VTSA summer school 2015

Exploiting SMT for Verification of Infinite-State Systems

2. Interpolation in SMT and in Verification

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Outline

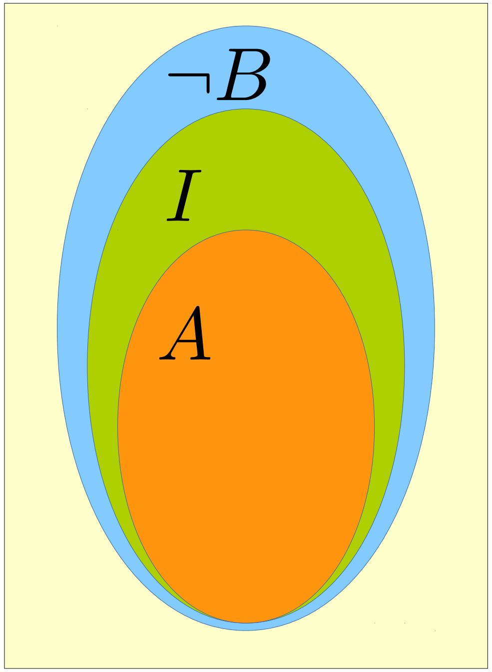
Introduction

Interpolants in Formal Verification

Computing interpolants in SMT

Introduction

- (Craig) Interpolant for an ordered pair (A, B) of formulae s.t. $A \wedge B \models_T \bot$ (or: $A \models_T \neg B$) is a formula I s.t.
  - $A \models_T I$
  - $I \wedge B \models_T \bot \quad (I \models_T \neg B)$
  - All the uninterpreted (in T) symbols of I are shared between A and B



- Why are interpolants useful?
  - Overapproximation of A relative to B
    - Overapprox. of $\exists_{\{x \notin B\}} \vec{x}.A$
  - "Local" explanation of why A is inconsistent with B

Importance of interpolation

Several important applications in formal verification:

- Approximate image computation for model checking of infinite-state systems
- Predicate discovery for Counterexample-Guided Abstraction Refinement
- Approximation of transition relation for infinite-state systems
- An alternative to (lazy) predicate abstraction for program verification
- Automatic generation of loop invariants
- ...

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Background

Symbolic transition systems

- State variables X
- Initial states formula $I(X)$
- Transition relation formula $T(X, X')$

- A state σ is an assignment to the state vars $\bigwedge_{x_i \in X} x_i = v_i$
- A path of the system S is a sequence of states $\sigma_0, \ldots, \sigma_k$ such that $\sigma_0 \models I$ and $\sigma_i, \sigma'_{i+1} \models T$
- A *k*-step (symbolic) unrolling of S is a formula

$$\boxed{I(X^0) \wedge \bigwedge_{i=0}^{k-1} T(X^i, X^{i+1})}$$

  - Encodes all possible paths of length up to *k*
- A state property is a formula P over X
  - Encodes all the states σ such that $\sigma \models P$

Forward reachability checking

- Forward image computation
  - Compute all states reachable from σ in one transition:

$$\boxed{\mathrm{Img}(\sigma(X)) := \exists X . \sigma(X) \wedge T(X, X')[X/X']}$$

- Prove that a set of states $\mathrm{Bad}(X)$ is not reachable:

$$R(X) := I(X)$$

$$\mathrm{Bad}(X)$$

$$\mathrm{Img}(R(X))$$

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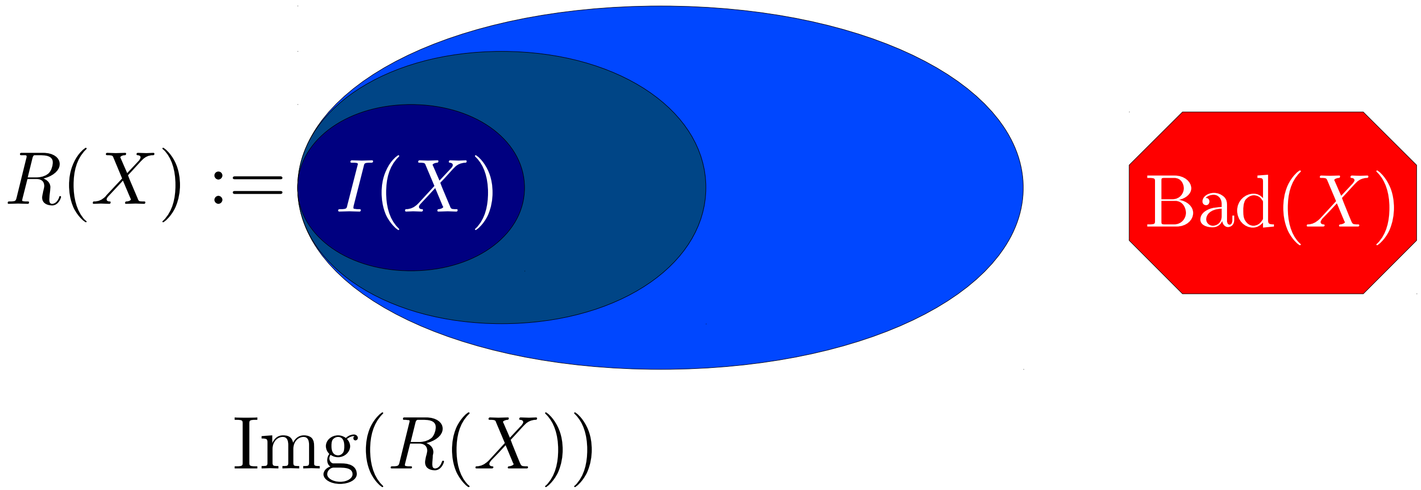

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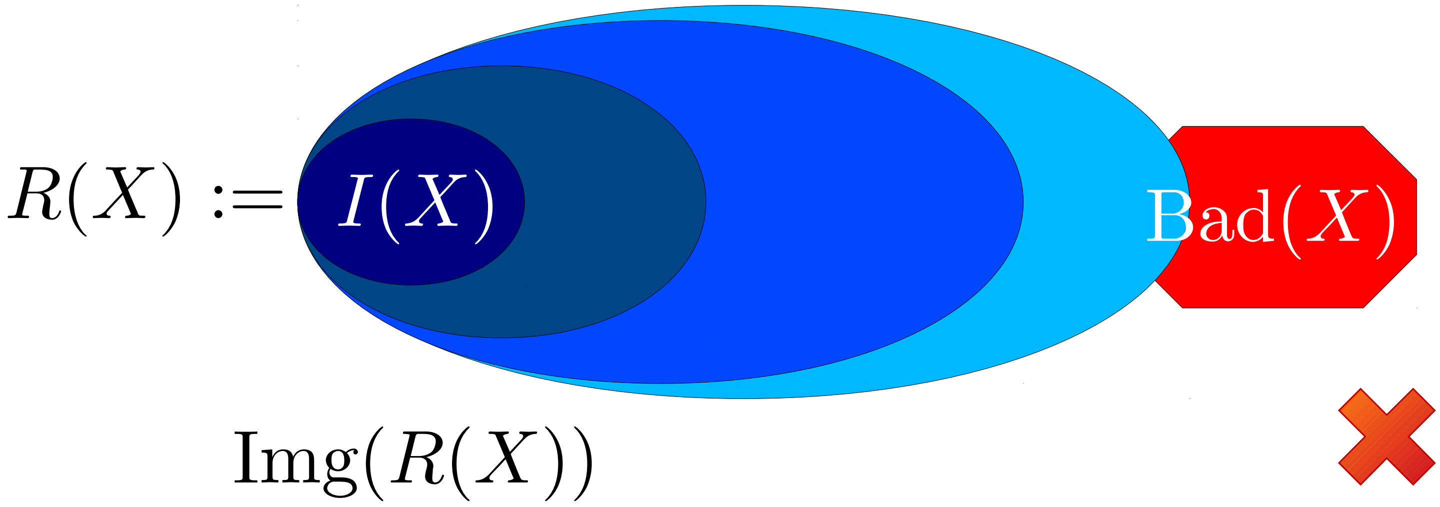

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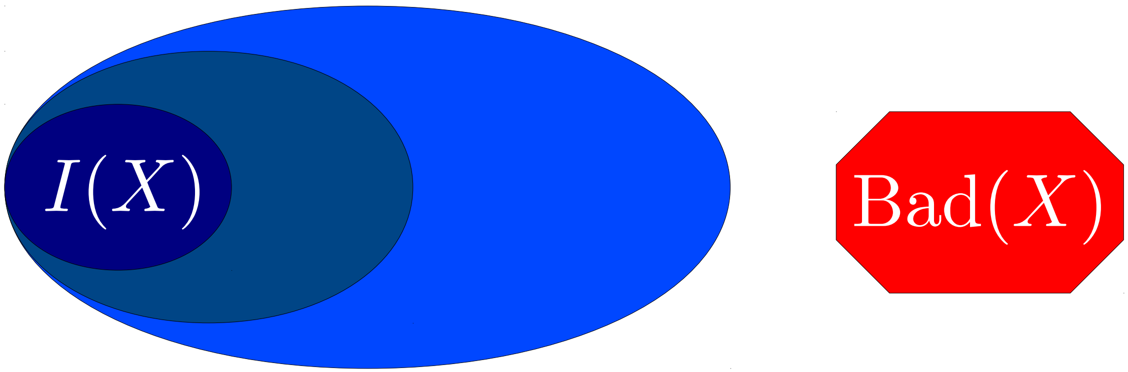

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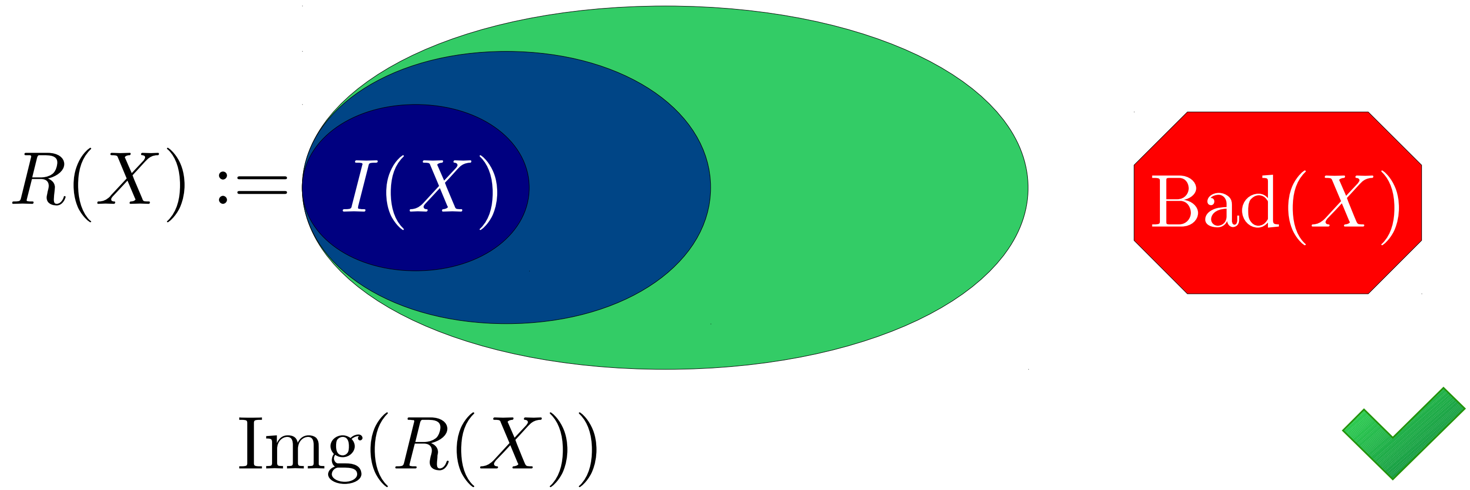


Interpolation-based reachability

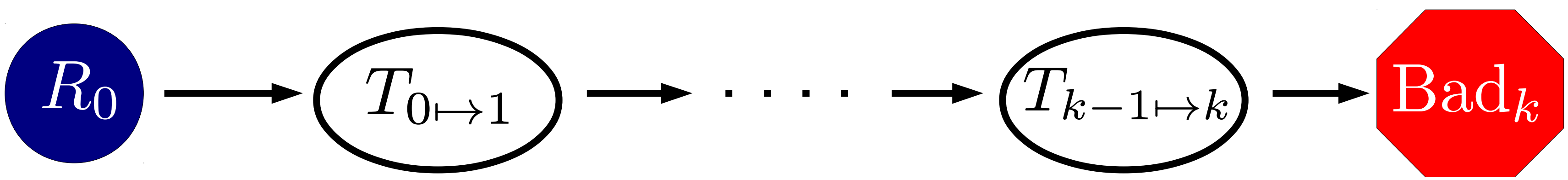
- Image computation requires quantifier elimination, which is typically very expensive (both in theory and in practice)

- Interpolation-based algorithm (McMillan CAV'03): use interpolants to overapproximate image computation
  - much more efficient than the previous algorithm
    - interpolation is often much cheaper than quantifier elimination
    - abstraction (overapproximation) accelerates convergence

  - termination is still guaranteed for finite-state systems

Interpolation-based reachability

- Set $R(X) := I(X)$
- Check satisfiability of $R_0 \wedge \bigwedge_{i=0}^{k-1} T_i \wedge \mathrm{Bad}_k$



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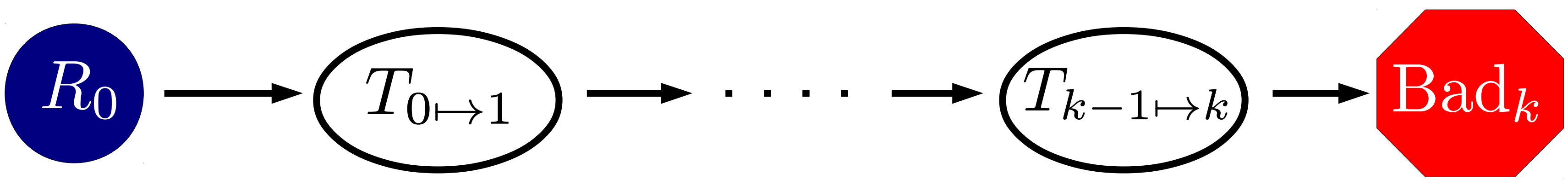

- If SAT:
  - If $R \equiv I$, return REACHABLE
  - else, increase *k* and repeat

the unrolling hits Bad

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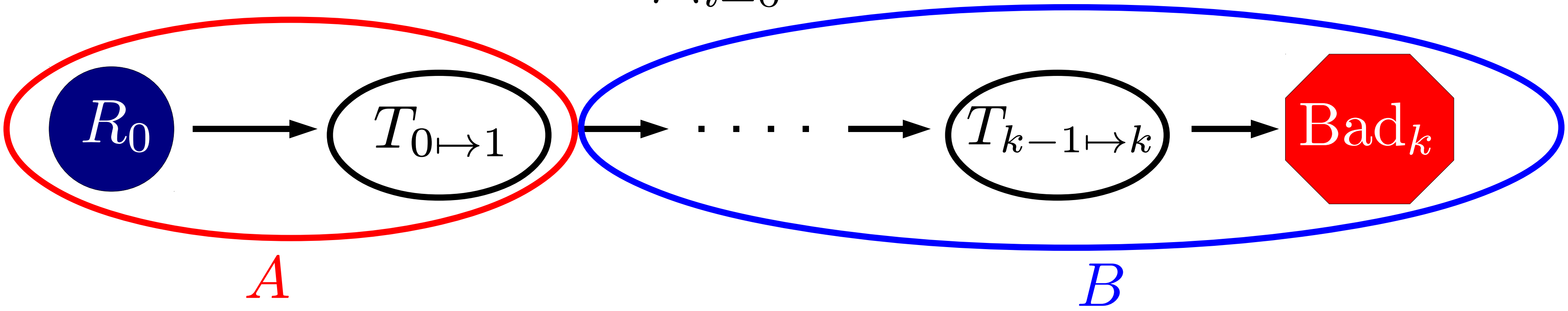

- If UNSAT:
  - Set $\varphi(X) := \mathrm{Interpolant}(A, B)[X'/X]$

φ is an abstraction of the forward image guided by the property

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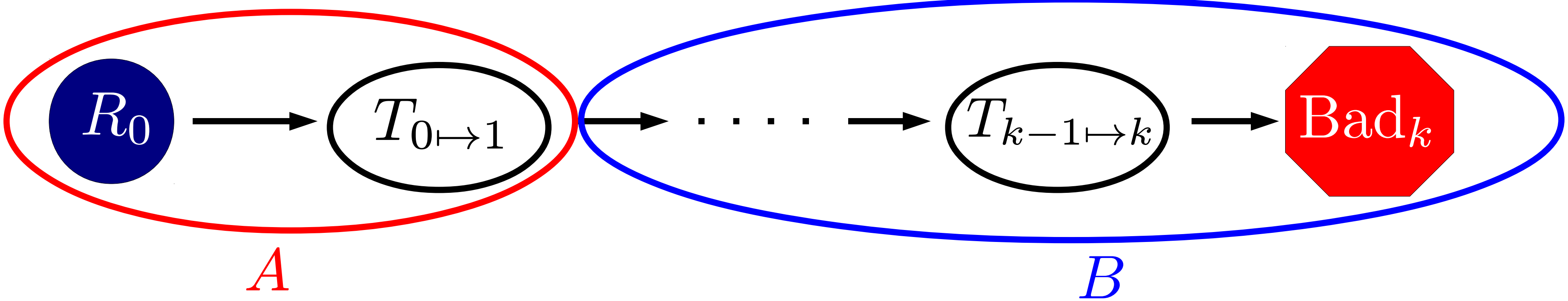


- If UNSAT:
  - Set $\varphi(X) := \mathrm{Interpolant}(A, B)[X'/X]$

    φ is an abstraction of the forward image guided by the property

  - If $\varphi \models R$, return UNREACHABLE

    fixpoint found

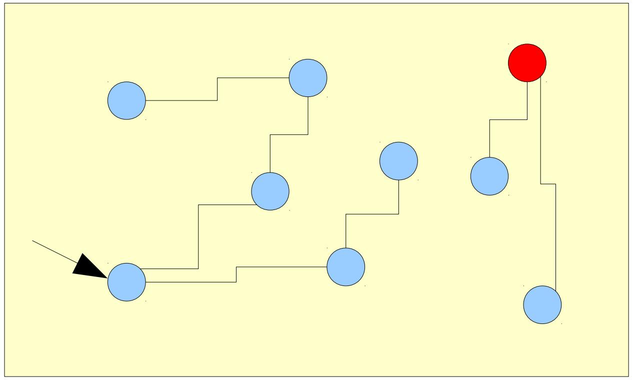
  - else, set $R(X) := R(X) \vee \varphi(X)$ and continue

Interpolation-based Abstraction Refinement

(Lazy) Predicate abstraction

- Given a Transition System $S := (I, T)$ and predicates $\mathbb{P}$
  - Abstract initial states
    $$\widehat{I(X)}_{\mathbb{P}} := \exists X.(I(X) \wedge \bigwedge_{p \in \mathbb{P}}(x_p \leftrightarrow p(X))[p(X)/x_p]$$
  - Abstract forward image
    $$\widehat{\mathrm{Img}}(\varphi(X))_{\mathbb{P}} := \exists X, X', \vec{x_p}.(\varphi(X) \wedge T(X, X') \wedge \bigwedge_{p \in \mathbb{P}}(x_p \leftrightarrow p(X) \wedge x'_p \leftrightarrow p(X'))[p(X)/x'_p]$$

- Standard technique applied in many verification tools
  - In conjunction with counterexample-guided refinement (CEGAR)
    - Extract new predicates from spurious counterexamples and compute a more precise abstraction



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The strongest boolean combination of predicates in $\mathbb{P}$ that is implied by $\mathrm{Img}(\varphi(X))$

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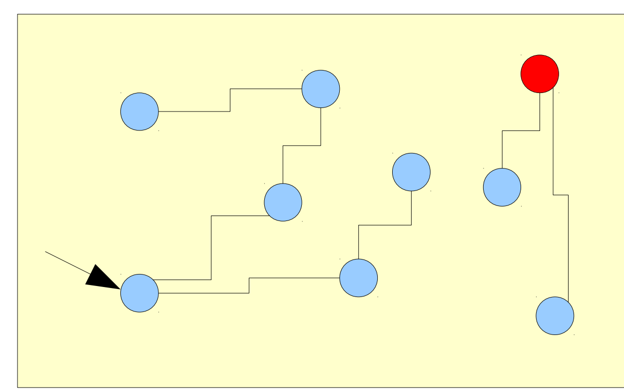

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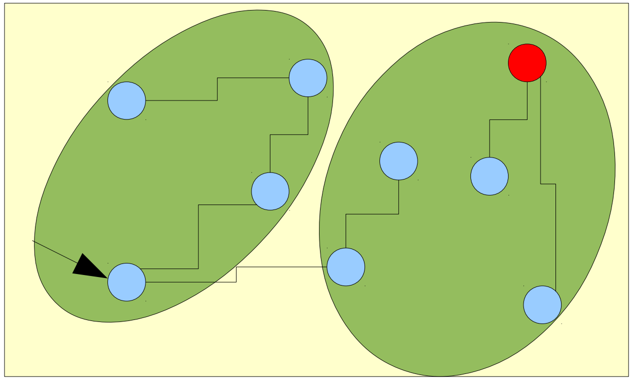

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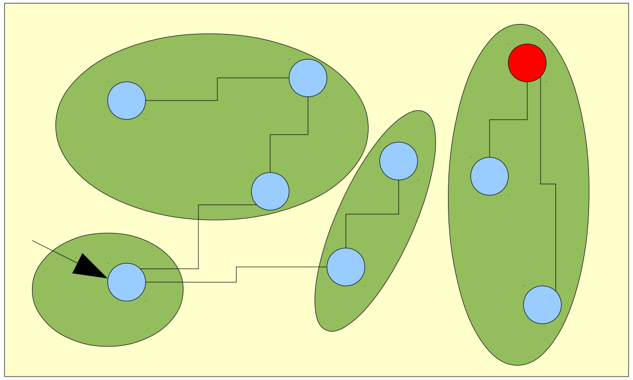
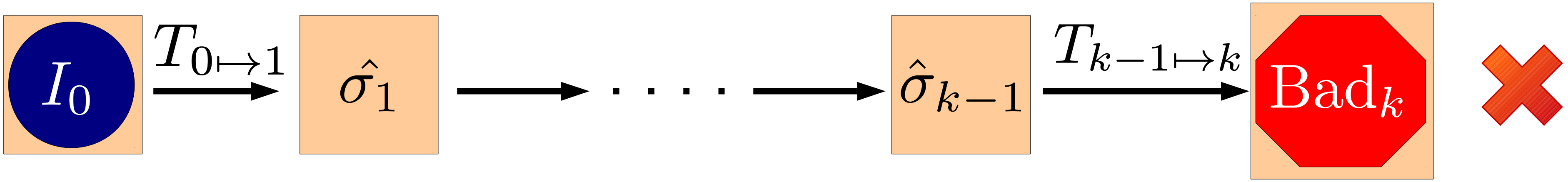


