_v^T \mathbf{x}_v - (\mathbf{x}_u^T \mathbf{x}_v)^2} \, \mathrm{d}u \, \mathrm{d}v \\ &= \sqrt{EG - F^2} \mathrm{d}u \, \mathrm{d}v \end{aligned}$$

Normal Curvature

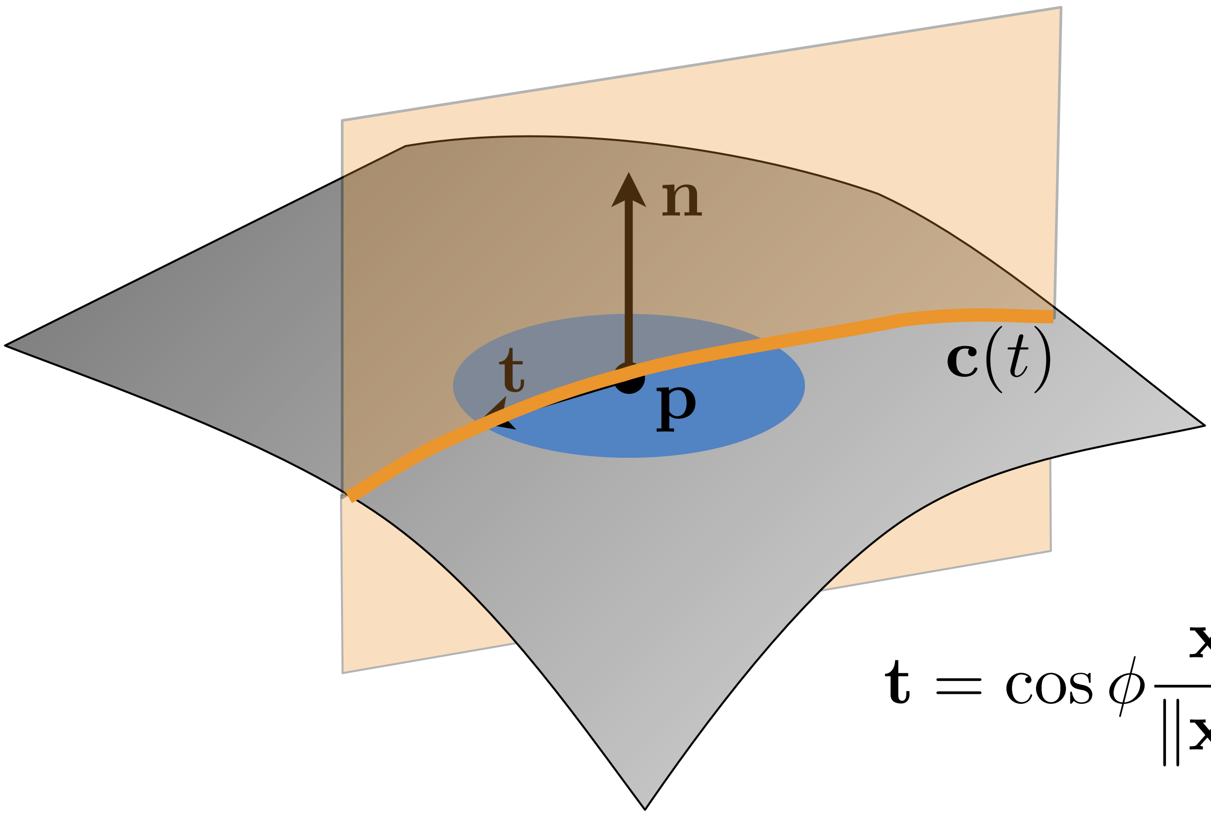
Tangent vector t...



$$\mathbf{t} = \cos\phi \frac{\mathbf{x}_u}{\|\mathbf{x}_u\|} + \sin\phi \frac{\mathbf{x}_v}{\|\mathbf{x}_v\|}$$

Normal Curvature

.. defines intersection plane, yielding curve $\mathbf{c}(t)$



$$\mathbf{t} = \cos\phi \frac{\mathbf{x}_u}{\|\mathbf{x}_u\|} + \sin\phi \frac{\mathbf{x}_v}{\|\mathbf{x}_v\|}$$

Normal Curvature

Normal curvature $\kappa_n(\mathbf{t})$ is defined as curvature of the normal curve $\mathbf{c}(t)$ at point $\mathbf{p} = \mathbf{x}(u, v)$.

With second fundamental form

$$\mathbf{II} = \begin{pmatrix} e & f \\ f & g \end{pmatrix} := \begin{pmatrix} \mathbf{x}_{uu}^T\mathbf{n} & \mathbf{x}_{uv}^T\mathbf{n} \\ \mathbf{x}_{uv}^T\mathbf{n} & \mathbf{x}_{vv}^T\mathbf{n} \end{pmatrix}$$

normal curvature can be computed as

$$\kappa_n(\bar{\mathbf{t}}) = \frac{\bar{\mathbf{t}}^T\,\mathbf{II}\,\bar{\mathbf{t}}}{\bar{\mathbf{t}}^T\,\mathbf{I}\,\bar{\mathbf{t}}} = \frac{ea^2 + 2fab + gb^2}{Ea^2 + 2Fab + Gb^2}$$

$$\begin{aligned} \mathbf{t} &= a\mathbf{x}_u + b\mathbf{x}_v \\ \bar{\mathbf{t}} &= (a, b) \end{aligned}$$

Surface Curvature(s)

Principal curvatures

- Maximum curvature $\kappa_1 = \max_{\phi} \kappa_n(\phi)$
- Minimum curvature $\kappa_2 = \min_{\phi} \kappa_n(\phi)$
- Euler theorem: $\kappa_n(\phi) = \kappa_1 \cos^2 \phi + \kappa_2 \sin^2 \phi$
- Corresponding *principal directions* $\mathbf{e}_1$, $\mathbf{e}_2$ are orthogonal

Special curvatures

- Mean curvature $H = \frac{\kappa_1 + \kappa_2}{2}$
- Gaussian curvature $K = \kappa_1 \cdot \kappa_2$

Laplace Operator

Laplace operator

gradient operator

2nd partial derivatives

$$\Delta f = \operatorname{div} \nabla f = \sum_i \frac{\partial^2 f}{\partial x_i^2}$$

function in Euclidean space

divergence operator

Cartesian coordinates

Laplace-Beltrami Operator

Extension of Laplace to functions on manifolds

Laplace-Beltrami

gradient operator

$$\Delta_{\mathcal{S}} f = \mathrm{div}_{\mathcal{S}} \nabla_{\mathcal{S}} f$$

function on manifold $\mathcal{S}$

divergence operator

Laplace-Beltrami Operator

Extension of Laplace to functions on manifolds

Laplace-Beltrami

gradient operator

mean curvature

$$\Delta_{\mathcal{S}} \mathbf{x} = \mathrm{div}_{\mathcal{S}} \nabla_{\mathcal{S}} \mathbf{x} = -2H\mathbf{n}$$

coordinate function

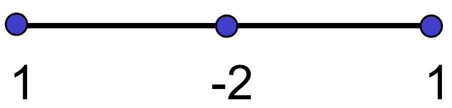
divergence operator

surface normal

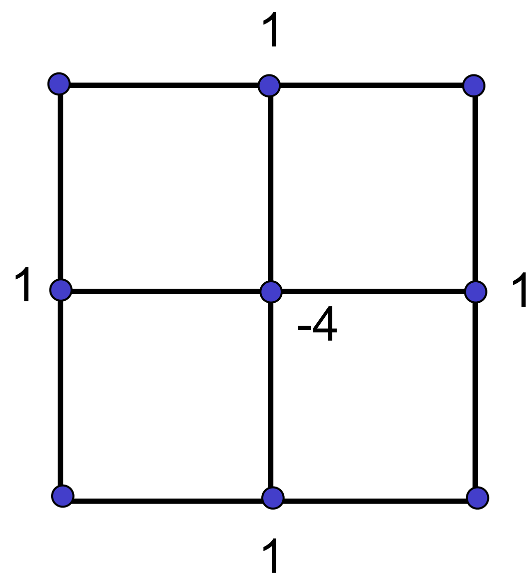
Laplace Operator on Meshes?