Interpolation-based Abstraction Refinement

- An abstract cex path $\hat{\sigma}_0, \ldots, \hat{\sigma}_k$ (wrt. $\mathbb{P}$) might be spurious
  - Because abstraction is overapproximating



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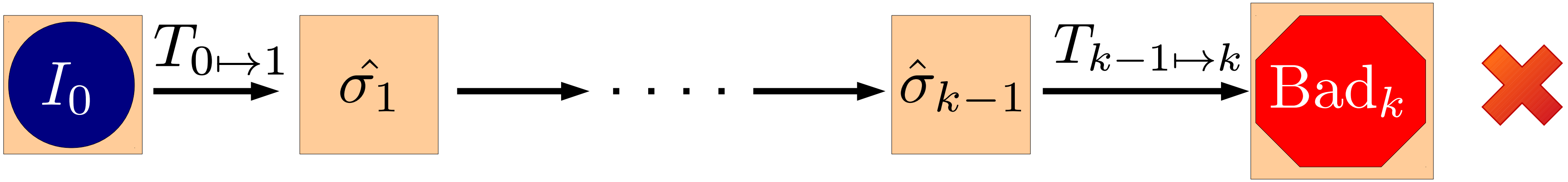

- Compute a sequence of interpolants $\varphi_0, \ldots, \varphi_{k-1}$

  such that $T_{i \mapsto i+1} \wedge \varphi_i \models \varphi_{i+1}$ for all $i \in [0, k-1)$

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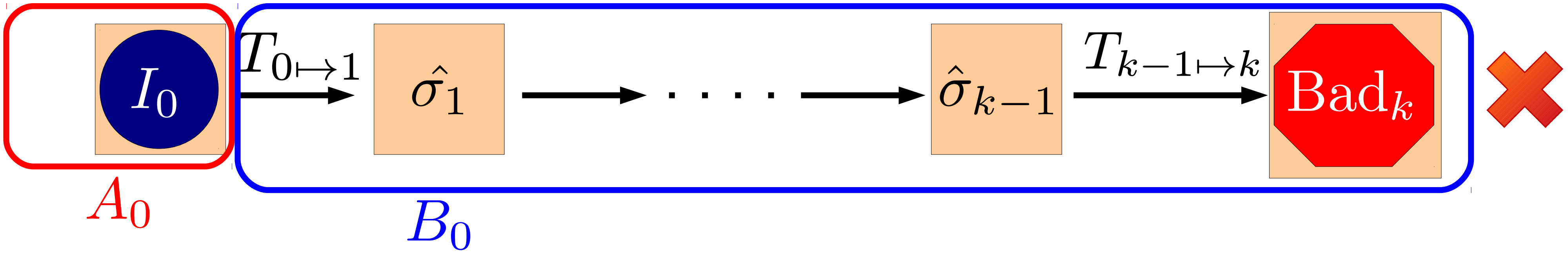

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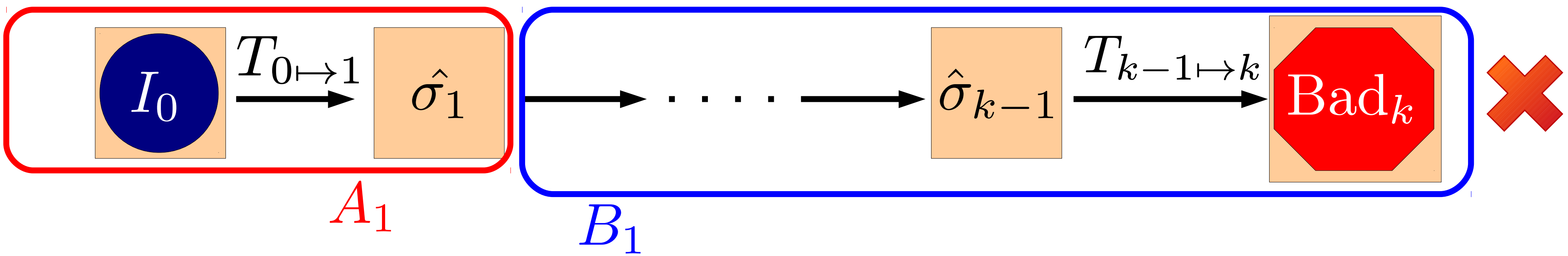

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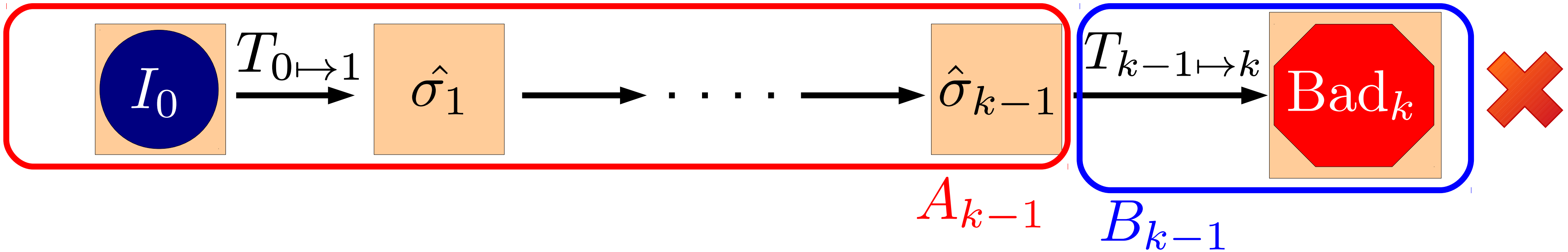

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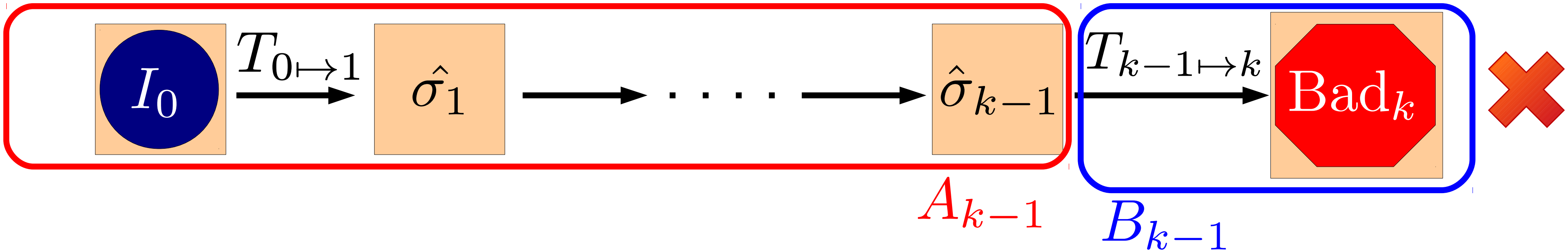

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  such that $T_{i \mapsto i+1} \wedge \varphi_i \models \varphi_{i+1}$ for all $i \in [0, k-1)$
- Let $\mathbb{P}_{\text{new}}$ be the set of all the predicates in $\varphi_0, \ldots, \varphi_{k-1}$
- Set $\mathbb{P}' := \mathbb{P} \cup \mathbb{P}_{\text{new}}$
- **Theorem:** $\hat{\sigma}_0, \ldots, \hat{\sigma}_k$ is not an abstract cex path wrt. $\mathbb{P}'$

Proof sketch

- φ_i is an overapproximation of the states reachable in *i* steps, compatible with the abstract trace $\hat{\sigma}_0, \ldots, \hat{\sigma}_i$
- φ_i is also incompatible with the rest of the abstract trace $\hat{\sigma}_{i+1}, \ldots, \hat{\sigma}_k$ (since it is an interpolant)
- By the requirement that $T_{i \mapsto i+1} \wedge \varphi_i \models \varphi_{i+1}$

  it follows that $\mathrm{Img}(\varphi_i) \models \varphi_{i+1}$
- Therefore, $\mathrm{Img}(\underbrace{\ldots}_{k-2} \mathrm{Img}(\varphi_0)) \models \varphi_{k-1}$ and $\mathrm{Img}(\varphi_{k-1}) \models \bot$

  (since the trace is spurious)
- Since we add all the atomic predicates of $\varphi_0, \ldots, \varphi_{k-1}$ to $\mathbb{P}'$ and the abstraction is precise wrt. $\mathbb{P}'$, then

$$\widehat{\mathrm{Img}}(\underbrace{\ldots}_{k-1} \widehat{\mathrm{Img}}(\varphi_0)_{\mathbb{P}'})_{\mathbb{P}'} \models \bot$$

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Efficient interpolation in SAT

- Interpolants for Boolean CNF formulae *(A, B)* can be computed from resolution refutations in linear time
- Traverse the resolution proof, annotating each node with a partial interpolant *I*
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  - For each leaf node (input clause) C in the proof:
    - If $C \in A$, set $I := \bigvee\{l \in C \mid \mathrm{var}(l) \in B\}$
    - Otherwise ($C \in B$), set $I := \top$

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    - Otherwise ($C \in B$), set $I := \top$
  - For each inner node (resolution) with parents $\varphi \vee l$ and $\psi \vee \neg l$ and annotations I_1 and I_2
    - If $\mathrm{var}(l) \in B$, set $I := I_1 \wedge I_2$; otherwise, set $I := I_1 \vee I_2$

Example

$$A := (x \vee \neg y_1) \wedge (\neg x \vee \neg y_2) \wedge y_1$$

$$B := (\neg y_1 \vee y_2) \wedge (y_1 \vee z) \wedge \neg z$$

$$\dfrac{\dfrac{y_1 \vee z \qquad \dfrac{\dfrac{\dfrac{x \vee \neg y_1 \qquad \neg x \vee \neg y_2}{\neg y_1 \vee \neg y_2} \qquad y_1}{\neg y_2} \qquad \neg y_1 \vee y_2}{\neg y_1}}{z} \qquad \neg z}{\perp}$$

Example

$$\color{red}{A} := (x \lor \neg y_1) \land (\neg x \lor \neg y_2) \land y_1$$

$$\color{blue}{B} := (\neg y_1 \lor y_2) \land (y_1 \lor z) \land \neg z$$

$$\dfrac{x \lor \neg y_1 \ \boxed{\neg y_1} \qquad \neg x \lor \neg y_2 \ \boxed{\neg y_2}}{\neg y_1 \lor \neg y_2 \ \boxed{\neg y_1 \lor \neg y_2}}$$

$$\dfrac{\neg y_1 \lor \neg y_2 \ \boxed{\neg y_1 \lor \neg y_2} \qquad y_1 \ \boxed{y_1}}{\neg y_2 \ \boxed{(\neg y_1 \lor \neg y_2) \land y_1}}$$

$$\dfrac{\neg y_2 \ \boxed{(\neg y_1 \lor \neg y_2) \land y_1} \qquad \neg y_1 \lor y_2 \ \boxed{\top}}{\neg y_1 \ \boxed{(\neg y_1 \lor \neg y_2) \land y_1}}$$

$$\dfrac{y_1 \lor z \ \boxed{\top} \qquad \neg y_1 \ \boxed{(\neg y_1 \lor \neg y_2) \land y_1}}{z \ \boxed{(\neg y_1 \lor \neg y_2) \land y_1}}$$

$$\dfrac{z \ \boxed{(\neg y_1 \lor \neg y_2) \land y_1} \qquad \neg z \ \boxed{\top}}{\bot \ \boxed{(\neg y_1 \lor \neg y_2) \land y_1}}$$

Proof of correctness

- By induction on the structure of the resolution refutation

- Lemma: for each annotated node $C\ [I]$, we have

  1) $A \models I \vee \bigvee\{l \in C \mid \mathrm{var}(l) \not\in B\}$

  2) $B \wedge I \models \vee \bigvee\{l \in C \mid \mathrm{var}(l) \in B\}$

  3) *I* contains only variables that occur in both *A* and *B*

- Then as a corollary, for the root $\perp\ [I]$, *I* is an interpolant

- The lemma trivially holds for leaf nodes (check)

Proof of correctness – resolution steps

Resolution step with parents $(\varphi \vee l)\ [I_1]$ and $(\psi \vee \neg l)\ [I_2]$

- Case $\mathrm{var}(l) \in B$

1) By ind. hyp $A \models I_1 \vee \bigvee\{p \in \varphi \mid \mathrm{var}(p) \notin B\}$ and
$A \models I_2 \vee \bigvee\{p \in \psi \mid \mathrm{var}(p) \notin B\}$

Therefore $A \models (I_1 \wedge I_2) \vee \bigvee\{p \in \varphi \wedge \psi \mid \mathrm{var}(p) \notin B\}$

2) By inductive hypotesis $B \wedge I_1 \models \bigvee\{p \in \varphi \vee l \mid \mathrm{var}(p) \in B\}$
which means $B \models \neg I_1 \vee \bigvee\{p \in \varphi \vee l \mid \mathrm{var}(p) \in B\}$
Similarly, $B \models \neg I_2 \vee \bigvee\{p \in \psi \vee \neg l \mid \mathrm{var}(p) \in B\}$
By resolution on $\mathrm{var}(l)$, then

$$B \models \neg I_1 \vee \neg I_2 \vee \bigvee\{p \in \varphi \vee \psi \mid \mathrm{var}(p) \in B\}$$

3) Trivial by the inductive hypothesis

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Therefore $B \models \neg I_1 \vee \bigvee\{p \in \varphi \vee \psi \mid \mathrm{var}(p) \in B\}$ and
$B \models \neg I_2 \vee \bigvee\{p \in \varphi \vee \psi \mid \mathrm{var}(p) \in B\}$

and so $B \wedge (I_1 \vee I_2) \models \bigvee\{p \in \varphi \vee \psi \mid \mathrm{var}(p) \in B\}$

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- Resolution refutations in SMT:

Boolean part (ground resolution) **+** *T*-specific part for conjunctions of constraints (negated *T*-lemmas)

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T-specific interpolation for conjunctions only

Theory interpolation only for sets of *T*-literals

- Annotation for a *T*-lemma *C*:

$$I := T\text{-interpolant}\left(\bigwedge\{l \in \neg C \mid \mathrm{var}(l) \notin B\},\ \bigwedge\{l \in \neg C \mid \mathrm{var}(l) \in B\}\right)$$

Equality (EUF)

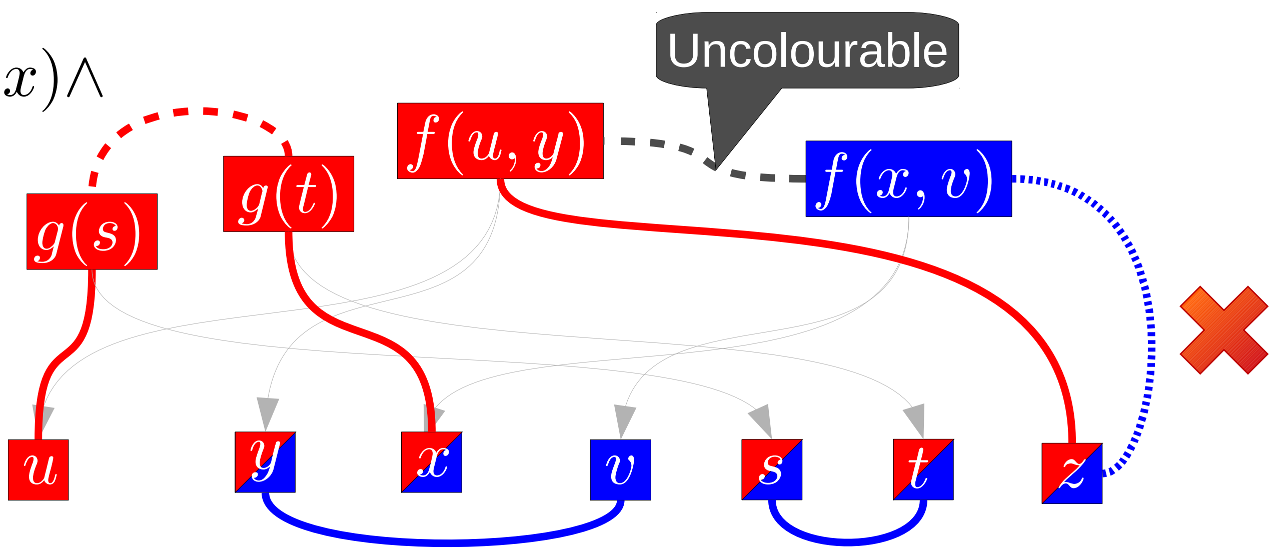
- Interpolants from coloured congruence graphs
  - Nodes with colours:
    - ■ if term occurs in A
    - ■ if term occurs in B
    - ◪ if term is shared
  - Edges with colours of the nodes they connect
    - Uncolorable edge: connects nodes of two different colours
  - Always possible to obtain a coloured graph
    - (by introducing new nodes)

Equality (EUF)

- Interpolants from coloured congruence graphs
  - Nodes with colours: (red) if term occurs in *A*; (blue) if term occurs in *B*; (red/blue) if term is shared
  - Edges with colours of the nodes they connect
    - Uncolorable edge: connects nodes of two different colours
  - Always possible to obtain a coloured graph
    - (by introducing new nodes)

$$A := (u = g(s)) \wedge (g(t) = x) \wedge (f(u, y) = z)$$

$$B := (v = y) \wedge (s = t) \wedge \neg(f(x, v) = z)$$



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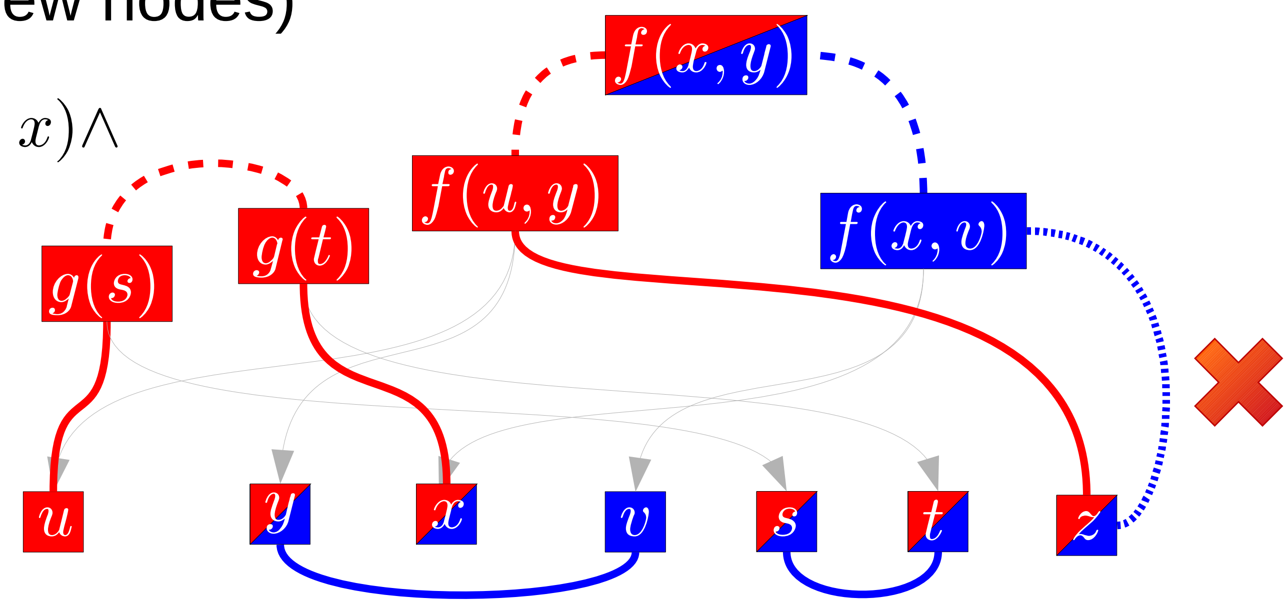
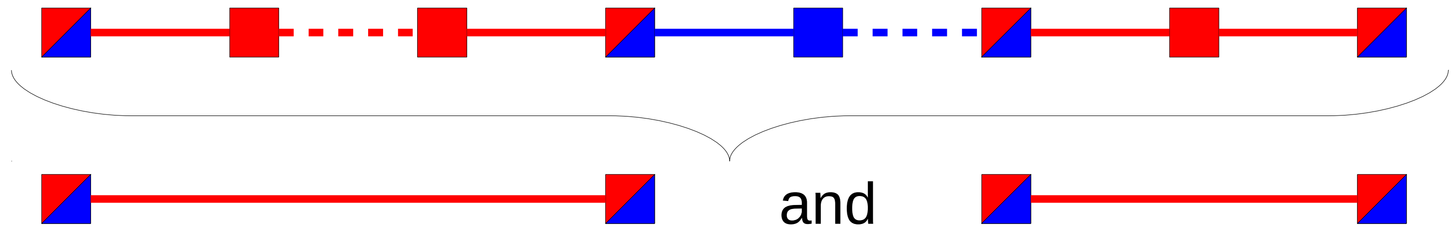


Interpolation algorithm (sketch)

- Start from disequality edge
- Compute summaries for *A*-paths with shared endpoints