Extend finite differences to meshes?

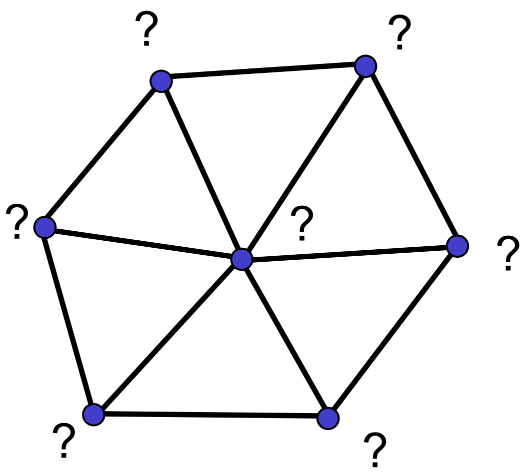
- What weights per vertex / edge?



1D grid



2D grid

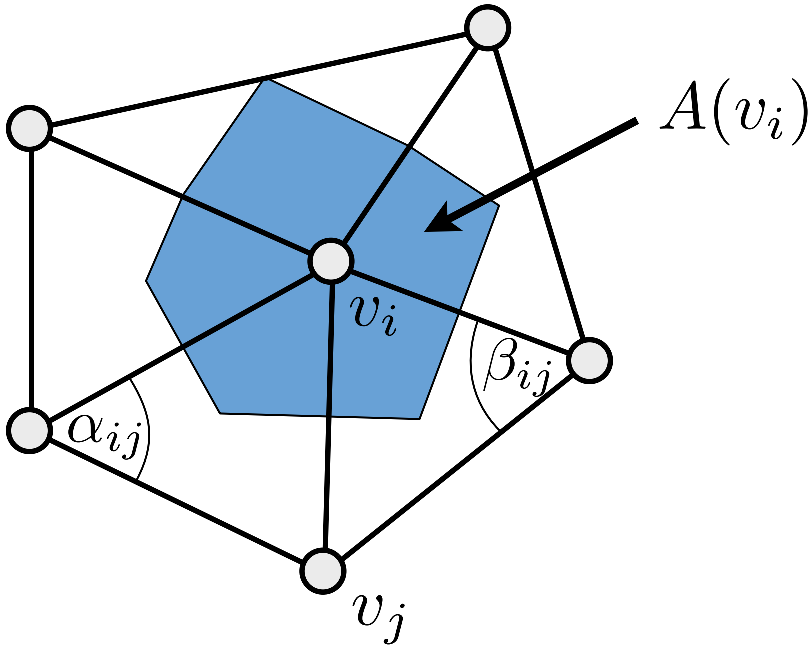


2D/3D mesh

Discrete Laplace-Beltrami

Cotangent discretization

$$\Delta_{\mathcal{S}} f(v_i) := \frac{1}{2A(v_i)} \sum_{v_j \in \mathcal{N}_1(v_i)} (\cot \alpha_{ij} + \cot \beta_{ij}) \left(f(v_j) - f(v_i) \right)$$



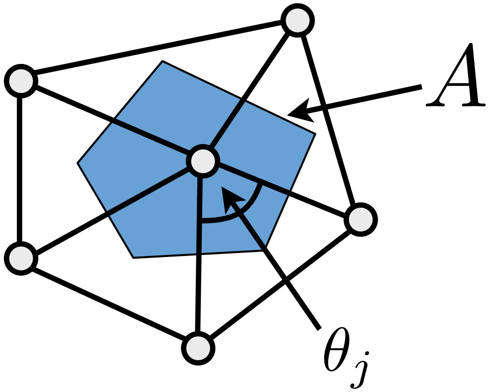
Discrete Curvatures

Mean curvature (absolute value)

$$H = \frac{1}{2} \|\Delta_{\mathcal{S}} \mathbf{x}\|$$

Gaussian curvature

$$K = (2\pi - \sum_j \theta_j)/A$$



Principal curvatures

$$\kappa_1 = H + \sqrt{H^2 - K} \qquad \kappa_2 = H - \sqrt{H^2 - K}$$

Physically-Based Deformation

Non-linear stretching & bending energies

$$\int_{\Omega} k_s \underbrace{\left\| \mathbf{I} - \mathbf{I}' \right\|^2}_{\text{stretching}} + k_b \underbrace{\left\| \mathbf{II} - \mathbf{II}' \right\|^2}_{\text{bending}} \, \mathrm{d}u\mathrm{d}v$$

Linearize energies

$$\int_{\Omega} k_s \underbrace{\left(\|\mathbf{d}_u\|^2 + \|\mathbf{d}_v\|^2 \right)}_{\text{stretching}} + k_b \underbrace{\left(\|\mathbf{d}_{uu}\|^2 + 2\,\|\mathbf{d}_{uv}\|^2 + \|\mathbf{d}_{vv}\|^2 \right)}_{\text{bending}} \mathrm{d}u\mathrm{d}v$$

Physically-Based Deformation

Minimize linearized bending energy

$$E(\mathbf{d}) \; = \; \int_{\mathcal{S}} \left\|\mathbf{d}_{uu}\right\|^2 + 2\left\|\mathbf{d}_{uv}\right\|^2 + \left\|\mathbf{d}_{vv}\right\|^2 \mathrm{d}\mathcal{S}$$

$f(x) \rightarrow \min$

Variational calculus, Euler-Lagrange PDE

$$\Delta^2 \mathbf{d} \; := \; \mathbf{d}_{uuuu} + 2\mathbf{d}_{uuvv} + \mathbf{d}_{vvvv} \; = \; 0$$

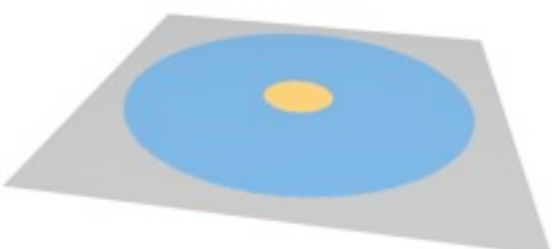
$f'(x) = 0$

➡ "Best" deformation that satisfies constraints

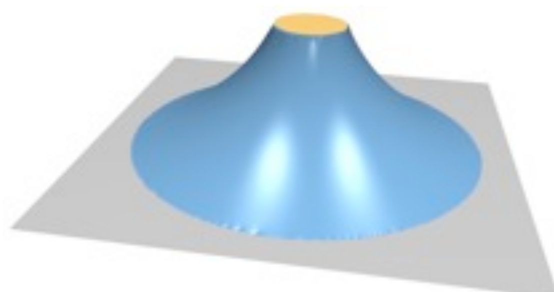
Deformation Energies

$$\Delta \mathbf{p} = 0$$

$$\Delta^2 \mathbf{p} = 0$$

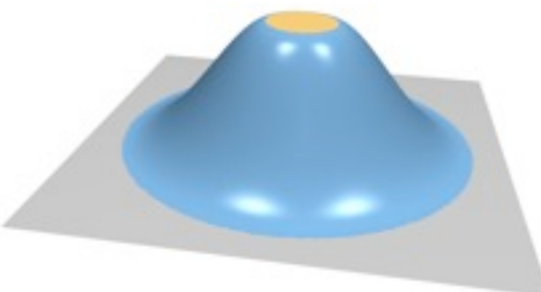


Initial state



$$\Delta \mathbf{d} = 0$$

(Membrane)



$$\Delta^2 \mathbf{d} = 0$$

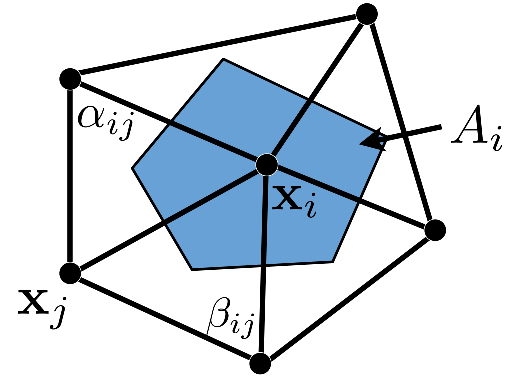
(Thin plate)