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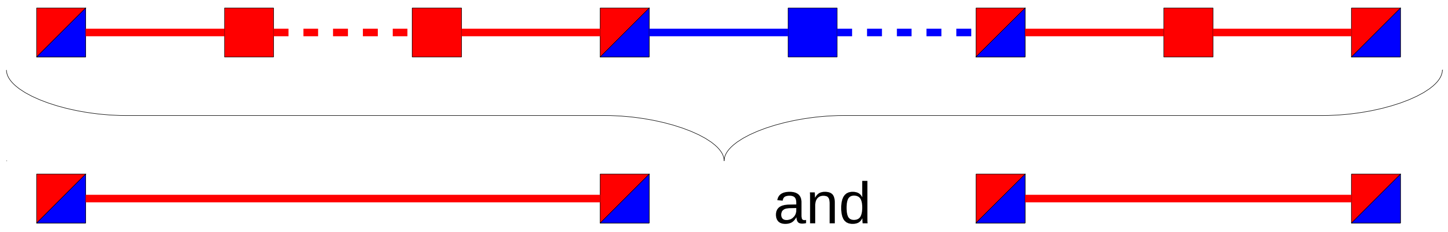
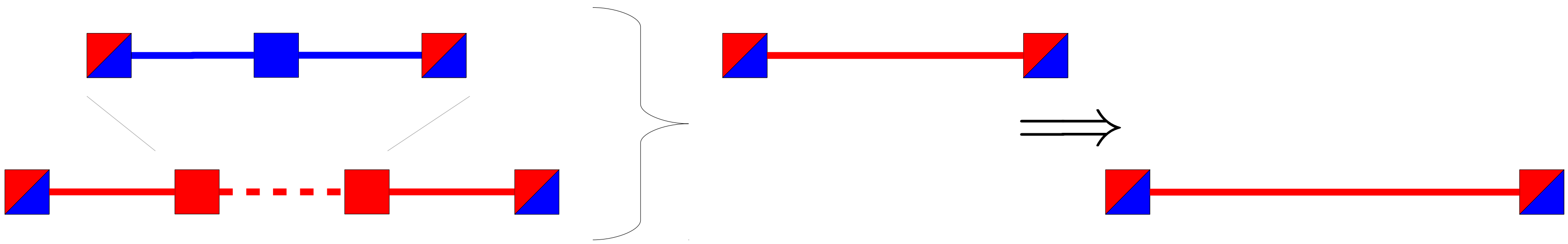


- If an *A*-summary involves a congruence edge, compute summaries recursively on function arguments
  - Use *B*-summaries as premises for the *A*-summary



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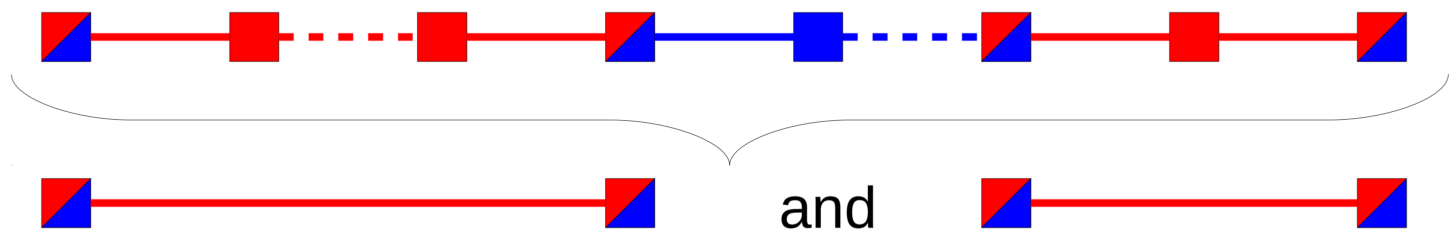

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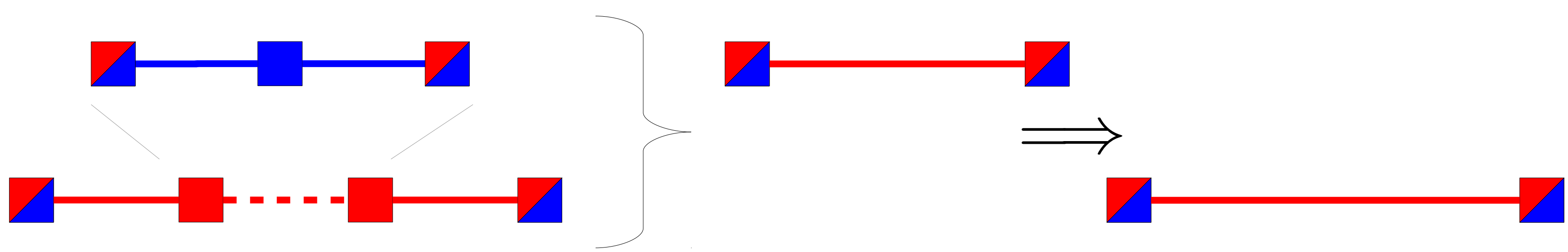

- (Several cases to consider)

Example

$$A := (u = g(s)) \wedge (g(t) = x) \wedge$$
$$(f(u, y) = z)$$

$$B := (v = y) \wedge (s = t) \wedge$$
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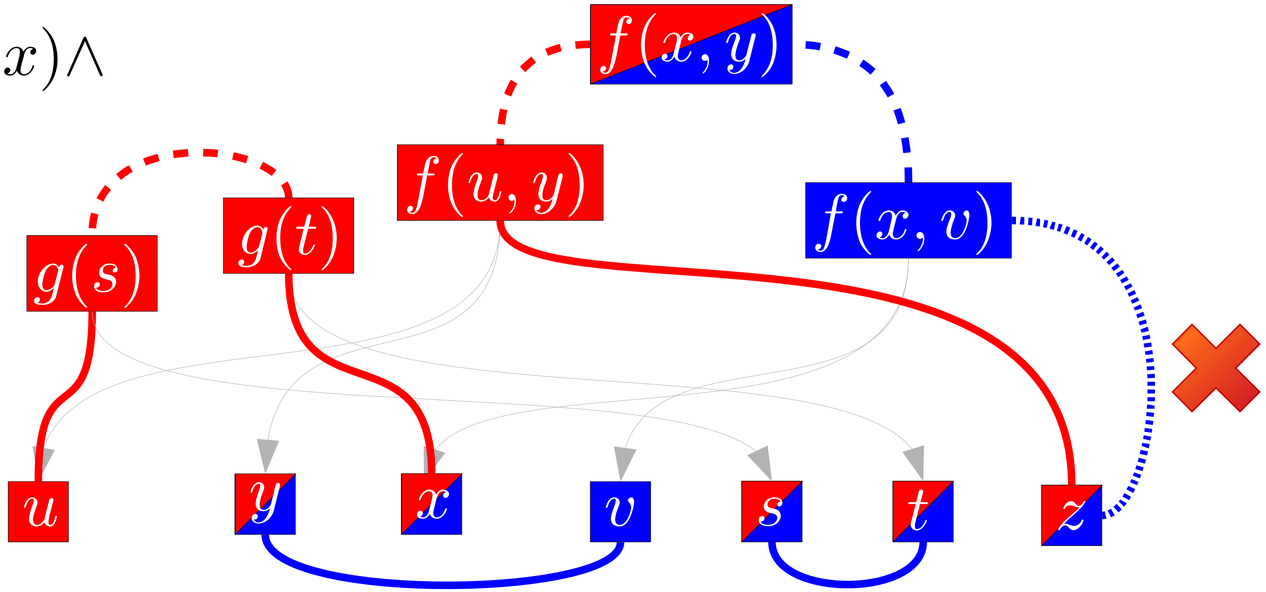

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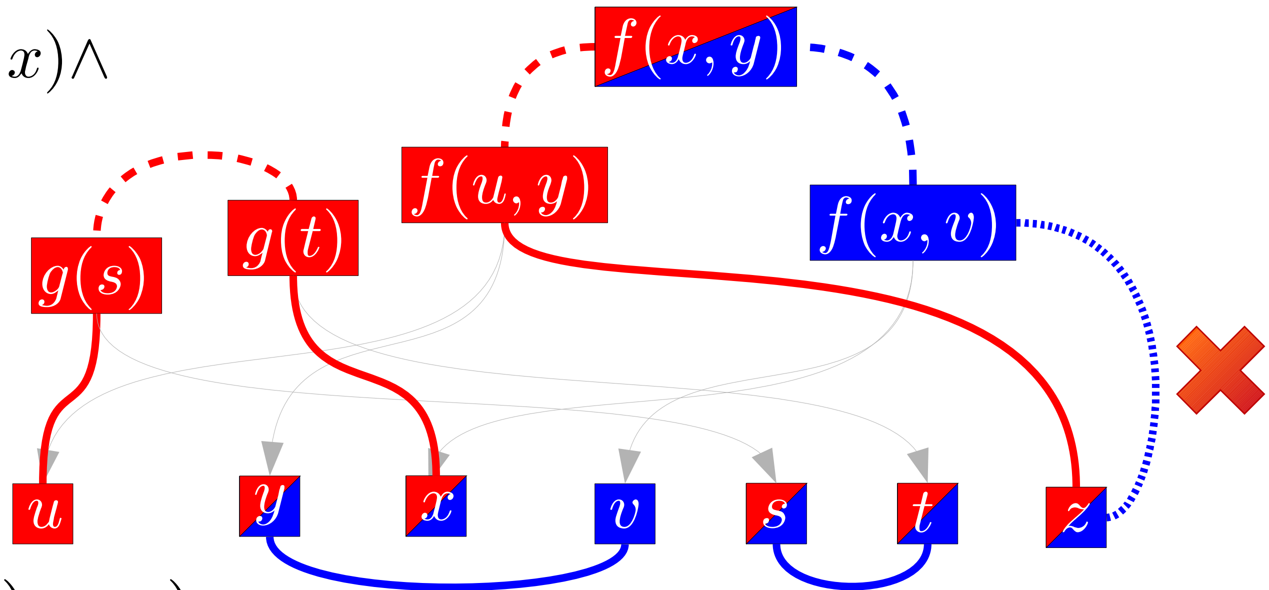
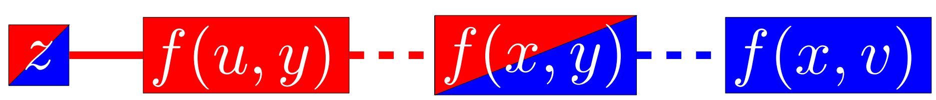


- Start from $\neg(f(x, v) = z)$
- *A*-summaries for 

 $z = f(x, y)$

Example

$A := (u = g(s)) \wedge (g(t) = x) \wedge$
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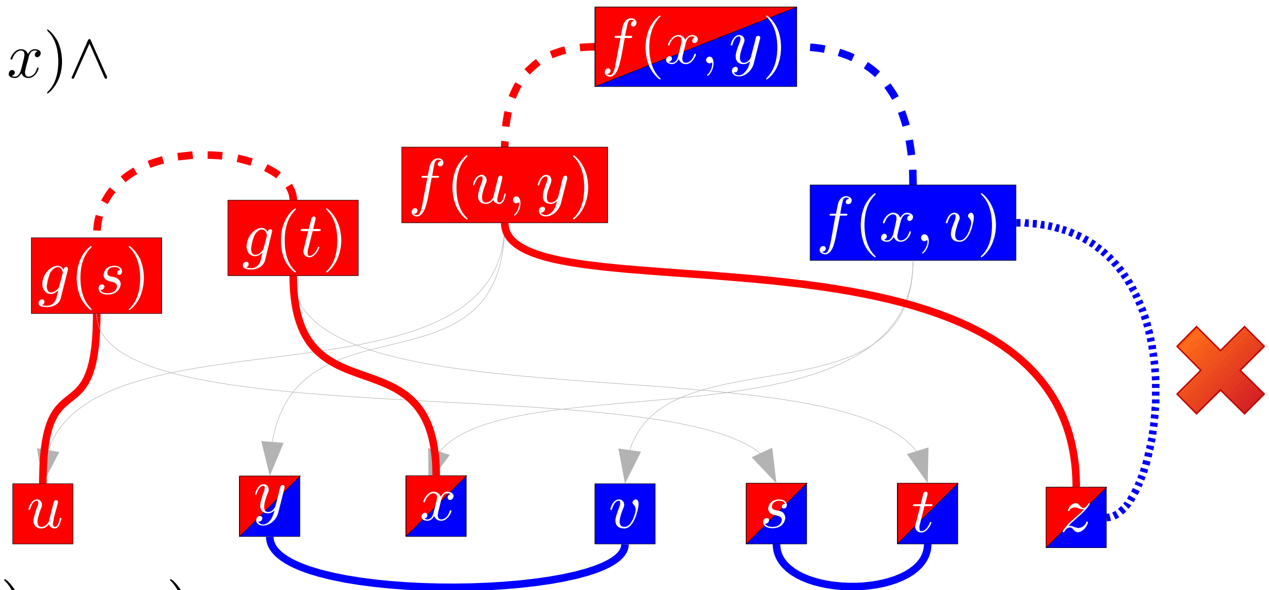

- Start from $\neg(f(x,v) = z)$
- *A*-summaries for z — $f(u,y)$ - - - $f(x,y)$ - - - $f(x,v)$ $\Big\}\ z = f(x,y)$
- Recurse on edge $f(u,y)$ - - - $f(x,y)$
  - Path u — $g(s)$ - - - $g(t)$ — x $\Big\}\ \top$

Example

$A := (u = g(s)) \wedge (g(t) = x) \wedge$
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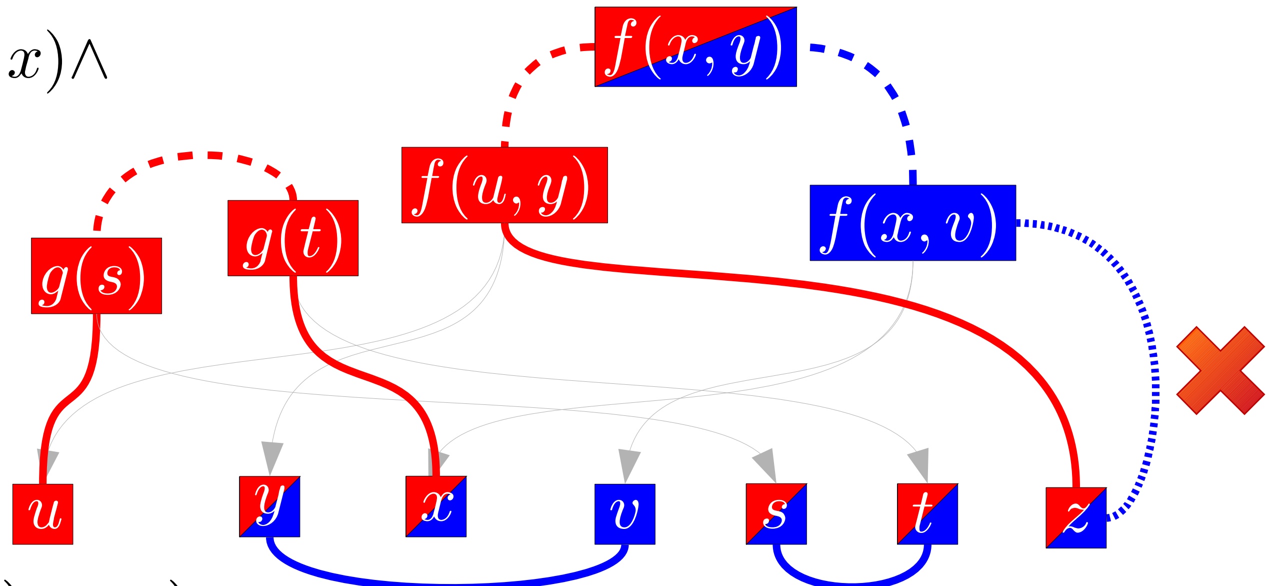

- Start from $\neg(f(x, v) = z)$
- A-summaries for $\left.\begin{array}{l} z \text{ — } f(u,y) \text{ - - } f(x,y) \text{ - - } f(x,v) \end{array}\right\}\ z = f(x, y)$
- Recurse on edge $f(u,y)$ - - - $f(x,y)$
  - Path $\left.\begin{array}{l} u \text{ — } g(s) \text{ - - - } g(t) \text{ — } x \end{array}\right\}\ \top$
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      - Path s — t , B-summary: $(s = t)$

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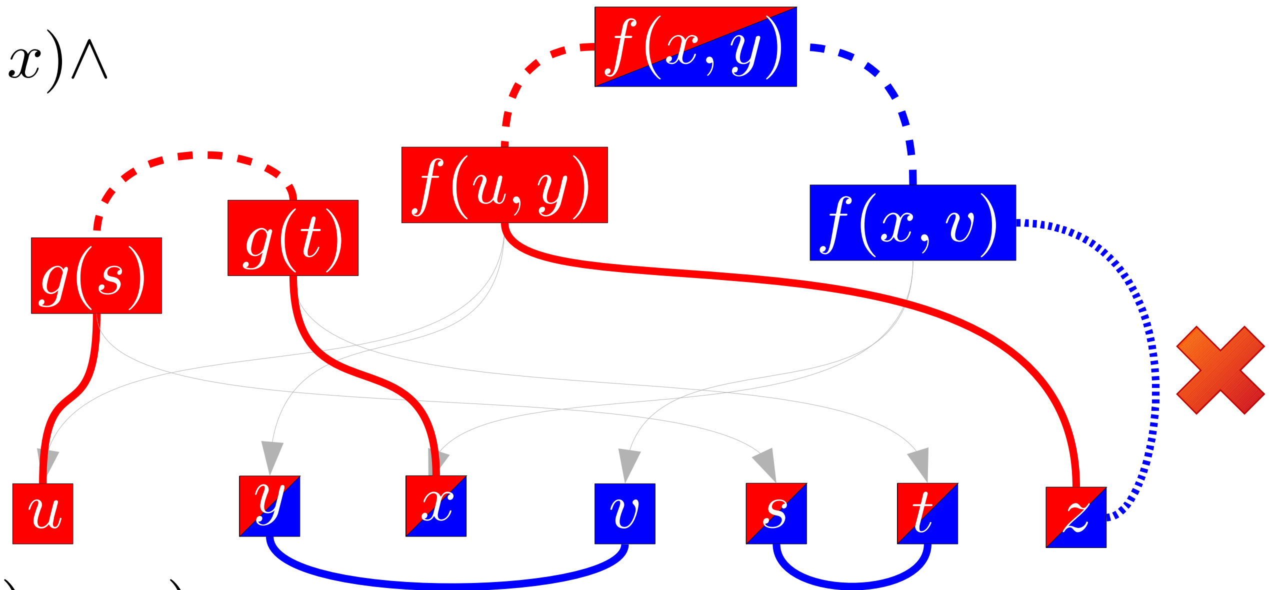

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  - Recurse on edge $g(s)$ - - - $g(t)$
    - Path s — t , *B*-summary: $(s = t)$

- Interpolant: $(s = t) \implies (z = f(x, y))$

Linear Rational Arithmetic (LRA)

- Interpolants from proofs of unsatisfiability of a system of inequalities $\sum_i a_i x_i \leq c$
- Proof of unsatisfiability: linear combination of inequalities with positive coefficients to derive a contradiction ($0 \leq c$ with $c < 0$)
- Interpolant obtained out of the proof by combining inequalities from *A* (using the same coefficients)
- Proof of unsatisfiability generated from the Simplex

Example

$$A := \underbrace{(3x_2 - x_1 \le 1)}_{s_1}, \underbrace{(0 \le x_1 + x_2)}_{s_2} \qquad B := \underbrace{(3 \le x_3 - 2x_1)}_{s_3}, \underbrace{(2x_3 \le 1)}_{s_4}$$

tableau

$$\begin{array}{rcl} s_1 &=& 3x_2 - x_1 \\ s_2 &=& x_1 + x_2 \\ s_3 &=& x_3 - 2x_1 \\ s_4 &=& 2x_3 \end{array}$$

bounds

$$\begin{array}{rcccl} -\infty &\le& s_1 &\le& 1 \\ 0 &\le& s_2 &\le& \infty \\ 3 &\le& s_3 &\le& \infty \\ -\infty &\le& s_4 &\le& 1 \end{array}$$

candidate solution β

$$\begin{array}{rcl} x_1 &\mapsto& 0 \\ x_2 &\mapsto& 0 \\ x_3 &\mapsto& 0 \\ s_1 &\mapsto& 0 \\ s_2 &\mapsto& 0 \\ s_3 &\mapsto& 0 \\ s_4 &\mapsto& 0 \end{array}$$

Example

$$\color{red}{A} := \underbrace{(3x_2 - x_1 \le 1)}_{s_1}, \underbrace{(0 \le x_1 + x_2)}_{s_2} \qquad \color{blue}{B} := \underbrace{(3 \le x_3 - 2x_1)}_{s_3}, \underbrace{(2x_3 \le 1)}_{s_4}$$

tableau

$$\begin{array}{lcl} x_3 & = & -\frac{1}{2}s_1 + \frac{3}{2}s_2 + s_3 \\ x_2 & = & \frac{1}{4}s_1 + \frac{1}{4}s_2 \\ x_1 & = & -\frac{1}{4}s_1 + \frac{3}{4}s_2 \\ \boxed{s_4 = -s_1 + 3s_2 + 2s_3} \end{array}$$

bounds

$$\begin{array}{ccccc} -\infty & \le & \boxed{s_1 \le 1} \\ \boxed{0 \le s_2} & \le & \infty \\ \boxed{3 \le s_3} & \le & \infty \\ -\infty & \le & \boxed{s_4 \le 1} \end{array}$$

candidate solution β

$$\begin{array}{lcr} x_1 & \mapsto & -\frac{1}{4} \\ x_2 & \mapsto & \frac{1}{4} \\ x_3 & \mapsto & \frac{5}{2} \\ s_1 & \mapsto & 1 \\ s_2 & \mapsto & 0 \\ s_3 & \mapsto & 3 \\ s_4 & \mapsto & 5 \end{array}$$

No suitable variable for pivoting!
Conflict

Example

$$A := \underbrace{(3x_2 - x_1 \le 1)}_{s_1}, \underbrace{(0 \le x_1 + x_2)}_{s_2} \qquad B := \underbrace{(3 \le x_3 - 2x_1)}_{s_3}, \underbrace{(2x_3 \le 1)}_{s_4}$$

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Proof:

$$\dfrac{\dfrac{\dfrac{1 \cdot (2x_3 \le 1) \qquad 1 \cdot (3x_2 - x_1 \le 1)}{(2x_3 + 3x_2 - x_1 \le 2)} \qquad 3 \cdot (0 \le x_1 + x_2)}{(2x_3 - 4x_1 \le 2)} \qquad 2 \cdot (3 \le x_3 - 2x_1)}{(0 \le -4)}$$

Example

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tableau | bounds | candidate solution β

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Interpolant:

$$\dfrac{\dfrac{\dfrac{— \qquad \textcolor{red}{1 \cdot (3x_2 - x_1 \leq 1)}}{(3x_2 - x_1 \leq 1)} \qquad \textcolor{red}{3 \cdot (0 \leq x_1 + x_2)}}{(-4x_1 \leq 1)} \qquad —}{\boxed{(-4x_1 \leq 1)}}$$

Linear Integer Arithmetic (LIA)

- Constraints of the form

$$\sum_i c_i x_i + c \bowtie 0, \qquad \bowtie \in \{\leq, =\}$$

- In general, no quantifier-free interpolation for LIA

Example: $A := (y - 2x = 0) \quad B := (y - 2z - 1 = 0)$

The only interpolant is: $\exists w.(y = 2w)$

- Solution: extend the signature to include modular equations (divisibility predicates)

$$(t + c =_d 0) \equiv \exists w.(t + c = d \cdot w), \quad d \in \mathbb{Z}^{>0}$$

The interpolant now becomes: $(y =_2 0)$

SMT(LIA) with modular equations

- Modular equations can be eliminated via preprocessing:
  - Replace every atom $a := (t + c =_d 0)$ with a fresh Boolean variable p_a
  - Add the 4 clauses

$$p_a \rightarrow (t + c - dw_1 = 0)$$

$$\neg p_a \rightarrow (t + c - dw_1 - w_2 = 0)$$

$$(-w_2 + 1 \leq 0)$$

$$(w_2 - d + 1 \leq 0)$$

where w_1, w_2 are fresh integer variables

Interpolants from LIA-proofs

- Cutting-plane proof system: complete proof system for LIA

$$\text{Hyp}\ \frac{-}{t \leq 0} \qquad\qquad \text{Comb}\ \frac{t_1 \leq 0 \qquad t_2 \leq 0}{c_1 \cdot t_1 + c_2 \cdot t_2 \leq 0}, c_1, c_2 > 0$$

$$\text{Div}\ \frac{\sum_i c_i x_i + c \leq 0}{\sum_i \frac{c_i}{d} x_i + \lceil \frac{c}{d} \rceil \leq 0}, d > 0 \text{ divides the } c_i\text{'s}$$

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LRA rules

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- Interpolation by annotating proof rules
  - Annotation: a set of pairs $\{\langle t_i \leq 0, \bigwedge_j (t_{ij} = 0) \rangle\}_i$
  - When $\perp$ is derived, then

$$I := \bigvee_i (t_i \leq 0 \wedge \bigwedge_j \text{ExistElim}(x_i \not\in B).(t_{ij} = 0))$$

is the computed interpolant

Interpolants from cutting-plane proofs

- Annotations for Hyp and Comb from McMillan (same as LRA)

$$\text{Hyp}\ \frac{-}{t \leq 0\ [\{\boxed{\langle t \leq 0,}\top\rangle\}]} \quad t' = \begin{cases} t & \text{if } t \leq 0 \in A \\ 0 & \text{if } t \leq 0 \in B \end{cases}$$

$$\text{Comb}\ \frac{t_1 \leq 0\ [I_1] \qquad t_2 \leq 0\ [I_2]}{c_1 \cdot t_1 + c_2 \cdot t_2 \leq 0\ [I]}$$

$$I := \{\langle c_1 t'_i + c_2 t'_j \leq 0, E_i \wedge E_j\rangle \mid \langle t'_i, E_i\rangle \in I_1, \langle t'_j, E_j\rangle \in I_2\}$$

- k-Strengthen rule of [Brillout et al. IJCAR'10]

$$\text{Str.}\ \frac{\sum_i c_i x_i + c \leq 0\ [\{\langle t \leq 0, \top\rangle\}]}{\sum_i c_i x_i + d \cdot \lceil \frac{c}{d} \rceil \leq 0\ [I]}, d > 0 \text{ divides the } c_i\text{'s}$$

$$I := \{\langle (t + n \leq 0),\ (t + n = 0)\rangle \mid 0 \leq n < d \cdot \lceil \tfrac{c}{d} \rceil - c\} \cup \{\langle (t + d \cdot \lceil \tfrac{c}{d} \rceil - c \leq 0), \top\rangle\}$$

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Example

$$A := \begin{cases} -y - 4x - 1 \leq 0 \\ y + 4x \leq 0 \end{cases} \qquad B := \begin{cases} -y - 4z + 1 \leq 0 \\ y + 4z - 2 \leq 0 \end{cases}$$

$$\dfrac{y + 4x \leq 0 \qquad -y - 4z + 1 \leq 0}{4x - 4z + 1 \leq 0}$$

$$-y - 4x - 1 \leq 0 \qquad y + 4z - 2 \leq 0$$

$$\dfrac{4x - 4z + 1 + 3 \leq 0 \qquad -4x + 4z - 3 \leq 0}{(1 \leq 0) \equiv \bot}$$

Example – with annotations

$$A := \begin{cases} -y - 4x - 1 \leq 0 \\ y + 4x \leq 0 \end{cases} \qquad B := \begin{cases} -y - 4z + 1 \leq 0 \\ y + 4z - 2 \leq 0 \end{cases}$$

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Example – with annotations

$$A := \begin{cases} -y - 4x - 1 \leq 0 \\ y + 4x \leq 0 \end{cases} \qquad B := \begin{cases} -y - 4z + 1 \leq 0 \\ y + 4z - 2 \leq 0 \end{cases}$$

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Interpolant: $(y =_4 0) \vee (y + 1 =_4 0)$

Drawback of Strengthen

- Interpolation of Strengthen creates potentially very big disjunctions
  - Linear in the strengthening factor $k := d\lceil \frac{c}{d} \rceil - c$
  - Can be exponential in the size of the proof

Example: $A := \begin{cases} -y - 4x - 1 \leq 0 \\ y + 4x \leq 0 \end{cases}$ $B := \begin{cases} -y - 4z + 1 \leq 0 \\ y + 4z - 2 \leq 0 \end{cases}$

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Drawback of Strengthen

- Interpolation of Strengthen creates potentially very big disjunctions
  - Linear in the strengthening factor $k := d\lceil \frac{c}{d} \rceil - c$
  - Can be exponential in the size of the proof

Example:

$$A := \begin{cases} -y - 2nx - n + 1 \leq 0 \\ y + 2nx \leq 0 \end{cases} \quad B := \begin{cases} -y - 2nz + 1 \leq 0 \\ y + 2nz - n \leq 0 \end{cases}$$

Interpolant: $(y =_{2n} 0) \vee (y + 1 =_{2n} 0) \vee \ldots \vee (y =_{2n} n - 1)$

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Interpolant: $(y =_{2n} 0) \vee (y + 1 =_{2n} 0) \vee \ldots \vee (y =_{2n} n - 1)$

- The problem are AB-mixed cuts:

$$\text{Strengthen}\ \frac{\sum_{x_i \notin B} c_i x_i + \sum_{y_j \notin A} c_j y_j + c \leq 0}{\sum_{x_i \notin B} c_i x_i + \sum_{y_j \notin A} c_j y_j + d \cdot \lceil \frac{c}{d} \rceil \leq 0}$$

Interpolation with ceilings

- Idea: use a different extension of the signature of LIA, and extend also its domain
  - Introduce the ceiling function $\lceil \cdot \rceil$ [Pudlák '97]
  - Allow non-variable terms to be non-integers (e.g. $\frac{x}{2}$)
- Much simpler interpolation procedure
  - Proof annotations are single inequalities $(t \leq 0)$

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- Much simpler interpolation procedure
  - Proof annotations are single inequalities $(t \leq 0)$

$$\text{Hyp}\ \frac{-}{t \leq 0\ [t' \leq 0]} \qquad \text{Comb}\ \frac{t_1 \leq 0\ [t_1' \leq 0] \qquad t_2 \leq 0\ [t_2' \leq 0]}{c_1 \cdot t_1 + c_2 \cdot t_2 \leq 0\ [c_1 \cdot t_1' + c_2 \cdot t_2' \leq 0]}$$

$$\text{Div}\ \frac{\begin{array}{c}\sum_{y_j \notin B} a_j y_j + \sum_{z_k \notin A} b_k z_k + \sum_{x_i \in A \cap B} c_i x_i + c \\ {[\sum_{y_j \notin B} a_j y_j + \sum_{x_i \in A \cap B} c_i' x_i + t']}\end{array}}{\begin{array}{c}\sum_{y_j \notin B} \frac{a_j}{d} y_j + \sum_{z_k \in B} \frac{b_k}{d} z_k + \sum_{x_i \in A \cap B} \frac{c_i}{d} x_i + \lceil \frac{c}{d} \rceil \\ {[\sum_{y_j \notin B} \frac{a_j}{d} y_j + \lceil \frac{\sum_{x_i \in A \cap B} c_i' x_i + t'}{d} \rceil]}\end{array}} \qquad d > 0 \text{ divides } a_j, b_k, c_i$$

Interpolation with ceilings - example

- No blowup of interpolants wrt. the size of the proofs

$$A := \begin{cases} -y - 2nx - n + 1 \leq 0 \\ y + 2nx \leq 0 \end{cases} \qquad B := \begin{cases} -y - 2nz + 1 \leq 0 \\ y + 2nz - n \leq 0 \end{cases}$$

$$\dfrac{\dfrac{\dfrac{y + 2nx \leq 0 \qquad -y - 2nz + 1 \leq 0}{2nx - 2nz + 1 \leq 0}}{2n \cdot (x - z + 1 \leq 0)} \qquad \dfrac{-y - 2nx - n + 1 \leq 0 \qquad y + 2nz - n \leq 0}{-2nx + 2nz - 2n + 1 \leq 0}}{(1 \leq 0) \equiv \bot}$$

Interpolation with ceilings - example

- No blowup of interpolants wrt. the size of the proofs

$$A := \begin{cases} -y - 2nx - n + 1 \leq 0 \\ y + 2nx \leq 0 \end{cases} \qquad B := \begin{cases} -y - 2nz + 1 \leq 0 \\ y + 2nz - n \leq 0 \end{cases}$$

$$\dfrac{\dfrac{\dfrac{\begin{array}{cc} y + 2nx \leq 0 & -y - 2nz + 1 \leq 0 \\ [y + 2nx \leq 0] & [0 \leq 0] \end{array}}{\begin{array}{c} 2nx - 2nz + 1 \leq 0 \\ [y + 2nx \leq 0] \end{array}}}{\begin{array}{c} 2n \cdot (x - z + 1 \leq 0) \\ [x + \lceil \frac{y}{2n} \rceil \leq 0] \end{array}} \quad \dfrac{\begin{array}{cc} -y - 2nx - n + 1 \leq 0 & y + 2nz - n \leq 0 \\ [-y - 2nx - n + 1 \leq 0] & [0 \leq 0] \end{array}}{\begin{array}{c} -2nx + 2nz - 2n + 1 \leq 0 \\ [-y - 2nx - n + 1 \leq 0] \end{array}}}{\begin{array}{c} (1 \leq 0) \equiv \bot \\ [2n \lceil \frac{y}{2n} \rceil - y - n + 1 \leq 0] \end{array}}$$

Interpolation with ceilings - example

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$$A := \begin{cases} -y - 2nx - n + 1 \leq 0 \\ y + 2nx \leq 0 \end{cases} \qquad B := \begin{cases} -y - 2nz + 1 \leq 0 \\ y + 2nz - n \leq 0 \end{cases}$$

$$\dfrac{\dfrac{\dfrac{\begin{array}{cc} y + 2nx \leq 0 & -y - 2nz + 1 \leq 0 \\ [y + 2nx \leq 0] & [0 \leq 0] \end{array}}{\begin{array}{c} 2nx - 2nz + 1 \leq 0 \\ [y + 2nx \leq 0] \end{array}}}{\begin{array}{c} 2n \cdot (x - z + 1 \leq 0) \\ [x + \lceil \frac{y}{2n} \rceil \leq 0] \end{array}} \qquad \dfrac{\begin{array}{cc} -y - 2nx - n + 1 \leq 0 & y + 2nz - n \leq 0 \\ [-y - 2nx - n + 1 \leq 0] & [0 \leq 0] \end{array}}{\begin{array}{c} -2nx + 2nz - 2n + 1 \leq 0 \\ [-y - 2nx - n + 1 \leq 0] \end{array}}}{(1 \leq 0) \equiv \bot}$$

Interpolant: $[2n\lceil \frac{y}{2n} \rceil - y - n + 1 \leq 0]$

SMT(LIA) with ceilings

- Like modular equations, also ceilings can be eliminated via preprocessing
  - Replace every term $\lceil t \rceil$ with a fresh integer variable $x_{\lceil t \rceil}$
  - Add the 2 unit clauses (encoding the meaning of ceiling: $\lceil t \rceil - 1 < t \leq \lceil t \rceil$)

$$(l \cdot x_{\lceil t \rceil} - l \cdot t + l \leq 0)$$

$$(l \cdot t - l \cdot x_{\lceil t \rceil} \leq 0)$$

where l is the least common multiple of the denominators of the coefficients in t

Bit-vectors (BV)

- Interpolation for bit-vectors is hard
  - Only some limited work done so far
- Most efficient solvers use eager encoding into SAT, which is efficient but not good for interpolation
  - Easy in principle, but not very useful interpolants
- Try to exploit lazy bit-blasting to incorporate BV into DPLL(T)

Interpolation via Bit-Blasting

- Interpolation via bit-blasting is easy…
  - From A_{BV} and B_{BV} generate $A_{\texttt{Bool}}$ and $B_{\texttt{Bool}}$

    Each var x of width *n* encoded with *n* Boolean vars $b_1^x \ldots b_n^x$
  - Generate a Boolean interpolant $I_{\texttt{Bool}}$ for $(A_{\texttt{Bool}}, B_{\texttt{Bool}})$
  - Replace every variable b_i^x in $I_{\texttt{Bool}}$ with the bit-selection $x[i]$ and every Boolean connective with the corresponding bit-wise connective: $\wedge \mapsto \&, \quad \vee \mapsto |, \quad \neg \mapsto \sim$

- ...but quite impractical
  - Generates “ugly” interpolants
  - Word-level structure of the original problem completely lost
    - How to apply word-level simplifications?

Interpolation via Bit-Blasting - Example

$$A \stackrel{\text{def}}{=} (a_{[8]} * b_{[8]} = 15_{[8]}) \wedge (a_{[8]} = 3_{[8]})$$

$$B \stackrel{\text{def}}{=} \neg(b_{[8]} \%_u c_{[8]} = 1_{[8]}) \wedge (c_{[8]} = 2_{[8]})$$

A word-level interpolant is:

$$I \stackrel{\text{def}}{=} (b_{[8]} * 3_{[8]} = 15_{[8]})$$

...but with bit-blasting we get:

$$I' \stackrel{\text{def}}{=} (b_{[8]}[0] = 1_{[1]}) \wedge ((b_{[8]}[0] \& \sim (((((((\sim b_{[8]}[7] \& \sim b_{[8]}[6]) \& \sim b_{[8]}[5]) \& \sim b_{[8]}[4]) \& \sim b_{[8]}[3]) \& b_{[8]}[2]) \& \sim b_{[8]}[1])) = 0_{[1]})$$

Alternative: lazy bit-blasting and DPLL(T)

- Exploit *lazy bit-blasting*
  - Bit-blast only BV-atoms, not the whole formula
  - Boolean skeleton of the formula handled by the "main" DPLL, like in DPLL(T)
  - Conjunctions of BV-atoms handled (via bit-blasting) by a "sub"-DPLL (DPLL-BV) that acts as a BV-solver

Standard Boolean Interpolation + BV-specific Interpolation for *conjunctions of constraints*

Interpolation for BV constraints

- A layered approach
- Apply in sequence a chain of procedures of increasing generality and cost
  - Interpolation in EUF
  - Interpolation via equality inlining
  - Interpolation via Linear Integer Arithmetic encoding
  - Interpolation via bit-blasting

Interpolation in EUF

- Treat all the BV-operators as uninterpreted functions
- Exploit cheap, efficient algorithms for solving and interpolating modulo EUF
  - Possible because we avoid bit-blasting upront!

Example:

$$A \stackrel{\text{def}}{=} (x_{1\,[32]} = 3_{[32]}) \wedge (x_{3\,[32]} = x_{1\,[32]} \cdot x_{2\,[32]})$$

$$B \stackrel{\text{def}}{=} (x_{4\,[32]} = x_{2\,[32]}) \wedge (x_{5\,[32]} = 3_{[32]} \cdot x_{4\,[32]}) \wedge \neg(x_{3\,[32]} = x_{5\,[32]})$$

$$I_{\mathsf{UF}} \stackrel{\text{def}}{=} x_3 = f^{\cdot}(f^3, x_2)$$

$$I_{\mathsf{BV}} \stackrel{\text{def}}{=} x_{3\,[32]} = 3_{[32]} \cdot x_{2\,[32]}$$

Interpolation via Equality Inlining

- Interpolation via quantifier elimination: given (A, B), an interpolant can be computed by eliminating quantifiers from $\exists_{x \notin B} A$ or from $\exists_{x \notin A} \neg B$

- In general, this can be very expensive for BV
  - Might require bit-blasting and can cause blow-up of the formula

- Cheap case: non-common variables occurring in “definitional” equalities

  Example: $(x = e) \wedge \varphi$ and x does not occur in e, then

$$\exists_x ((x = e) \wedge \varphi) \Longrightarrow \varphi[x \mapsto e]$$

Interpolation via Equality Inlining

- Inline definitional equalities until either all all non-common variables are removed, or a fixpoint is reached
  - Try both from A and $\neg B$
  - If one of them succeeds, we have an interpolant

Example: $A \stackrel{\text{def}}{=} (0_{[24]} :: (x_{4[8]} \cdot x_{5[8]}) \leq_s (0_{[24]} :: x_{1[8]} - 1_{[32]})) \wedge$
$(x_{2[8]} = x_{1[8]}) \wedge (x_{4[8]} = 192_{[8]}) \wedge (x_{5[8]} = 128_{[8]})$

$B \stackrel{\text{def}}{=} ((x_{3[8]} \cdot x_{6[8]}) = (-(0_{[24]} :: x_{2[8]}))[7:0]) \wedge$
$(x_{3[8]} <_u 1_{[8]}) \wedge (0_{[8]} \leq_u x_{3[8]}) \wedge (x_{6[8]} = 1_{[8]})$

Interpolation via Equality Inlining

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$$A \stackrel{\text{def}}{=} (0_{[24]} :: (x_{4\,[8]} \cdot x_{5\,[8]}) \leq_s (0_{[24]} :: x_{1\,[8]} - 1_{[32]})) \wedge \boxed{(x_{2\,[8]} = x_{1\,[8]})} \wedge \boxed{(x_{4\,[8]} = 192_{[8]})} \wedge \boxed{(x_{5\,[8]} = 128_{[8]})}$$

Definitional equalities

$$B \stackrel{\text{def}}{=} ((x_{3\,[8]} \cdot x_{6\,[8]}) = (-(0_{[24]} :: x_{2\,[8]}))[7:0]) \wedge (x_{3\,[8]} <_u 1_{[8]}) \wedge (0_{[8]} \leq_u x_{3\,[8]}) \wedge \boxed{(x_{6\,[8]} = 1_{[8]})}$$

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$B \stackrel{\text{def}}{=} ((x_{3\,[8]} \cdot x_{6\,[8]}) = (-(0_{[24]} :: x_{2\,[8]}))[7:0])\wedge$
$(x_{3\,[8]} <_u 1_{[8]}) \wedge (0_{[8]} \leq_u x_{3\,[8]}) \wedge (x_{6\,[8]} = 1_{[8]})$

Interpolation via Equality Inlining

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Example: $A \stackrel{\text{def}}{=} (0_{[24]} :: (192_{[8]} \cdot 128_{[8]}) \leq_s (0_{[24]} :: {x_2}_{[8]} - 1_{[32]}))$

$\wedge \qquad \wedge$

$B \stackrel{\text{def}}{=} (({x_3}_{[8]} \cdot {x_6}_{[8]}) = (-(0_{[24]} :: {x_2}_{[8]}))[7:0]) \wedge$
$({x_3}_{[8]} <_u 1_{[8]}) \wedge (0_{[8]} \leq_u {x_3}_{[8]}) \wedge ({x_6}_{[8]} = 1_{[8]})$

Interpolation via Equality Inlining

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  - If one of them succeeds, we have an interpolant

Example: $A \stackrel{\text{def}}{=} (0_{[24]} :: (192_{[8]} \cdot 128_{[8]}) \leq_s (0_{[24]} :: x_{2\,[8]} - 1_{[32]}))$

$$\wedge \qquad \wedge$$

$$I \stackrel{\text{def}}{=} (0_{32} \leq_s (0_{24} :: x_{2\,[8]} - 1_{[32]})$$

$$B \stackrel{\text{def}}{=} ((x_{3\,[8]} \cdot x_{6\,[8]}) = (-(0_{[24]} :: x_{2\,[8]}))[7:0]) \wedge$$
$$(x_{3\,[8]} <_u 1_{[8]}) \wedge (0_{[8]} \leq_u x_{3\,[8]}) \wedge (x_{6\,[8]} = 1_{[8]})$$

Interpolation via LIA Encoding

- Simple idea (in principle):
  - Encode a set of BV-constraints into an SMT(LIA)-formula
  - Generate a LIA-interpolant using existing algorithms
  - Map back to a BV-interpolant

- However, several problems to solve:
  - Efficiency
  - More importantly, soundness

Encoding BV into LIA

- Use well-known encodings from BV to SMT(LIA)
  - Encode each BV term $t_{[n]}$ as an integer variable x_t and the constraints $(0 \le x_t) \wedge (x_t \le 2^n - 1)$
  - Encode each BV operation as a LIA-formula.