Discretization

Laplace discretization

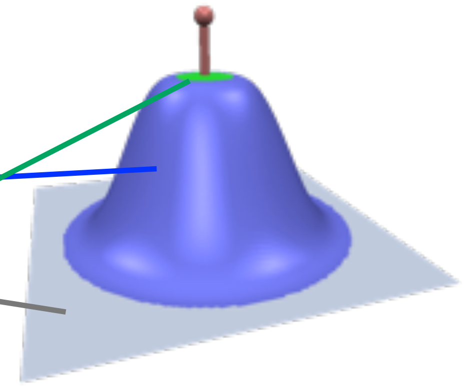
$$\Delta \mathbf{d}_i = \frac{1}{2A_i} \sum_{j \in \mathcal{N}_i} (\cot \alpha_{ij} + \cot \beta_{ij})(\mathbf{d}_j - \mathbf{d}_i)$$

$$\Delta^2 \mathbf{d}_i = \Delta(\Delta \mathbf{d}_i)$$



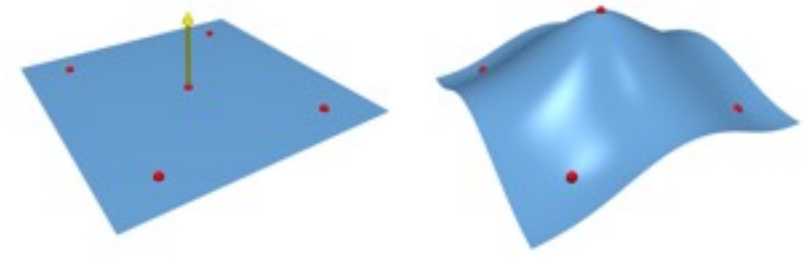
Sparse linear system

$$\underbrace{\begin{pmatrix} & \Delta^2 & \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{pmatrix}}_{=:\mathbf{M}} \begin{pmatrix} \vdots \\ \mathbf{d}_i \\ \vdots \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \delta \mathbf{h}_i \end{pmatrix}$$



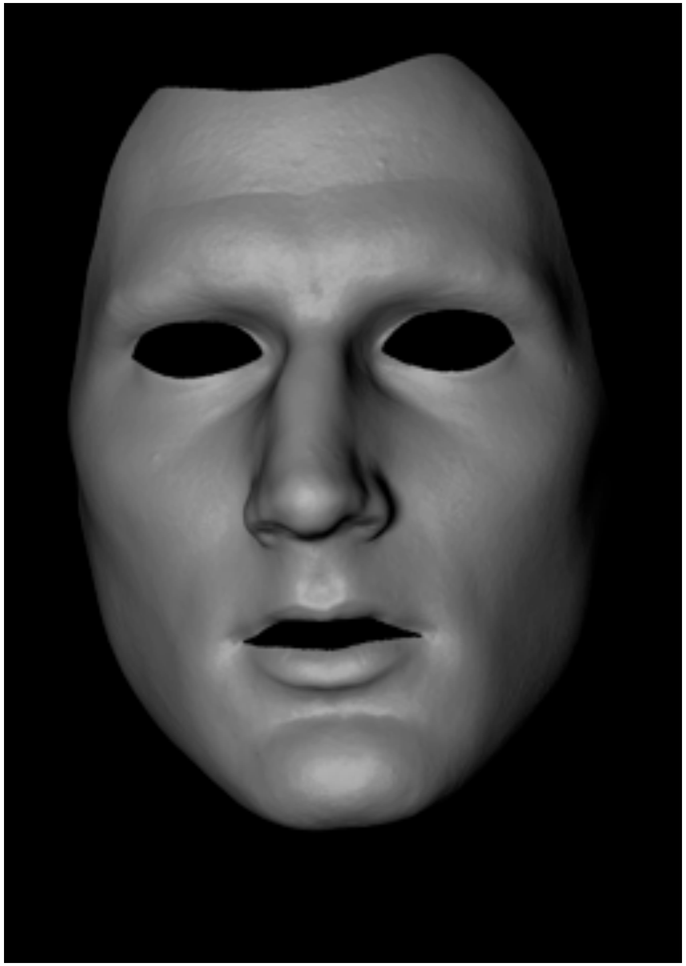
Linear Face Animation

MoCap markers control facial deformation



Minimize bending energy

- Solve linear system



Bickel et al.: *Multi-Scale Capture of Facial Geometry and Motion, SIGGRAPH 2007*