Examples:

$$t_{[i-j+1]} \stackrel{\text{def}}{=} t_{1[n]}[i:j] \Rightarrow (x_t = m) \wedge (x_{t_1} = 2^{i+1}h + 2^j m + l) \wedge$$
$$l \in [0, 2^i) \wedge m \in [0, 2^{i-j+1}) \wedge h \in [0, 2^{n-i-1})$$

$$t_{[n]} \stackrel{\text{def}}{=} t_{1[n]} + t_{2[n]} \Rightarrow (x_t = x_{t_1} + x_{t_2} - 2^n\sigma) \wedge (0 \le \sigma \le 1)$$

$$t_{[n]} \stackrel{\text{def}}{=} t_{1[n]} \cdot k \Rightarrow (x_t = k \cdot x_{t_1} - 2^n\sigma) \wedge (0 \le \sigma \le k)$$

From LIA-interpolants to BV-interpolants

- "Invert" the LIA encoding to get a BV interpolant
- *Unsound* in general
  - Issues due to overflow and (un)signedness of operations
- Our (very simple) solution: *check the interpolants*
  - Given a candidate interpolant $\hat{I}$, use our SMT(BV) solver to check the unsatisfiability of $(A \wedge \neg\hat{I}) \vee (B \wedge \hat{I})$
  - If successful, then $\hat{I}$ is an interpolant

From LIA- to BV-interpolants: examples

$$A \stackrel{\text{def}}{=} (y_{1[8]} = y_{5[4]} :: y_{5[4]}) \wedge (y_{1[8]} = y_{2[8]}) \wedge (y_{5[4]} = 1_{[4]})$$

$$B \stackrel{\text{def}}{=} \neg(y_{4[8]} + 1_{[8]} \leq_u y_{2[8]}) \wedge (y_{4[8]} = 1_{[8]})$$

Encoding into LIA:

$$\begin{aligned} A_{\text{LIA}} \stackrel{\text{def}}{=} & (x_{y_2} = 16x_{y_5} + x_{y_5}) \wedge (x_{y_1} = x_{y_2}) \wedge (x_{y_5} = 1) \wedge \\ & (x_{y_1} \in [0, 2^8)) \wedge (x_{y_2} \in [0, 2^8)) \wedge (x_{y_5} \in [0, 2^4)) \end{aligned}$$

$$\begin{aligned} B_{\text{LIA}} \stackrel{\text{def}}{=} & \neg(x_{y_4+1} \leq x_{y_2}) \wedge (x_{y_4+1} = x_{y_4} + 1 - 2^8\sigma) \wedge \\ & (x_{y_4} = 1) \wedge \\ & (x_{y_4+1} \in [0, 2^8)) \wedge (x_{y_4} \in [0, 2^8)) \wedge (0 \leq \sigma \leq 1) \end{aligned}$$

From LIA- to BV-interpolants: examples

$$A \stackrel{\text{def}}{=} ({y_1}_{[8]} = {y_5}_{[4]} :: {y_5}_{[4]}) \wedge ({y_1}_{[8]} = {y_2}_{[8]}) \wedge ({y_5}_{[4]} = 1_{[4]})$$

$$B \stackrel{\text{def}}{=} \neg({y_4}_{[8]} + 1_{[8]} \leq_u {y_2}_{[8]}) \wedge ({y_4}_{[8]} = 1_{[8]})$$

LIA-Interpolant:

$$I_{\text{LIA}} \stackrel{\text{def}}{=} (17 \leq x_{y_2})$$

BV-interpolant:

$$I \stackrel{\text{def}}{=} (17_{[8]} \leq_u {y_2}_{[8]})$$

Good!

From LIA- to BV-interpolants: examples

$$A \stackrel{\text{def}}{=} (y_{2[8]} = 81_{[8]}) \wedge (y_{3[8]} = 0_{[8]}) \wedge (y_{4[8]} = y_{2[8]})$$

$$B \stackrel{\text{def}}{=} (y_{13[16]} = 0_{[8]} :: y_{4[8]}) \wedge (255_{[16]} \leq_u y_{13[16]} + (0_{[8]} :: y_{3[8]}))$$

Encoding into LIA:

$$\begin{aligned} A_{\text{LIA}} \stackrel{\text{def}}{=} & (x_{y_2} = 81) \wedge (x_{y_3} = 0) \wedge (x_{y_4} = x_{y_2}) \wedge \\ & (x_{y_2} \in [0, 2^8)) \wedge (x_{y_3} \in [0, 2^8)) \wedge (x_{y_4} \in [0, 2^8)) \end{aligned}$$

$$\begin{aligned} B_{\text{LIA}} \stackrel{\text{def}}{=} & (x_{y_{13}} = 2^8 \cdot 0 + x_{y_4}) \wedge (255 \leq x_{y_{13}+(0::y_3)}) \wedge \\ & (x_{y_{13}+(0::y_3)} = x_{y_{13}} + 2^8 \cdot 0 + x_{y_3} - 2^{16}\sigma) \wedge \\ & (x_{y_{13}} \in [0, 2^{16})) \wedge (x_{y_{13}+(0::y_3)} \in [0, 2^{16})) \wedge \\ & (0 \leq \sigma \leq 1) \end{aligned}$$

From LIA- to BV-interpolants: examples

$$A \stackrel{\text{def}}{=} (y_{2[8]} = 81_{[8]}) \wedge (y_{3[8]} = 0_{[8]}) \wedge (y_{4[8]} = y_{2[8]})$$

$$B \stackrel{\text{def}}{=} (y_{13[16]} = 0_{[8]} :: y_{4[8]}) \wedge (255_{[16]} \leq_u y_{13[16]} + (0_{[8]} :: y_{3[8]}))$$

LIA-interpolant:

$$I_{\text{LIA}} \stackrel{\text{def}}{=} (x_{y_3} + x_{y_4} \leq 81)$$

BV-interpolant:

$$\hat{I} \stackrel{\text{def}}{=} (y_{3[8]} + y_{4[8]} \leq_u 81_{[8]})$$

Wrong!

$B \wedge \hat{I} \not\models \bot$

From LIA- to BV-interpolants: examples

$$A \stackrel{\text{def}}{=} (y_{2[8]} = 81_{[8]}) \wedge (y_{3[8]} = 0_{[8]}) \wedge (y_{4[8]} = y_{2[8]})$$

$$B \stackrel{\text{def}}{=} (y_{13[16]} = 0_{[8]} :: y_{4[8]}) \wedge (255_{[16]} \leq_u y_{13[16]} + (0_{[8]} :: y_{3[8]}))$$

LIA-interpolant:

$$I_{\text{LIA}} \stackrel{\text{def}}{=} (x_{y_3} + x_{y_4} \leq 81)$$

Addition might overflow in BV!

BV-interpolant:

$$\hat{I} \stackrel{\text{def}}{=} (y_{3[8]} + y_{4[8]} \leq_u 81_{[8]})$$

Wrong!

$B \wedge \hat{I} \not\models \bot$

From LIA- to BV-interpolants: examples

$$A \stackrel{\text{def}}{=} (y_{2[8]} = 81_{[8]}) \wedge (y_{3[8]} = 0_{[8]}) \wedge (y_{4[8]} = y_{2[8]})$$

$$B \stackrel{\text{def}}{=} (y_{13[16]} = 0_{[8]} :: y_{4[8]}) \wedge (255_{[16]} \leq_u y_{13[16]} + (0_{[8]} :: y_{3[8]}))$$

LIA-interpolant:

$$I_{\text{LIA}} \stackrel{\text{def}}{=} (x_{y_3} + x_{y_4} \leq 81)$$

Addition might overflow in BV!

BV-interpolant:

A correct interpolant would be

$$I \stackrel{\text{def}}{=} (0_{[1]} :: y_{3[8]} + 0_{[1]} :: y_{4[8]} \leq_u 81_{[9]})$$

Wrong!

$B \wedge \hat{I} \not\models \bot$

From LIA- to BV-interpolants: examples

$$A \stackrel{\text{def}}{=} \neg({y_4}_{[8]} + 1_{[8]} \leq_u {y_3}_{[8]}) \wedge ({y_2}_{[8]} = {y_4}_{[8]} + 1_{[8]})$$

$$B \stackrel{\text{def}}{=} ({y_2}_{[8]} + 1_{[8]} \leq_u {y_3}_{[8]}) \wedge ({y_7}_{[8]} = 3_{[8]}) \wedge ({y_7}_{[8]} = {y_2}_{[8]} + 1_{[8]})$$

Encoding into LIA:

$$\begin{aligned} A_{\text{LIA}} \stackrel{\text{def}}{=} & \neg(x_{y_4+1} \leq x_{y_3}) \wedge (x_{y_2} = x_{y_4+1}) \wedge \\ & (x_{y_4+1} = x_{y_4} + 1 - 2^8\sigma_1) \wedge \\ & (x_{y_2} \in [0, 2^8)) \wedge (x_{y_3} \in [0, 2^8)) \wedge (x_{y_4} \in [0, 2^8)) \wedge \\ & (x_{y_4+1} \in [0, 2^8)) \wedge (0 \leq \sigma_1 \leq 1) \end{aligned}$$

$$\begin{aligned} B_{\text{LIA}} \stackrel{\text{def}}{=} & (x_{y_2+1} \leq x_{y_3}) \wedge (x_{y_7} = 3) \wedge (x_{y_7} = x_{y_2+1}) \wedge \\ & (x_{y_2+1} = x_{y_2} + 1 - 2^8\sigma_2) \wedge \\ & (x_{y_7} \in [0, 2^8)) \wedge (x_{y_2+1} \in [0, 2^8)) \wedge (0 \leq \sigma_2 \leq 1) \end{aligned}$$

From LIA- to BV-interpolants: examples

$$A \stackrel{\text{def}}{=} \neg(y_{4[8]} + 1_{[8]} \leq_u y_{3[8]}) \wedge (y_{2[8]} = y_{4[8]} + 1_{[8]})$$

$$B \stackrel{\text{def}}{=} (y_{2[8]} + 1_{[8]} \leq_u y_{3[8]}) \wedge (y_{7[8]} = 3_{[8]}) \wedge (y_{7[8]} = y_{2[8]} + 1_{[8]})$$

LIA-interpolant:

$$I_{\text{LIA}} \stackrel{\text{def}}{=} (-255 \leq x_{y_2} - x_{y_3} + 256 \lfloor -1 \frac{x_{y_2}}{256} \rfloor)$$

BV-interpolant: (after fixing overflows)

$$\hat{I}' \stackrel{\text{def}}{=} (65281_{[16]} \leq_u (0_{[8]} :: y_{2[8]}) - (0_{[8]} :: y_{3[8]}) + 256_{[16]} \cdot (65535_{[16]} \cdot (0_{[8]} :: y_{2[8]}) /_u 256_{[16]}))$$

From LIA- to BV-interpolants: examples

$$\color{red}{A} \stackrel{\text{def}}{=} \neg(\color{red}{y_4}_{[8]} + 1_{[8]} \leq_u y_3{}_{[8]}) \wedge (y_2{}_{[8]} = \color{red}{y_4}_{[8]} + 1_{[8]})$$

$$\color{blue}{B} \stackrel{\text{def}}{=} (y_2{}_{[8]} + 1_{[8]} \leq_u y_3{}_{[8]}) \wedge (\color{blue}{y_7}_{[8]} = 3_{[8]}) \wedge (\color{blue}{y_7}_{[8]} = y_2{}_{[8]} + 1_{[8]})$$

LIA-interpolant:

$$I_{\text{LIA}} \stackrel{\text{def}}{=} (-255 \leq x_{y_2} - x_{y_3} + 256\lfloor -1\frac{x_{y_2}}{256} \rfloor)$$

BV-interpolant: (after fixing overflows)

$$\hat{I}' \stackrel{\text{def}}{=} (65281_{[16]} \leq_u (0_{[8]} :: y_2{}_{[8]}) - (0_{[8]} :: y_3{}_{[8]}) + 256_{[16]} \cdot (65535_{[16]} \cdot (0_{[8]} :: y_2{}_{[8]}) /_u 256_{[16]}))$$

In this case, the problem is also the sign

Still Wrong!

From LIA- to BV-interpolants: examples

$$A \stackrel{\text{def}}{=} \neg({y_4}_{[8]} + 1_{[8]} \leq_u {y_3}_{[8]}) \wedge ({y_2}_{[8]} = {y_4}_{[8]} + 1_{[8]})$$

$$B \stackrel{\text{def}}{=} ({y_2}_{[8]} + 1_{[8]} \leq_u {y_3}_{[8]}) \wedge ({y_7}_{[8]} = 3_{[8]}) \wedge ({y_7}_{[8]} = {y_2}_{[8]} + 1_{[8]})$$

LIA-interpolant:

$$I_{\text{LIA}} \stackrel{\text{def}}{=} (-255 \leq x_{y_2} - x_{y_3} + 256\lfloor -1\frac{x_{y_2}}{256} \rfloor)$$

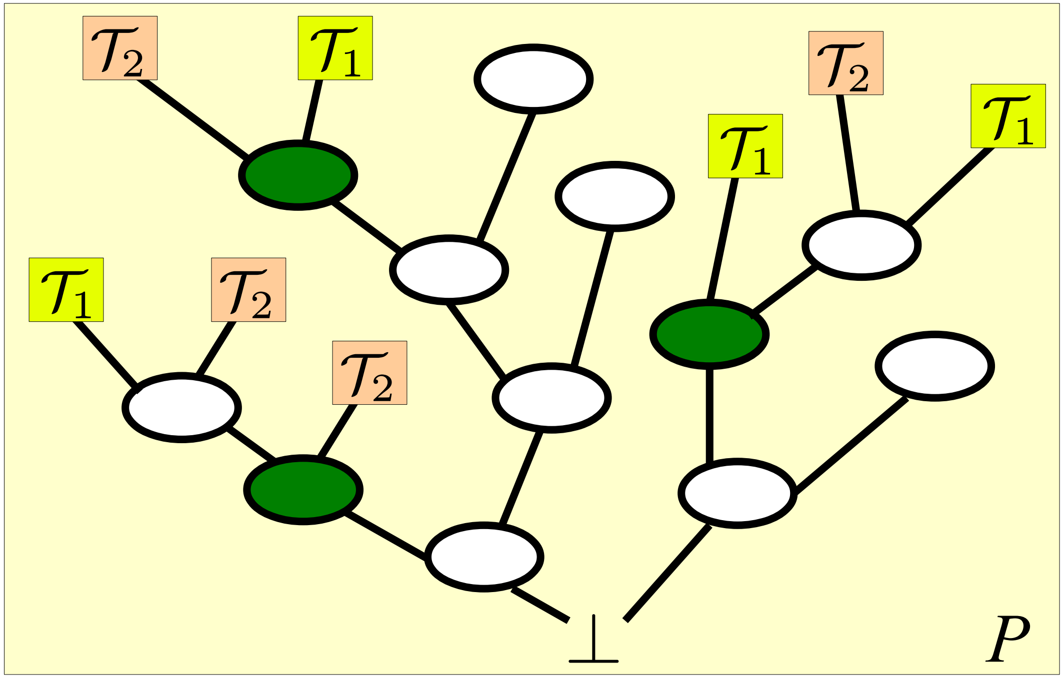
BV-interpolant:

$$\begin{aligned} I \stackrel{\text{def}}{=} &(65281_{[16]} \leq_s (0_{[8]} :: {y_2}_{[8]}) - (0_{[8]} :: {y_3}_{[8]}) + \\ &256_{[16]} \cdot (65535_{[16]} \cdot (0_{[8]} :: {y_2}_{[8]}) /_u 256_{[16]})) \end{aligned}$$

Correct interpolant

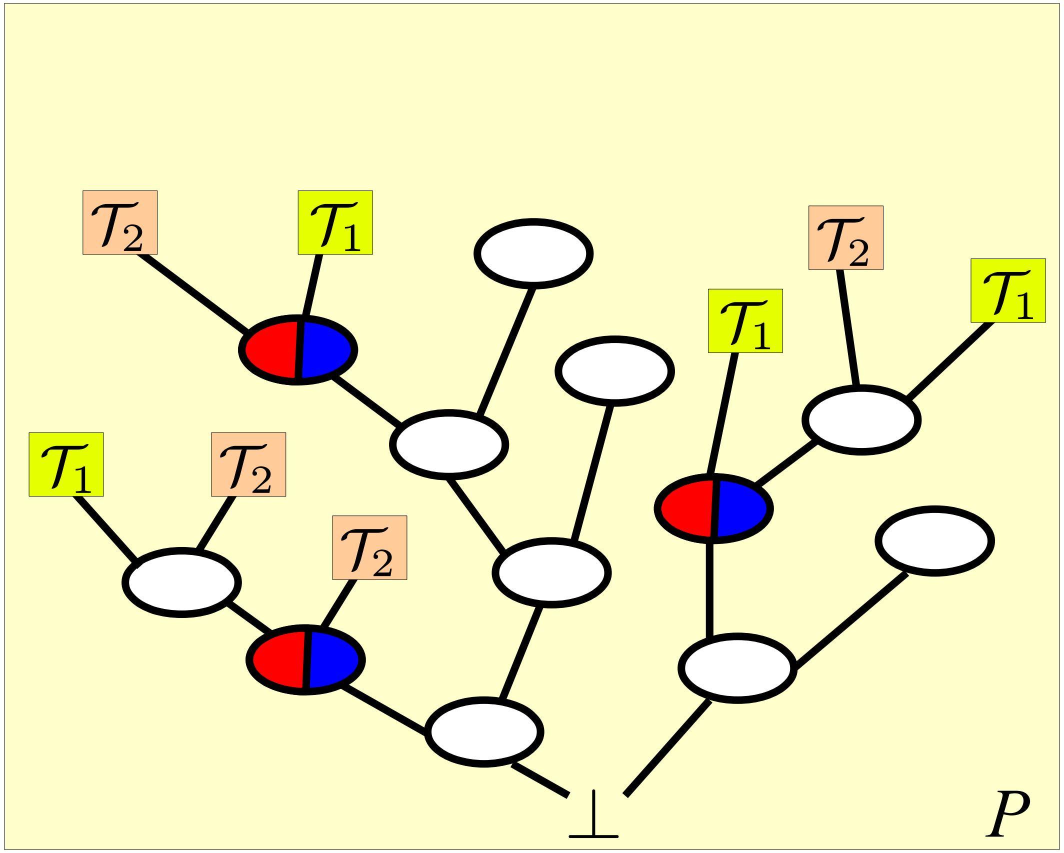
Interpolation in combined theories

- Delayed Theory Combination (DTC): use the DPLL engine to perform theory combination
  - Independent $\mathcal{T}_i$-solvers, that interact only with DPLL
  - How: Boolean search space augmented with interface equalities
    - Equalities between variables shared by the two theories
- Combination of theories encoded directly in the proof of unsatisfiability P
  - $\mathcal{T}_i$-lemmas for the individual theories
  - ***P*** contains interface equalities



Interpolation in combined theories

- Problem for interpolation:
  - Some interface equalities ($x = y$) are AB-mixed: $x \not\in B,\ y \not\in A$
  - *Interpolation procedures don't work with AB-mixed terms*
- Solution: *Split AB-mixed equalities occurring in P, and fix the proof*
  - How: Split each $\mathcal{T}$-lemma $\eta \vee (x = y)$ into $(\eta \vee (x = t)) \wedge \eta \vee (t = y)$ with $t \in A \cap B$ using available algorithms
  - $\mathcal{T}_i$'s must be equality-interpolating and convex
  - Propagate the changes throughout P



Interpolation in combined theories

- Problem for interpolation:
  - Some interface equalities ($x = y$) are AB-mixed: $x \notin B,\ y \notin A$
  - *Interpolation procedures don't work with AB-mixed terms*
- Solution: *Split AB-mixed equalities occurring in P, and fix the proof*
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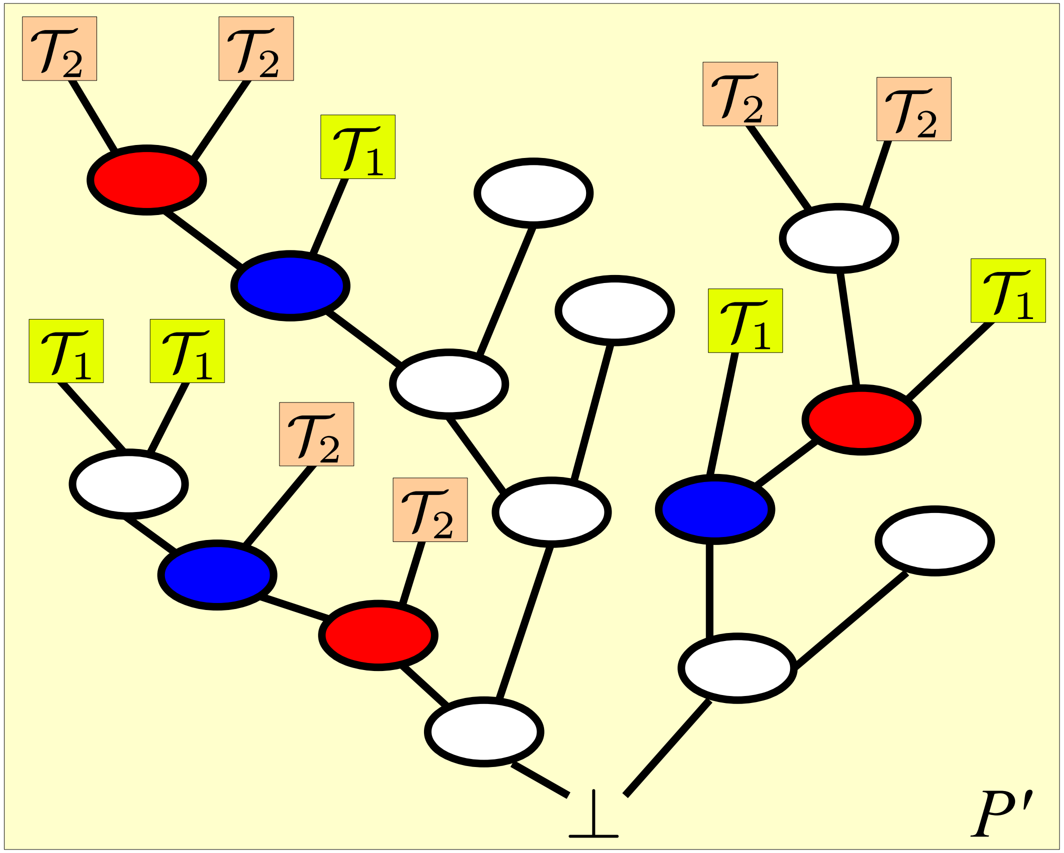
  - $\mathcal{T}_i$'s must be equality-interpolating and convex
  - Propagate the changes throughout P