Surface-Based Deformation

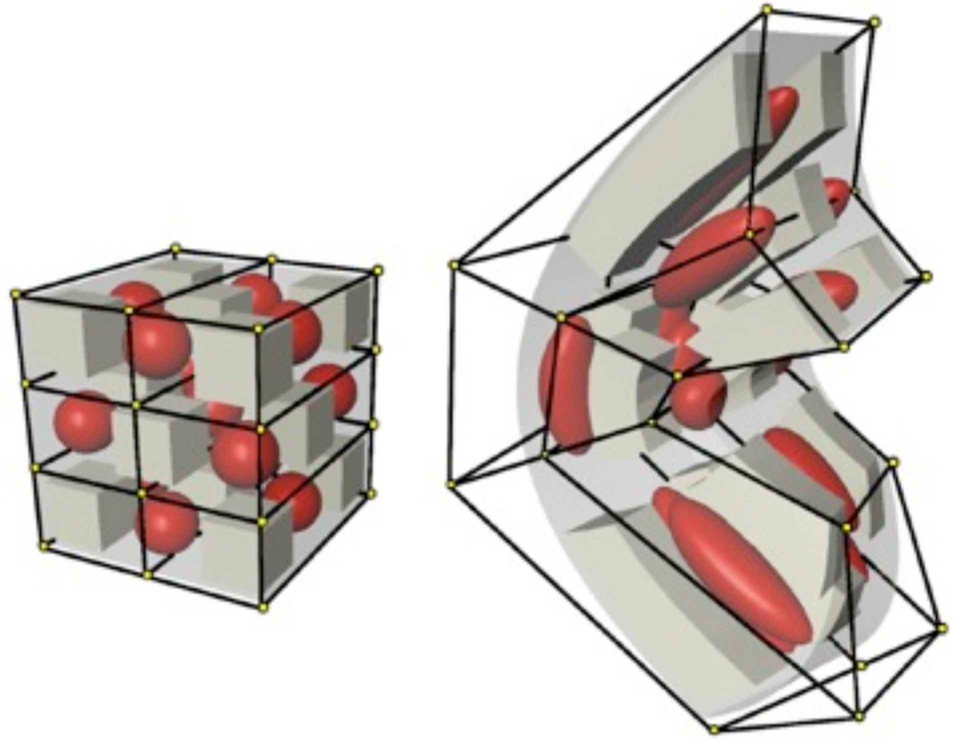
Problems with

- Highly complex models
- Topological and geometric inconsistencies

Freeform Deformation

Deform object's bounding box

- Implicitly deforms embedded objects



Volumetric Energy Minimization

Minimize similar energies to surface case

$$\int_{\mathbb{R}^3} \|\mathbf{d}_{uu}\|^2 + \|\mathbf{d}_{uv}\|^2 + \ldots + \|\mathbf{d}_{ww}\|^2 \;\mathrm{d}V \;\rightarrow\; \min$$

Radial Basis Functions

Represent deformation by RBFs

$$\mathbf{d}(\mathbf{x}) = \sum_j \mathbf{w}_j \cdot \varphi\left(\|\mathbf{c}_j - \mathbf{x}\|\right) + \mathbf{p}(\mathbf{x})$$

Triharmonic basis function $\varphi(r) = r^3$

- C^2 boundary constraints
- Highly smooth / fair interpolation

$$\int_{\mathbb{R}^3} \|\mathbf{d}_{uuu}\|^2 + \|\mathbf{d}_{vuu}\|^2 + \ldots + \|\mathbf{d}_{www}\|^2 \,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w \;\rightarrow\; \min$$