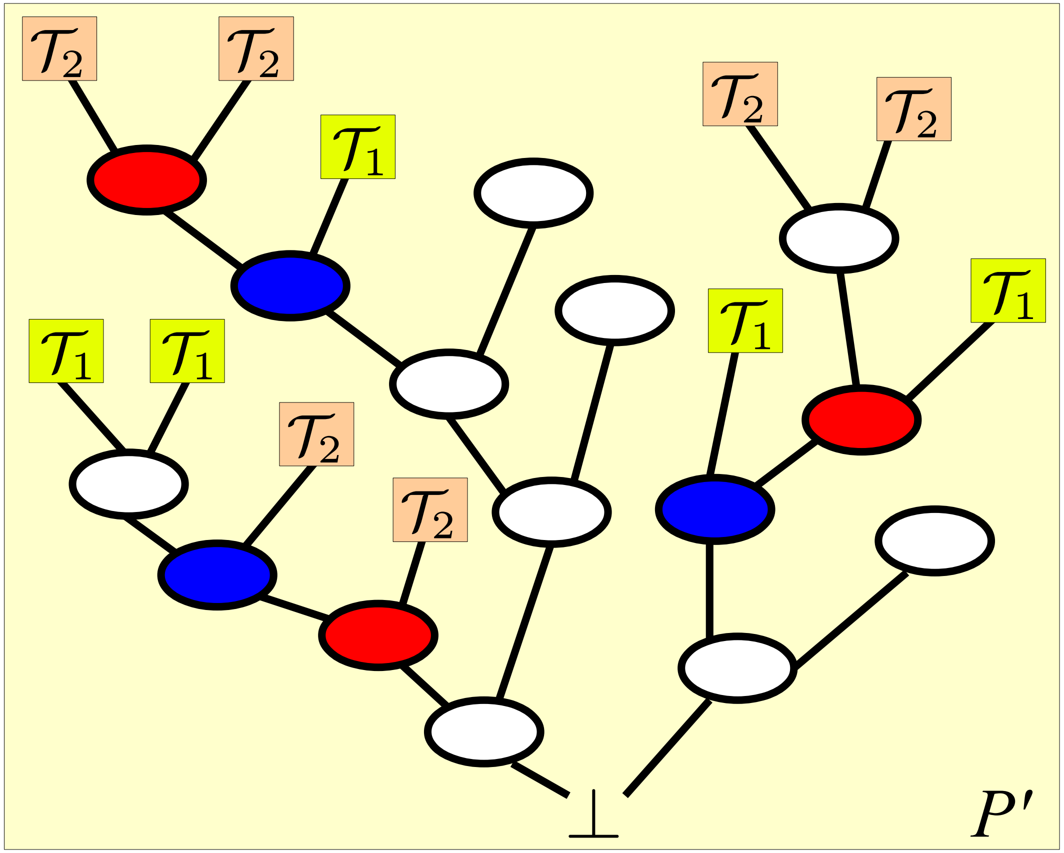


Interpolation in combined theories

- Problem for interpolation:
  - Some interface equalities ($x = y$) are AB-mixed: $x \not\in B,\ y \not\in A$
  - *Interpolation procedures don't work with AB-mixed terms*
- Solution: *Split AB-mixed equalities occurring in P, and fix the proof*
  - How: Split each $\mathcal{T}$-lemma

Problem: splitting can cause exponential blow-up in P

Solution: control the kind of proofs generated by DPLL, so that the splitting can be performed **efficiently** (ie-local proofs)



Interpolation in combined theories

- After splitting AB-mixed equalities, we can compute an interpolant as usual
  - *Nothing special needed for theory combination!*
  - Because theory combination is encoded in the proof, we can reuse the Boolean interpolation algorithm
- Features:
  - No need of ad-hoc interpolant combination procedures
  - Exploit state-of-the-art SMT solvers, based on (variants of) DTC
  - Split only when necessary

Example

$$A := (a_1 = f(x_1)) \wedge (z - x_1 = 1) \wedge (a_1 + z = 0)$$

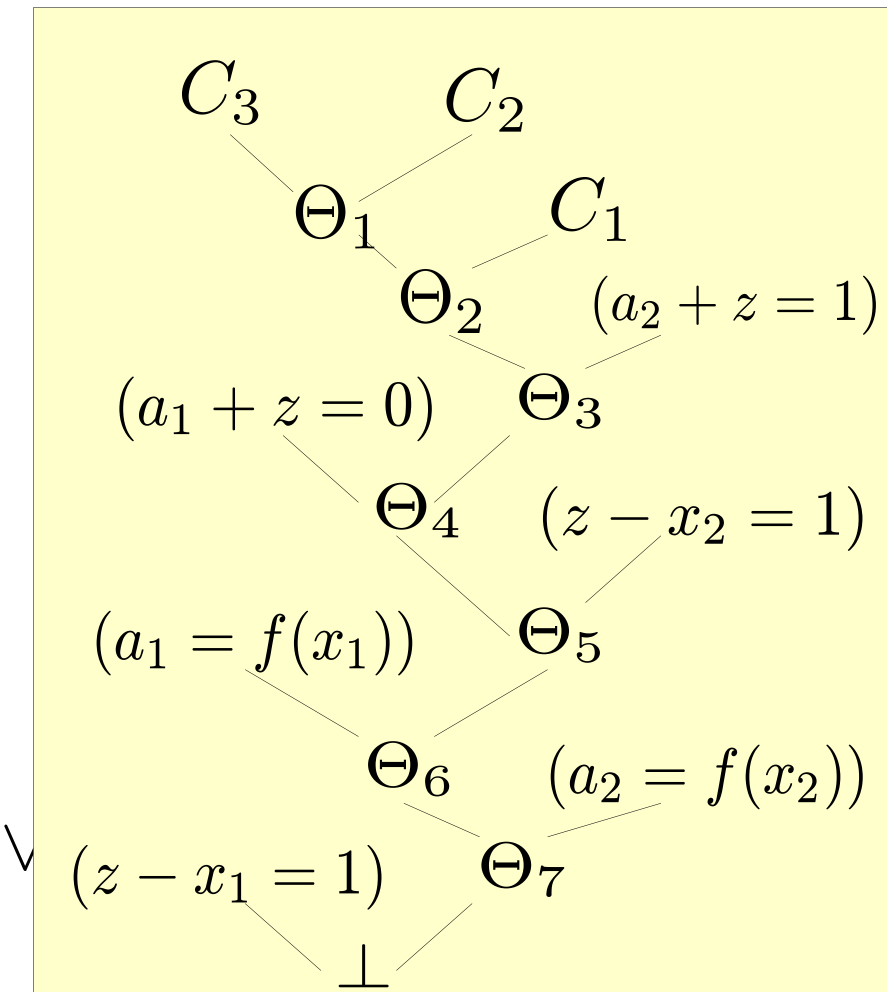
$$B := (a_2 = f(x_2)) \wedge (z - x_2 = 1) \wedge (a_2 + z = 1)$$

Example

$$A := (a_1 = f(x_1)) \wedge (z - x_1 = 1) \wedge (a_1 + z = 0)$$
$$B := (a_2 = f(x_2)) \wedge (z - x_2 = 1) \wedge (a_2 + z = 1)$$

T-lemmas:

$$C_1 \equiv (x_1 = x_2) \vee \neg(z - x_1 = 1) \vee \neg(z - x_2 = 1)$$
$$C_2 \equiv (a_1 = a_2) \vee \neg(a_2 = f(x_2)) \vee \neg(a_1 = f(x_1)) \vee \neg(x_1 = x_2)$$
$$C_3 \equiv \neg(a_1 + z = 0) \vee \neg(a_2 + z = 1) \vee \neg(a_1 = a_2)$$



Example

$$A := (a_1 = f(x_1)) \wedge (z - x_1 = 1) \wedge (a_1 + z = 0)$$

$$B := (a_2 = f(x_2)) \wedge (z - x_2 = 1) \wedge (a_2 + z = 1)$$

Pivot: $(a_1 = a_2)$

Pivot: $(x_1 = x_2)$

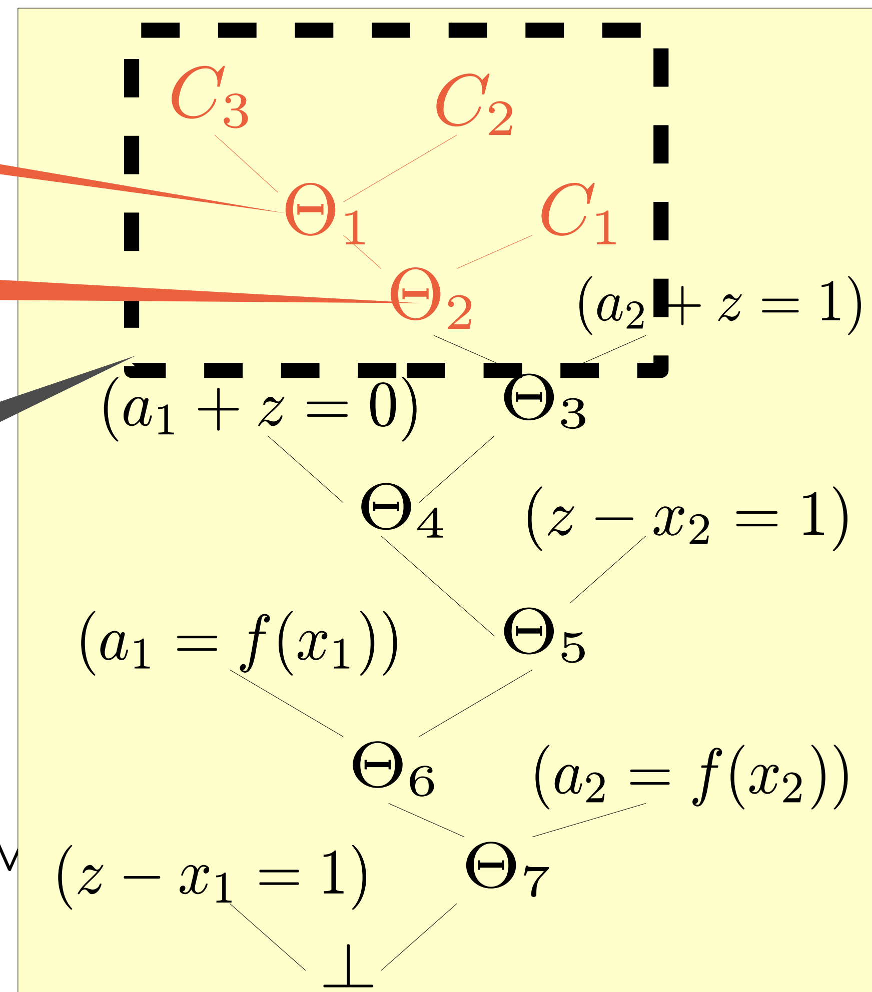
T-lemmas:

$$C_1 \equiv (x_1 = x_2) \vee \neg(z - x_1 = 1) \vee \neg(z - x_2 = 1)$$

subproof with int.eqs.

$$C_2 \equiv (a_1 = a_2) \vee \neg(a_2 = f(x_2)) \vee \neg(a_1 = f(x_1)) \vee \neg(x_1 = x_2)$$

$$C_3 \equiv \neg(a_1 + z = 0) \vee \neg(a_2 + z = 1) \vee \neg(a_1 = a_2)$$

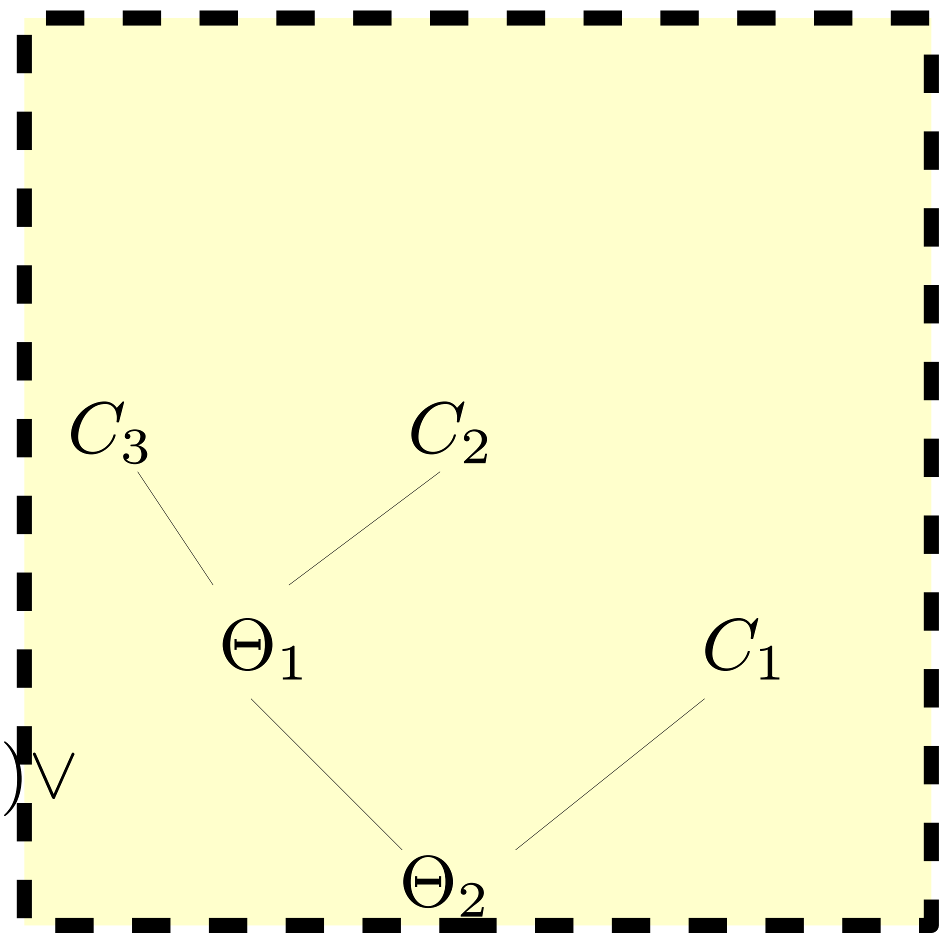


Example

$$A := (a_1 = f(x_1)) \wedge (z - x_1 = 1) \wedge (a_1 + z = 0)$$

$$B := (a_2 = f(x_2)) \wedge (z - x_2 = 1) \wedge (a_2 + z = 1)$$

P^{ie} subproof:



T-lemmas:

$$C_1 \equiv (x_1 = x_2) \vee \neg(z - x_1 = 1) \vee \neg(z - x_2 = 1)$$

$$C_2 \equiv (a_1 = a_2) \vee \neg(a_2 = f(x_2)) \vee \neg(a_1 = f(x_1)) \vee \neg(x_1 = x_2)$$

$$C_3 \equiv \neg(a_1 + z = 0) \vee \neg(a_2 + z = 1) \vee \neg(a_1 = a_2)$$

Example

$$A := (a_1 = f(x_1)) \wedge (z - x_1 = 1) \wedge (a_1 + z = 0)$$

$$B := (a_2 = f(x_2)) \wedge (z - x_2 = 1) \wedge (a_2 + z = 1)$$

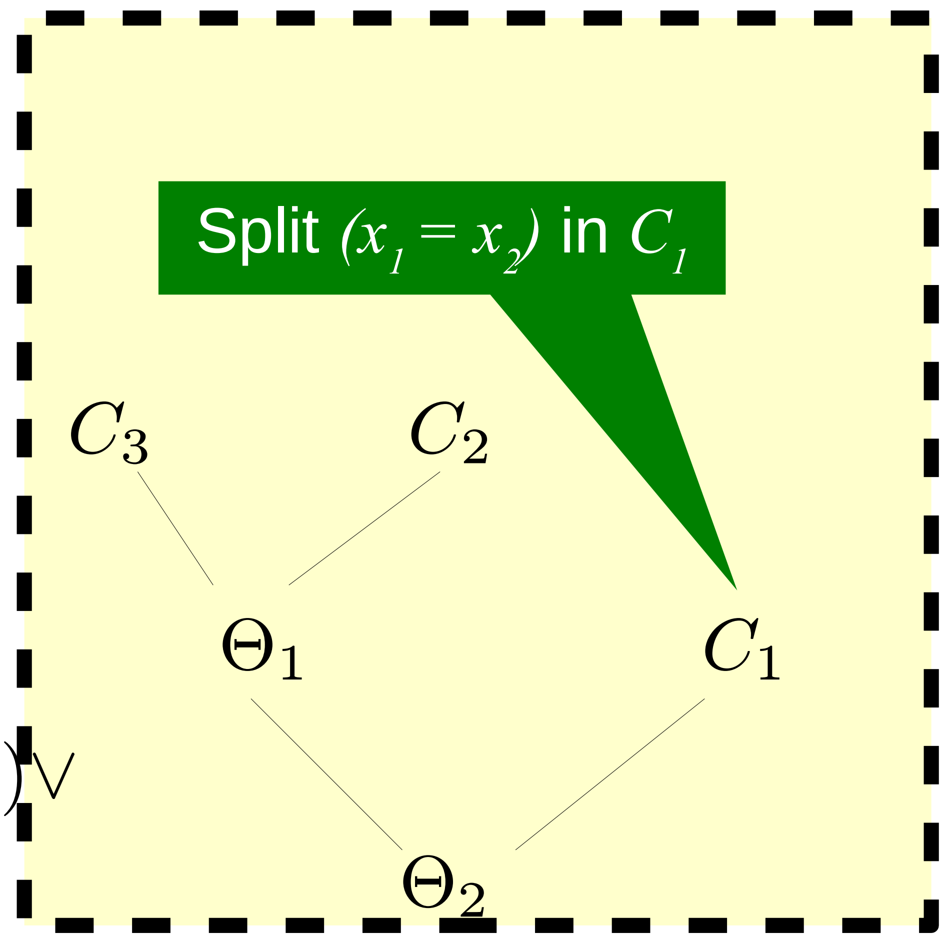
P^{ie} subproof:

T-lemmas:

$$\boxed{C_1 \equiv (x_1 = x_2) \vee \neg(z - x_1 = 1) \vee \neg(z - x_2 = 1)}$$

$$C_2 \equiv (a_1 = a_2) \vee \neg(a_2 = f(x_2)) \vee \neg(a_1 = f(x_1)) \vee \neg(x_1 = x_2)$$

$$C_3 \equiv \neg(a_1 + z = 0) \vee \neg(a_2 + z = 1) \vee \neg(a_1 = a_2)$$



Example

$$A := (a_1 = f(x_1)) \wedge (z - x_1 = 1) \wedge (a_1 + z = 0)$$

$$B := (a_2 = f(x_2)) \wedge (z - x_2 = 1) \wedge (a_2 + z = 1)$$

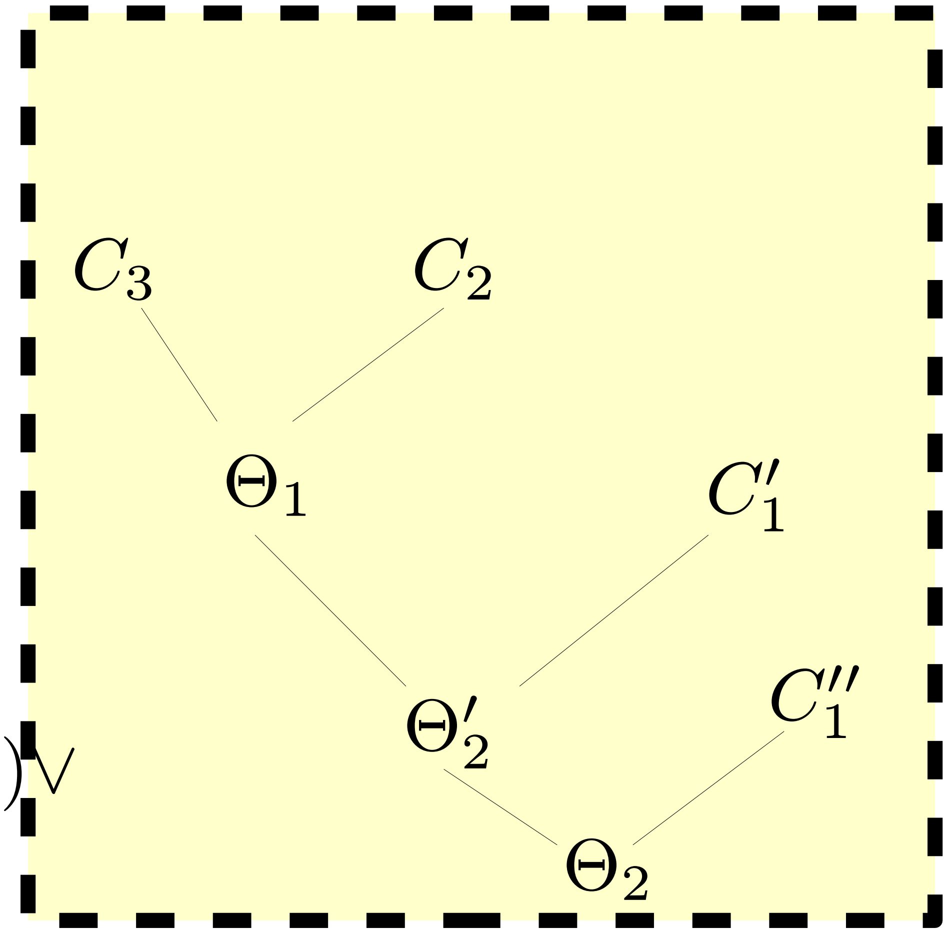
P^{ie} subproof:

$$C_1' \equiv (x_1 = z - 1) \vee \neg(z - x_1 = 1) \vee \neg(z - x_2 = 1)$$

$$C_1'' \equiv (z - 1 = x_2) \vee \neg(z - x_1 = 1) \vee \neg(z - x_2 = 1)$$

$$C_2 \equiv (a_1 = a_2) \vee \neg(a_2 = f(x_2)) \vee \neg(a_1 = f(x_1)) \vee \neg(x_1 = x_2)$$

$$C_3 \equiv \neg(a_1 + z = 0) \vee \neg(a_2 + z = 1) \vee \neg(a_1 = a_2)$$



Example

$$A := (a_1 = f(x_1)) \wedge (z - x_1 = 1) \wedge (a_1 + z = 0)$$

$$B := (a_2 = f(x_2)) \wedge (z - x_2 = 1) \wedge (a_2 + z = 1)$$

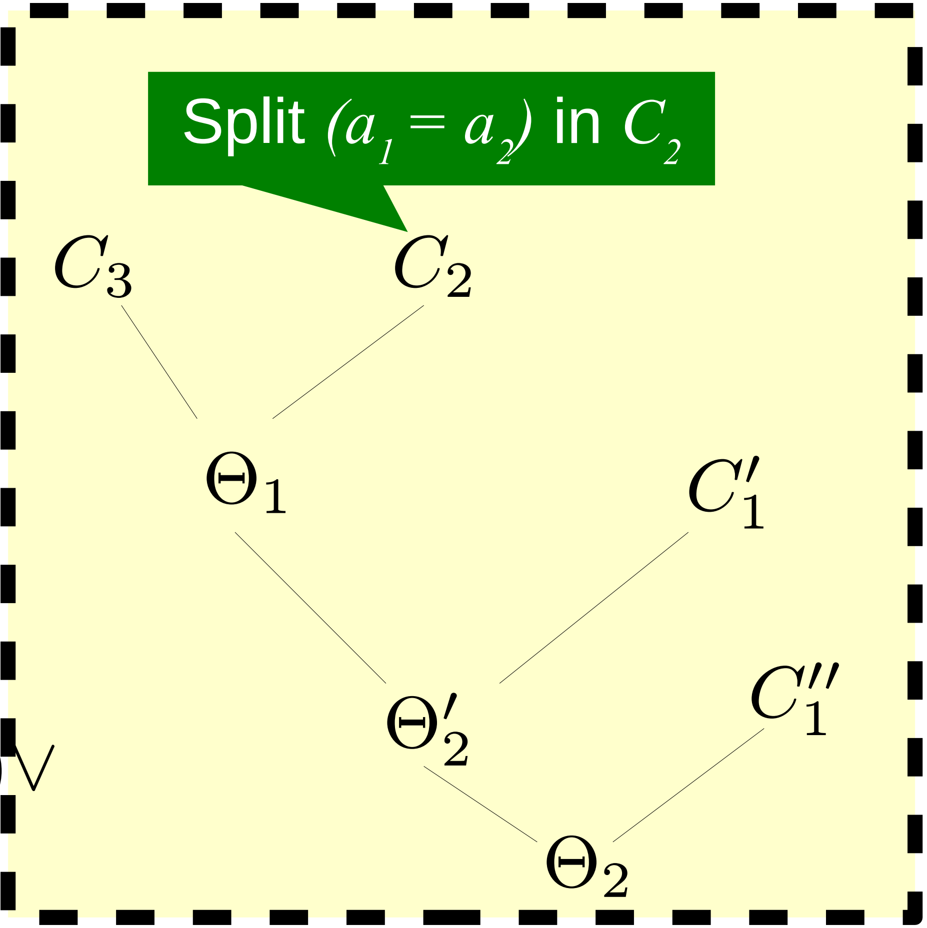
P^{ie} subproof:

$$C_1' \equiv (x_1 = z - 1) \vee \neg(z - x_1 = 1) \vee \neg(z - x_2 = 1)$$

$$C_1'' \equiv (z - 1 = x_2) \vee \neg(z - x_1 = 1) \vee \neg(z - x_2 = 1)$$

$$C_2 \equiv (a_1 = a_2) \vee \neg(a_2 = f(x_2)) \vee \neg(a_1 = f(x_1)) \vee \neg(x_1 = x_2)$$

$$C_3 \equiv \neg(a_1 + z = 0) \vee \neg(a_2 + z = 1) \vee \neg(a_1 = a_2)$$



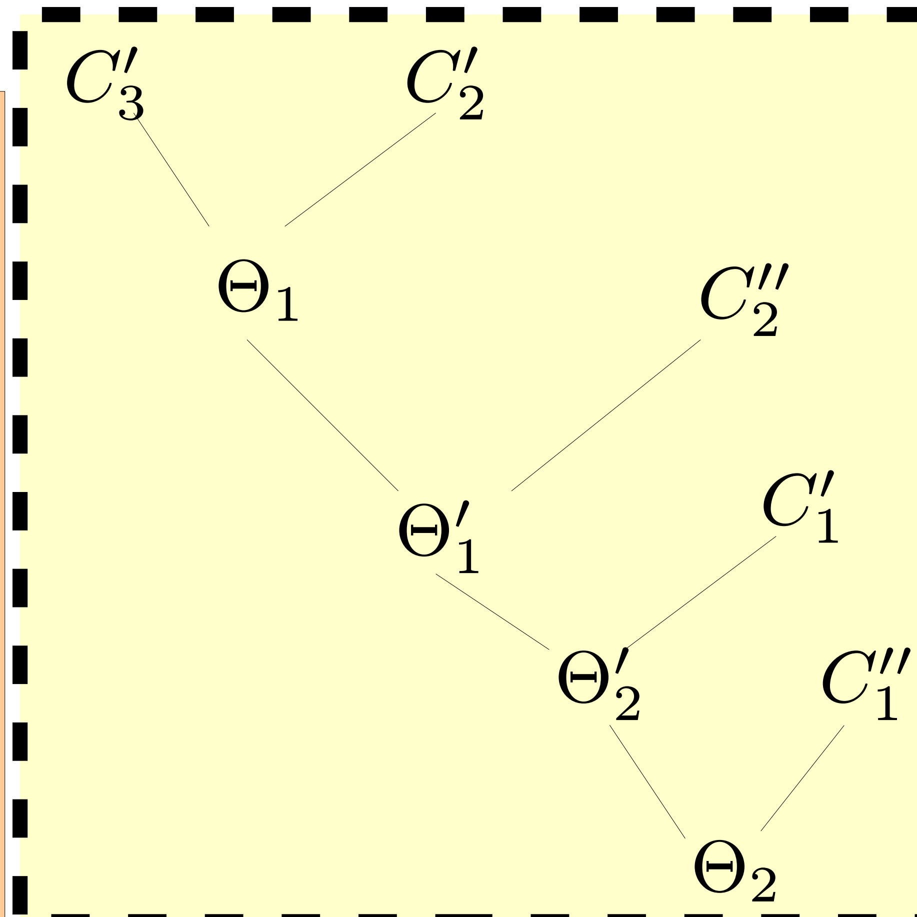
Example

$$\color{red}{A} := (a_1 = f(x_1)) \wedge (z - x_1 = 1) \wedge (a_1 + z = 0)$$

$$\color{blue}{B} := (a_2 = f(x_2)) \wedge (z - x_2 = 1) \wedge (a_2 + z = 1)$$

P^{ie} subproof:

$$\begin{aligned}
C_2' \equiv & (a_1 = f(z-1)) \vee \neg(a_2 = f(x_2)) \vee \\
& \neg(a_1 = f(x_1)) \vee \neg(x_1 = z - 1) \vee \\
& \neg(z - 1 = x_2) \\
C_2'' \equiv & (f(z-1) = a_2) \vee \neg(a_2 = f(x_2)) \vee \\
& \neg(a_1 = f(x_1)) \vee \neg(x_1 = z - 1) \vee \\
& \neg(z - 1 = x_2) \\
C_3' \equiv & \neg(a_1 + z = 0) \vee \neg(a_2 + z = 1) \vee \\
& \neg(a_1 = f(z-1)) \vee \neg(f(z-1) = a_2)
\end{aligned}$$



Proof Tree Preserving Interpolation

- [Christ, Hoenicke and Nutz, TACAS 2013]
- Interpolants with AB-mixed literals without proof rewriting
  - Replace AB-mixed terms $(s \leq t)$ with $(s \leq x) \wedge (x \leq t)$ in leaves, where x is a fresh purification variable
  - Eliminate the purification variable when resolving on $(s \leq t)$

$$\frac{C_1 \vee (s \leq t)\ [I_1(x)] \qquad C_2 \vee \neg(s \leq t)\ [I_2(x)]}{C_1 \vee C_2\ [I_3]}$$

- Advantages:
  - no need of proof rewriting
  - handles also for non-convex theories
- Drawbacks:
  - need *T*-specific interpolation rules for resolution steps
  - more complex interpolation system

From Binary to Sequence Interpolants

- An ordered sequence of formulae $F_1, \ldots, F_n$ such that $\bigwedge_i F_i \models \bot$
- We want a sequence of interpolants $I_1, \ldots, I_{n-1}$ such that
  - I_k is an interpolant for $(\bigwedge_{i=1}^{k} F_i, \bigwedge_{j=k+1}^{n} F_j)$
  - $F_k \wedge I_{k-1} \models I_k$ for all $k \in [2, n-1]$
- Needed in various applications (e.g. abstraction refinement)
- How to compute them?
  - In general, if we compute arbitrary binary interpolants for $(\bigwedge_{i=1}^{k} F_i, \bigwedge_{j=k+1}^{n} F_j)$, the second condition will not hold

A simple solution

- Compute I_1 as an interpolant of $(F_1, \bigwedge_{j=2}^{n} F_j)$
- Compute I_k as an interpolant of $(I_{k-1} \wedge F_k, \bigwedge_{j=k+1}^{n} F_j)$

- **Claim:** I_k is an interpolant for $(\bigwedge_{i=1}^{k} F_i, \bigwedge_{j=k+1}^{n} F_j)$
- Proof (sketch):
  - By ind.hyp. I_{k-1} is an interpolant for $(\bigwedge_{i=1}^{k-1} F_i, \bigwedge_{j=k}^{n} F_j)$ so $\bigwedge_{i=1}^{k-1} F_i \models I_{k-1}$ and $I_{k-1} \wedge F_k \wedge \bigwedge_{j=k+1}^{n} F_j \models \bot$

- Advantages:
  - simple to implement
  - can use any off-the-shelf binary interpolation
- Drawback: requires *n-1* SMT calls

A more efficient algorithm

- Compute an SMT proof of unsatisfiablity *P* for $\bigwedge_{i=1}^{n} F_i$
- Compute each $I_k := \text{Interpolant}(\bigwedge_{i=1}^{k} F_i, \bigwedge_{j=k+1}^{n} F_j)$ from the same proof *P*
- **Theorem:** $F_k \wedge I_{k-1} \models I_k$

A more efficient algorithm

- Compute an SMT proof of unsatisfiablity *P* for $\bigwedge_{i=1}^{n} F_i$
- Compute each $I_k := \mathrm{Interpolant}(\bigwedge_{i=1}^{k} F_i, \bigwedge_{j=k+1}^{n} F_j)$ from the same proof *P*

- **Theorem:** $F_k \wedge I_{k-1} \models I_k$
- Proof (sketch) – case *n=3*:
  - Let *C* be a node of *P* with partial interpolants *I'* and *I''* for the partitionings $(F_1, F_2 \wedge F_3)$ and $(F_1 \wedge F_2, F_3)$ resp. Then we can prove, by induction on the structure of *P*, that:

$$\boxed{I' \wedge F_2 \models I'' \vee \bigvee\{l \in C \mid \mathrm{var}(l) \not\in F_3\}}$$

  - The theorem then follows as a corollary
  - Works also for DTC-rewritten proofs

Selected bibliography

*DISCLAIMER: this is **very** incomplete. Apologies to missing authors/works*

- Interpolants in Formal Verification

  - McMillan. **Interpolation and SAT-based Model Checking**. CAV 2003
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Thank You

VTSA summer school 2015

Exploiting SMT for Verification of Infinite-State Systems

3. SMT-based Verification with IC3

Alberto Griggio
Fondazione Bruno Kessler – Trento, Italy

Outline

Introduction

IC3 for finite-state systems

SMT-based IC3 for infinite-state systems

IC3 for LTL verification

Introduction

- IC3 very successful SAT-based model checking algorithm
  - **I**ncremental **C**onstruction
  - of **I**nductive **C**lauses
  - for **I**ndubitable **C**orrectness
- Key principles:
  - Verification by induction
  - Inductive invariant built incrementally
    - by discovering (relatively-)inductive clauses
  - Exploiting efficient SAT solvers

Introduction

- IC3 has been further generalized to SMT in various ways

- We will look in some detail at one such generalization, called IC3 with Implicit Predicate Abstraction (IC3-IA)
  - Exploits several features of modern SMT solvers that we have discussed so far
    - Incremental solving
    - Assumptions and unsatisfiable cores
    - Interpolation

- A “hands-down” approach
  - We will build a (simple) real implementation on top of MathSAT

Proofs by Induction

- Given transition system $\langle I(X), T(X, X')\rangle$ and property $P(X)$
  - Base case (initiation):
    $$I(X) \models P(X)$$
  - Inductive step (consectution):
    $$P(X) \wedge T(X, X') \models P(X')$$

- Typically however, P is not inductive
  - Find an inductive invariant $Inv(X)$, stronger than P
    - $I(X) \models Inv(X)$
    - $Inv(X) \wedge T(X, X') \models Inv(X')$
    - $Inv(X) \models P(X)$

Outline

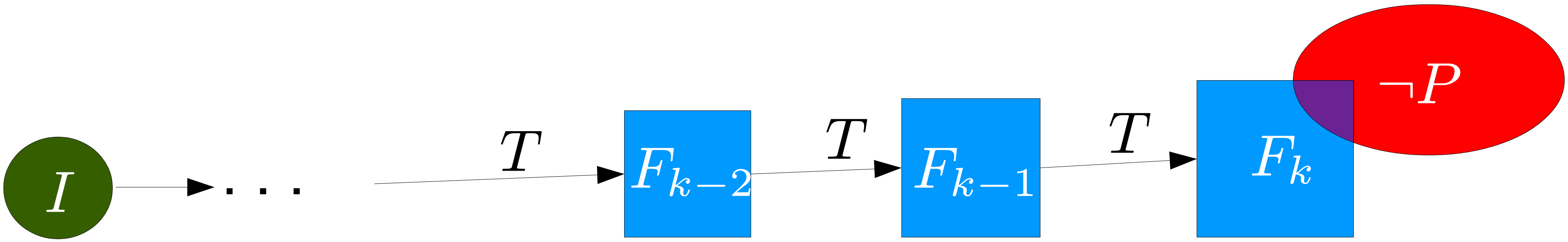
Introduction

IC3 for finite-state systems

SMT-based IC3 for infinite-state systems

IC3 for LTL verification

A (very) high level view of IC3



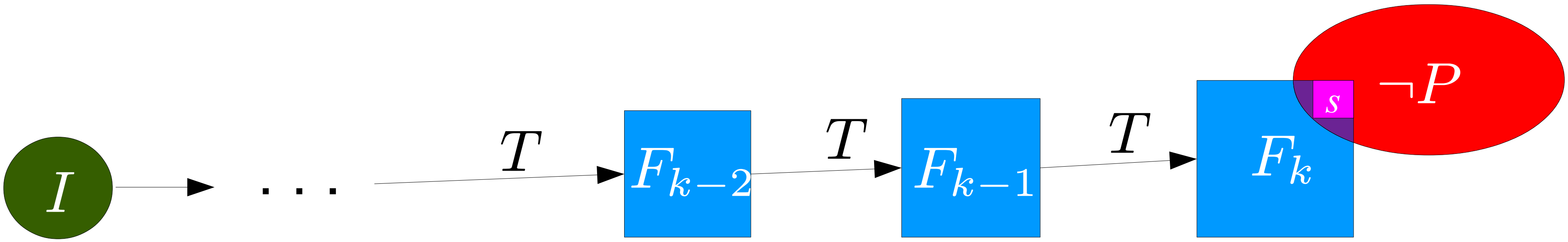
- Given a symbolic transition system and invariant property P, build an inductive invariant F s.t. $F \models P$
- Trace of formulae $F_0(X) \equiv I, \dots, F_k(X)$ s.t:
  - for $i > 0$, F_i is a set of clauses overapproximation of states reachable in up to i steps

  $F_{i+1} \subseteq F_i$ (so $F_i \models F_{i+1}$)

  $F_i \wedge T \models F'_{i+1}$

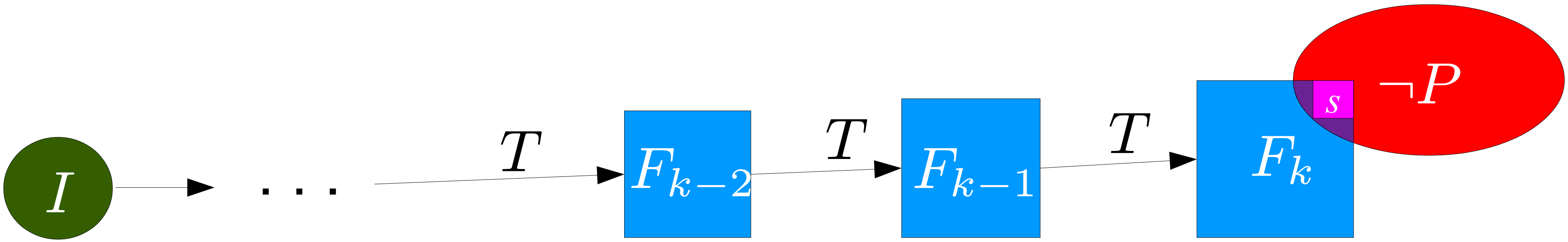
  for all $i < k$, $F_i \models P$

A (very) high level view of IC3



- Blocking phase: incrementally strengthen trace until $F_k \models P$
  - Get bad cube s
  - Call SAT solver on $F_{k-1} \wedge \neg s \wedge T \wedge s'$
    (i.e., check if $F_{k-1} \wedge \neg s \wedge T \models \neg s'$)

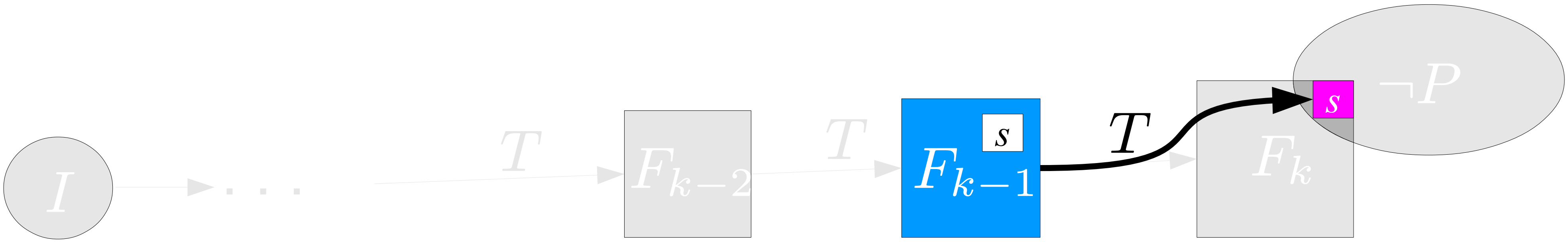
A (very) high level view of IC3



- Blocking phase: incrementally strengthen trace until $F_k \models P$
  - Get bad cube s
  - Call SAT solver on $F_{k-1} \wedge \neg s \wedge T \wedge s'$
    (i.e., check if $F_{k-1} \wedge \neg s \wedge T \models \neg s'$)

Check if s is inductive relative to F_{k-1}

A (very) high level view of IC3



- Blocking phase: incrementally strengthen trace until $F_k \models P$
  - Get bad cube s
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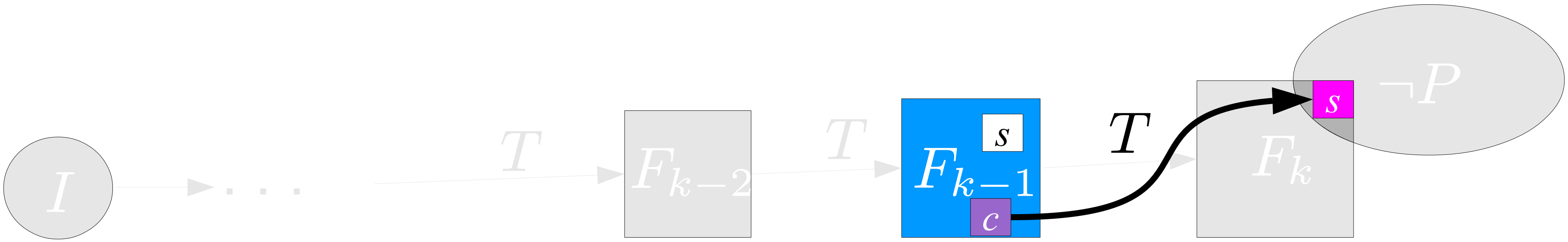
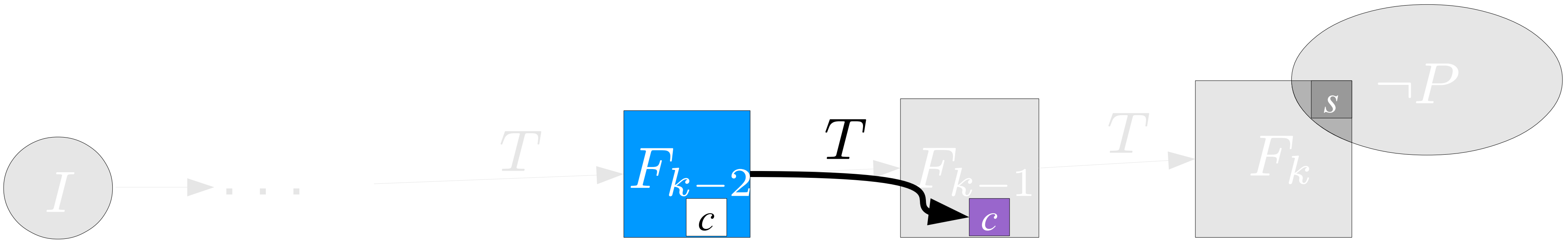


- Blocking phase: incrementally strengthen trace until $F_k \models P$
  - Get bad cube s
  - Call SAT solver on $F_{k-1} \wedge \neg s \wedge T \wedge s'$
    - SAT: s is reachable from $F_{k-1} \wedge \neg s$ in 1 step
    - Get a cube c in the preimage of s and try (recursively) to prove it unreachable from F_{k-2}, …
      - c is a counterexample to induction (CTI)