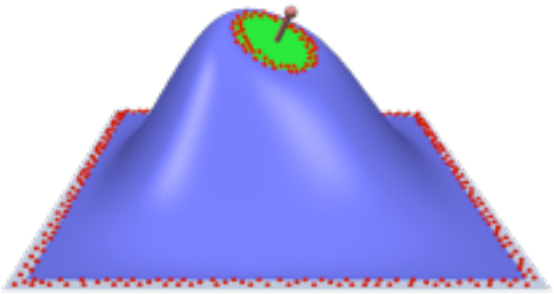
RBF Fitting

Represent deformation by RBFs

$$\mathbf{d}(\mathbf{x}) = \sum_j \mathbf{w}_j \cdot \varphi(\|\mathbf{c}_j - \mathbf{x}\|) + \mathbf{p}(\mathbf{x})$$

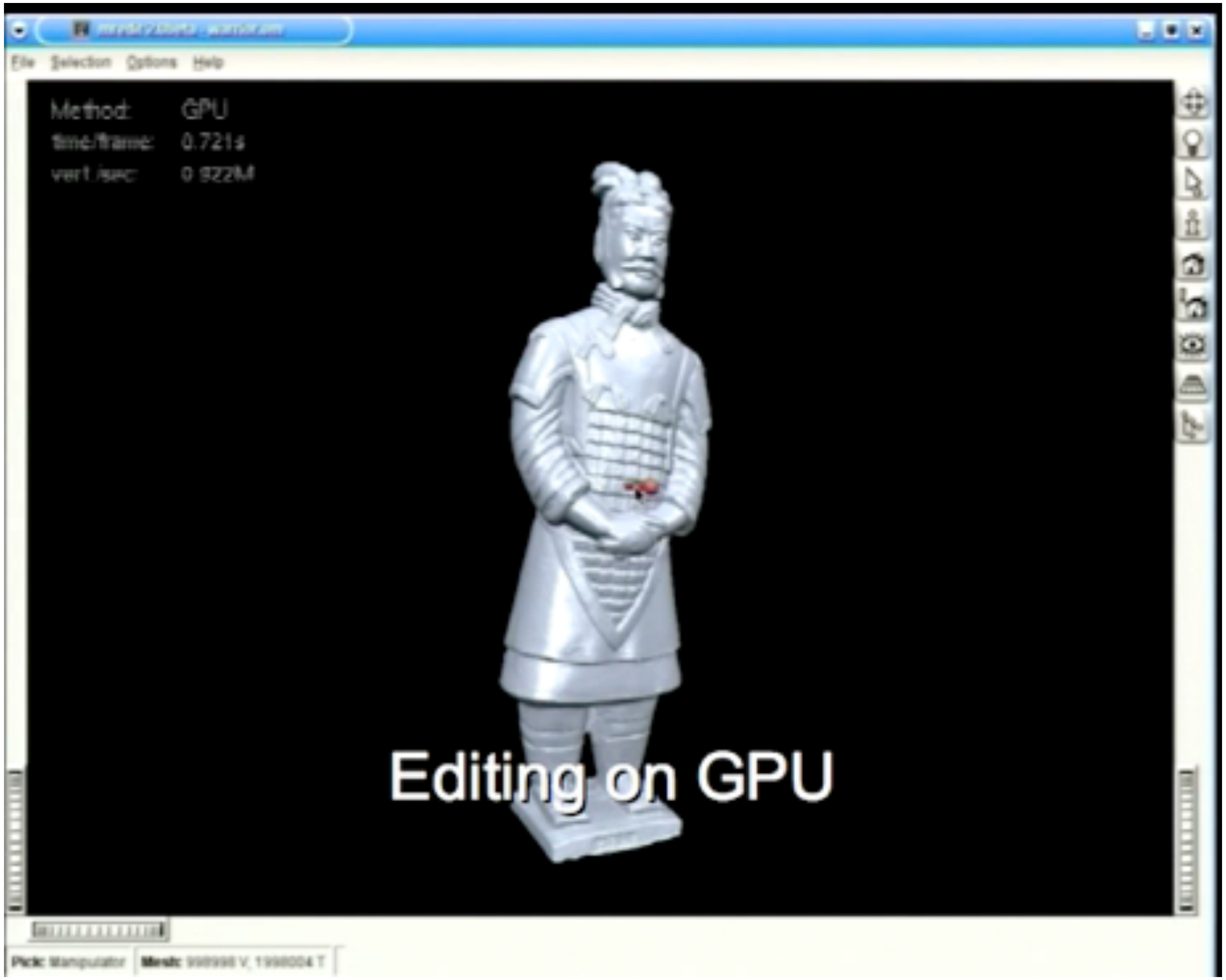
RBF fitting

- Interpolate displacement constraints
- Solve linear system for $\mathbf{w}_j$ and $\mathbf{p}$



RBF Deformation

1M vertices



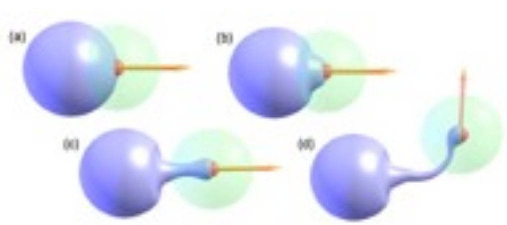
Botsch, Kobbelt: *Real-Time Shape Editing using Radial Basis Functions, Eurographics 2005*

Advanced Methods

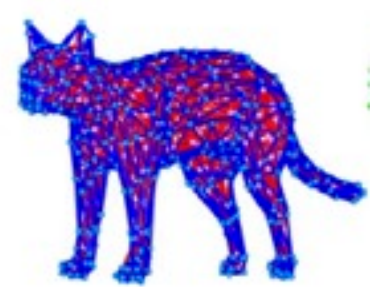
Sorkine, Alexa: *As-Rigid-As-Possible Surface Modeling, SGP 2007*

Sumner et al.: *Embedded Deformation for Shape Manipulation, SIGGRAPH 2007*

Botsch et al.: *Adaptive Space Deformations Based on Rigid Cells, Eurographics 2007*



von Funck et al.: *Vector Field Based Shape Deformations, SIGGRAPH 2006*



Zhou et al.: *Large Mesh Deformation Using the Volumetric Graph Laplacian, SIGGRAPH 2005*

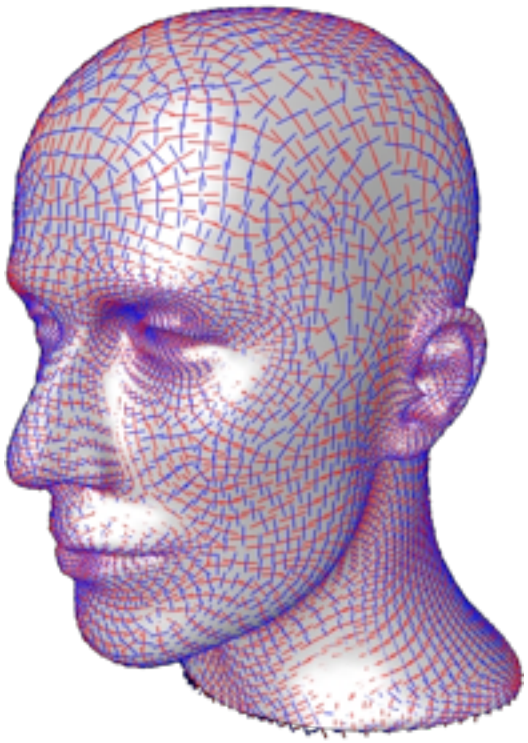
Literature

- Farin: *Curves and Surfaces for CAGD*, Morgan Kaufmann, 2001.
- Do Carmo: *Differential Geometry of Curves and Surfaces*, Prentice Hall, 1976.
- Meyer et al: *Discrete Differential-Geometry Operators for Triangulated 2-Manifolds*, VisMath 2002.
- Botsch & Sorkine, *"On linear variational surface deformation methods"*, TVCG 2007

Links

P. Alliez: *Estimating Curvature Tensors on Triangle Meshes* (source code)

- http://www-sop.inria.fr/geometrica/team/Pierre.Alliez/demos/curvature/



principal directions