If I is reached, counterexample found

A (very) high level view of IC3



- Blocking phase: incrementally strengthen trace until $F_k \models P$
  - Get bad cube s
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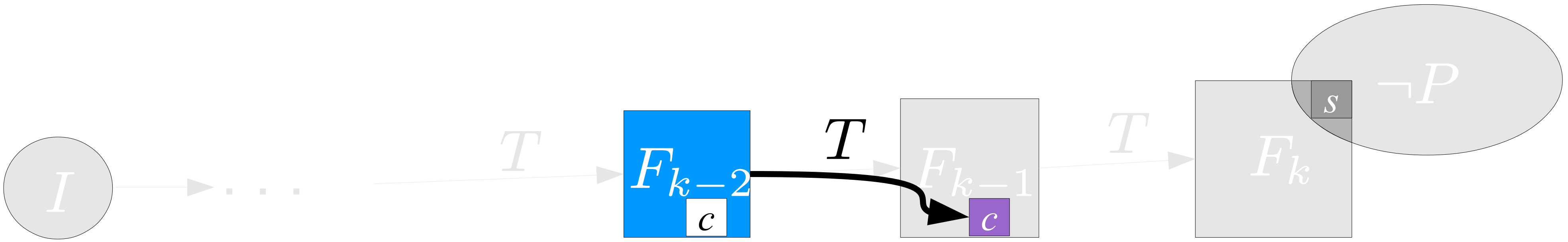
A (very) high level view of IC3



- Blocking phase: incrementally strengthen trace until $F_k \models P$
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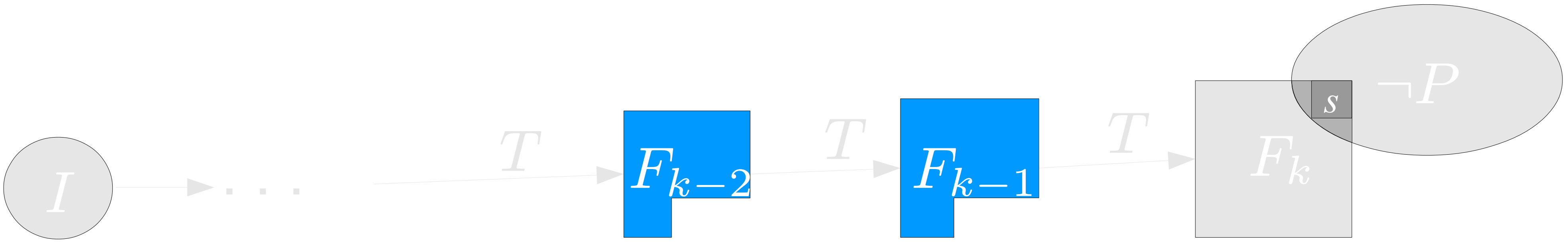
  - Call SAT solver on $F_{k-2} \wedge \neg s \wedge T \wedge s'$
    - UNSAT: $\neg c$ is inductive relative to F_{k-2} $\boxed{F_{k-2} \wedge \neg c \wedge T \models \neg c'}$
    - Generalize c to g and block by adding $\neg g$ to $F_{k-1}, F_{k-2}, \ldots, F_1$

A (very) high level view of IC3



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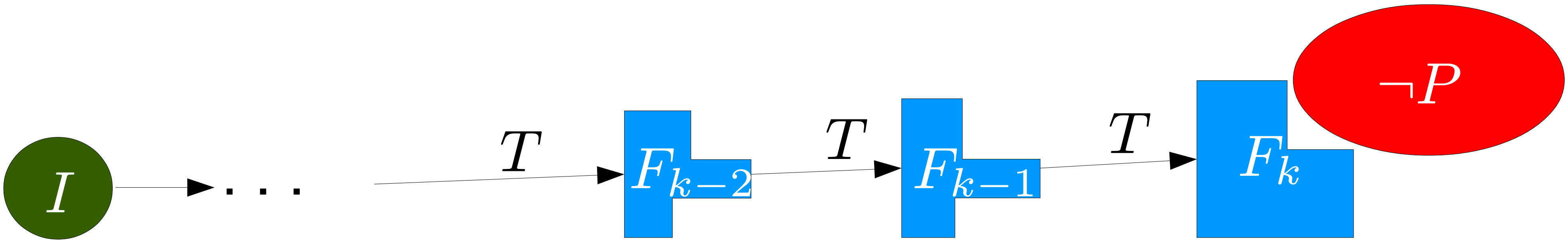
A (very) high level view of IC3



Propagation: extend trace to F_{k+1} and push forward clauses

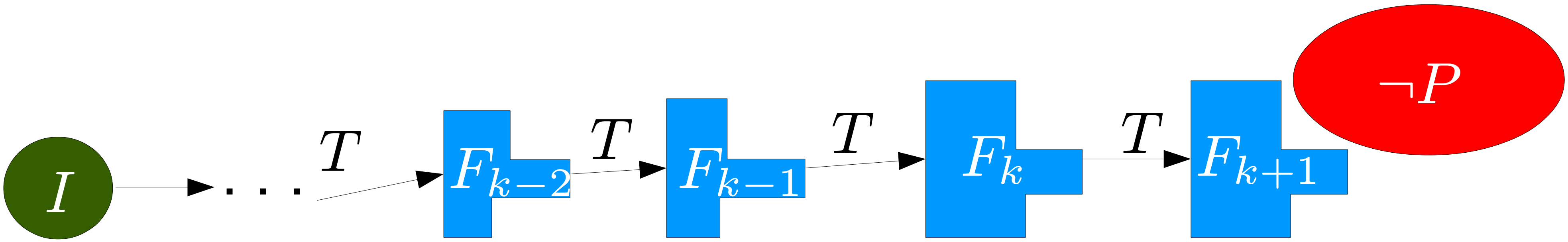
For each i and each clause $c \in F_i$:

Call SAT solver on $F_i \wedge T \wedge \neg c'$

If UNSAT, add c to F_{i+1}

$$\boxed{F_i \wedge T \models c'}$$

A (very) high level view of IC3



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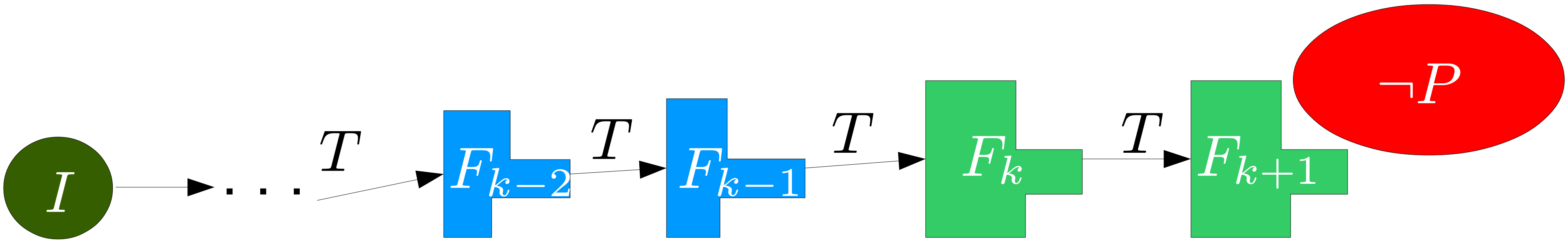

Propagation: extend trace to F_{k+1} and push forward clauses

For each i and each clause $c \in F_i$:

Call SAT solver on $F_i \wedge T \wedge \neg c'$

If UNSAT, add c to F_{i+1}

$\boxed{F_i \wedge T \models c'}$

If $\boxed{F_i \equiv F_{i+1}}$, P is proved,
otherwise start another round of blocking and propagation

IC3 pseudo-code

```
bool IC3(I, T, P):
    trace = [I]   # first elem of trace is init formula
    trace.push()  # add a new frame
    while True:
        # blocking phase
        while is_sat(trace.last() & ~P):
            c = extract_cube() # c |= trace.last() & ~P
            if not rec_block(c, trace.size()-1):
                return False # counterexample found

        # propagation phase
        trace.push()
        for i=1 to trace.size()-1:
            for each cube c in trace[i]:
                if not is_sat(trace[i] & ~c & T & c'):
                    trace[i+1].append(c)
            if trace[i] == trace[i+1]:
                return True # property proved
```

IC3 pseudo-code

```
bool rec_block(s, i):
    if i == 0:
        return False  # reached initial states
    while is_sat(trace[i-1] & ~s & T & s'):
        c = get_predecessor(i-1, T, s')
        if not rec_block(c, i-1):
            return False
    g = generalize(~s, i)
    trace[i].append(g)
    return True
```

Correctness (sketch)

- Consider the formula $F_{k-1} \wedge T \wedge s'$ where *s* is a bad cube
  - If UNSAT, then F_{k-1} is strong enough to block *s*
  - Since $F_i \wedge T \models F'_{i+1}$, then *s* is unreachable in *k* steps or less
  - Since $F_i \models F_{i+1}$, then we can add *s* to all $F_j, j \leq k$

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- Consider now the relative induction check $F_{k-1} \wedge \neg s \wedge T \wedge s'$
  - We know that $I \equiv F_0 \not\models s$ because $I \models P$ (base case)
  - Since $F_i \models F_{i+1}$, then we know that $\neg s$ holds up to *k*

Correctness (sketch)

- Consider the formula $F_{k-1} \wedge T \wedge s'$ where *s* is a bad cube
  - If UNSAT, then F_{k-1} is strong enough to block *s*
  - Since $F_i \wedge T \models F'_{i+1}$, then *s* is unreachable in *k* steps or less
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- Consider now the relative induction check $F_{k-1} \wedge \neg s \wedge T \wedge s'$
  - We know that $I \equiv F_0 \not\models s$ because $I \models P$ (base case)
  - Since $F_i \models F_{i+1}$, then we know that $\neg s$ holds up to *k*
- Propagation: for each $c \in F_i$, check $F_i \wedge T \wedge \neg c'$
  - we know that *c* holds up to *i*, if UNSAT then it holds up to *i+1*
  - since $F_i \models F_{i+1}$, $F_i \wedge T \models F'_{i+1}$ and $F_i \models P$,
    if $F_i \equiv F_{i+1}$ then the fixpoint is an inductive invariant

Inductive Clause Generalization

- Crucial step of IC3
- Given a relatively inductive clause $c \stackrel{\text{def}}{=} \{l_1, \ldots, l_n\}$ compute a generalization $g \subseteq c$ that is still inductive

$$\boxed{F_{i-1} \wedge T \wedge g \models g' \qquad (1)}$$

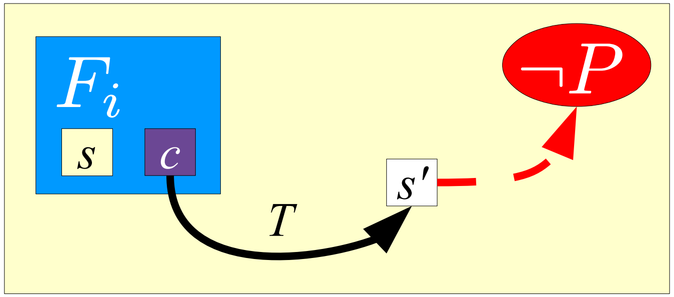
- Drop literals from c and check that (1) still holds
  - Accelerate with unsat cores returned by the SAT solver
    - Using SAT under assumptions
- However, make sure the base case still holds
  - If $I \not\models c \setminus \{l_j\}$, then l_j cannot be dropped

Simple iterative generalization

```
void indgen(c, i):
    done = False
    for iter = 1 to max_iters:
        if done:
            break
        done = True
        for each l in c:
            cand = c \ {l}
            if not is_sat(I & cand) and
               not is_sat(trace[i] & ~cand & T & cand'):
                c = get_unsat_core(cand)
                rest = cand \ c
                while is_sat(I & c):
                    l1 = rest.pop()
                    c.add(l1)
                done = False
                break
```

CTI computation

- When $F_i \land \neg s \land T \land s'$ is satisfiable:
  - s reaches $\neg P$ in k-i steps
  - s can be reached from F_i in 1 step
    - strengthen F_i by blocking cubes c in the preimage of s



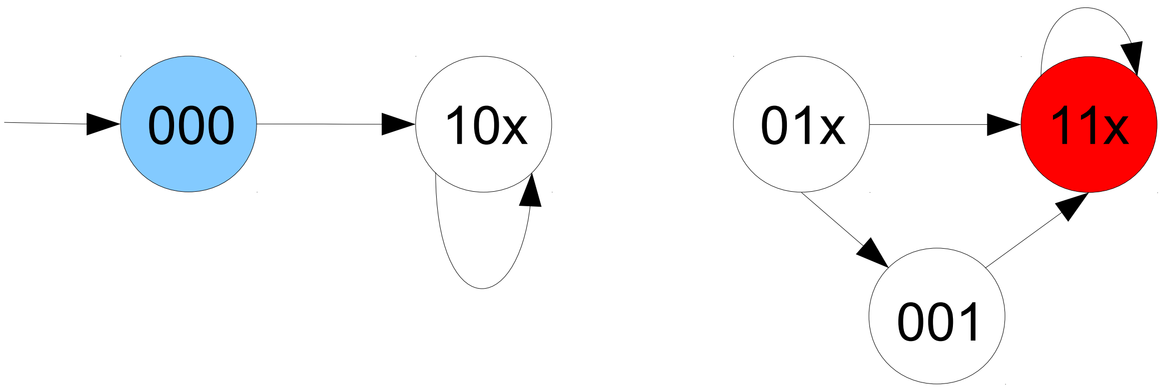
- Extract CTI c from the SAT assignment
  - And generalize to represent multiple bad predecessors
  - Use unsat cores, exploiting a functional encoding of the transition relation
    - If T is functional, then $c \land \text{inputs} \land T \models s'$
    - check $\text{inputs} \land T \land \neg s'$ under assumptions c

SAT-based CTI generalization

```
void generalize_cti(cti, inputs, next):
    for i = 1 to max_iters:
        b = is_sat(cti & inputs & T & ~next')
        assert not b # assume T to be functional
        c = get_unsat_core(cti)
        if should_stop(c, cti):
            break
        cti = c
```

Example

No counterexamples of length 0



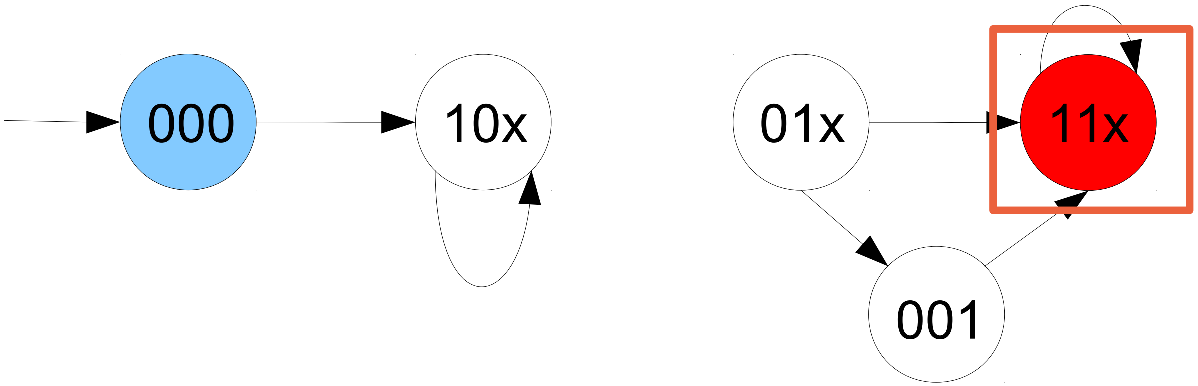
$$I = \neg x_1 \wedge \neg x_2 \wedge \neg x_3$$
$$P = \neg x_1 \vee \neg x_2$$

$$F_0 = I$$
$$F_1 = \top$$

[borrowed and adapted from F. Somenzi]

Example

Get bad cube $c = x_1 \wedge x_2$ in $F_1 \wedge \neg P$



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$$F_0 = I$$
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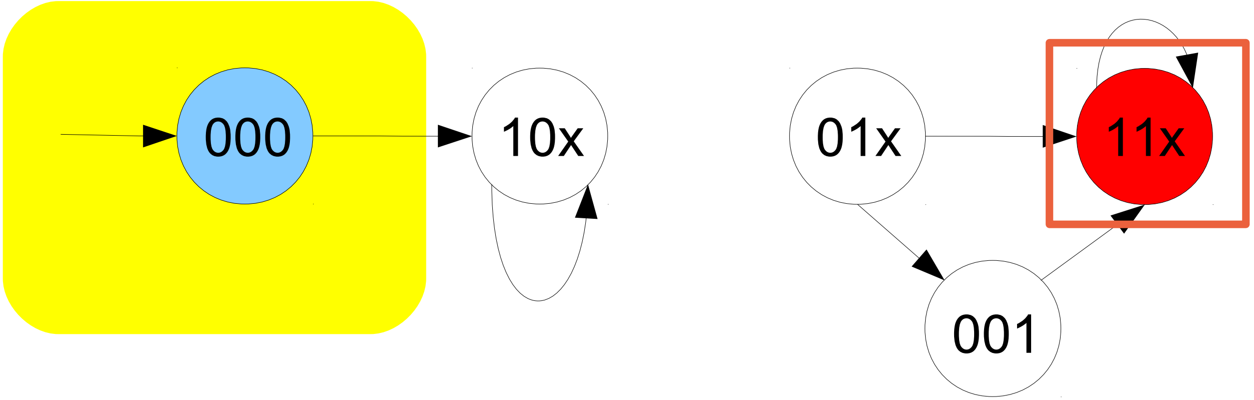
Example

Is $\neg c$ inductive relative to F_0 ? $\quad F_0 \wedge T \wedge \neg c \models \neg c'$



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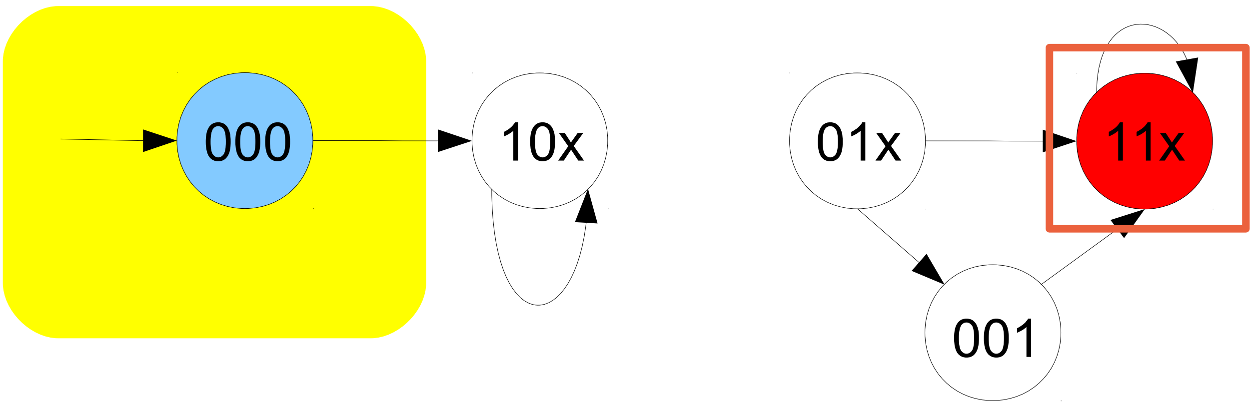
Example

Yes, generalize $\neg c = \neg x_1 \vee \neg x_2$



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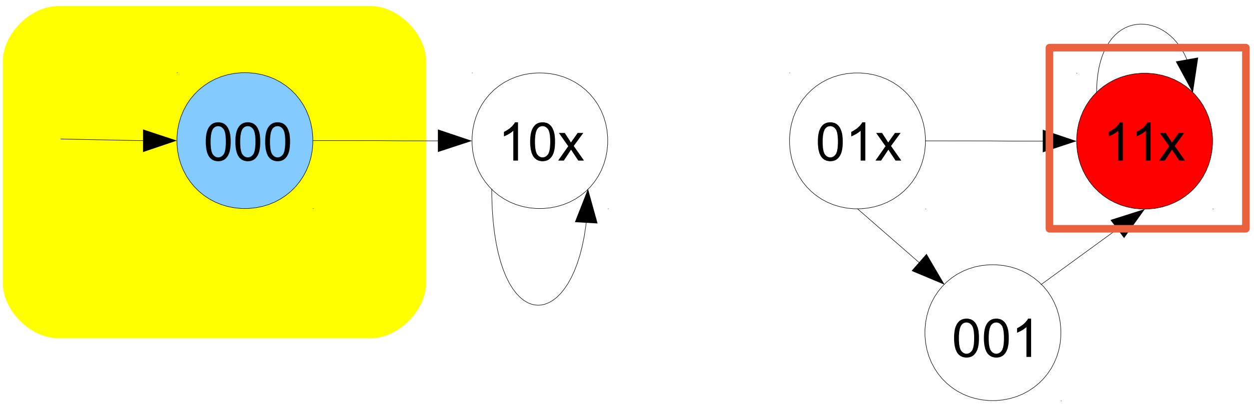

$$I = \neg x_1 \wedge \neg x_2 \wedge \neg x_3$$
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$$F_0 = I$$
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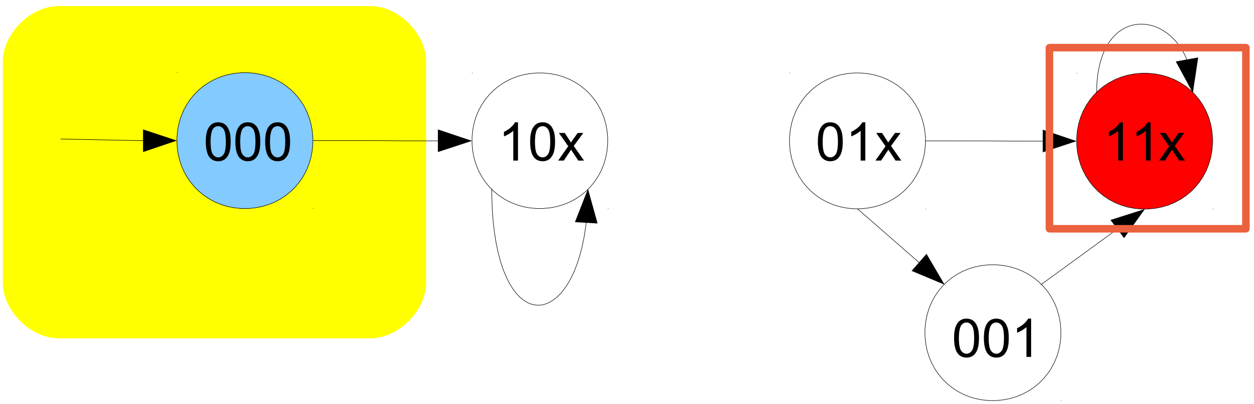
Try dropping $\neg x_2$

$$F_0 \wedge T \wedge \neg x_1 \not\models \neg x_1'$$

Example

Yes, generalize $\neg c = \neg x_1 \vee \neg x_2$



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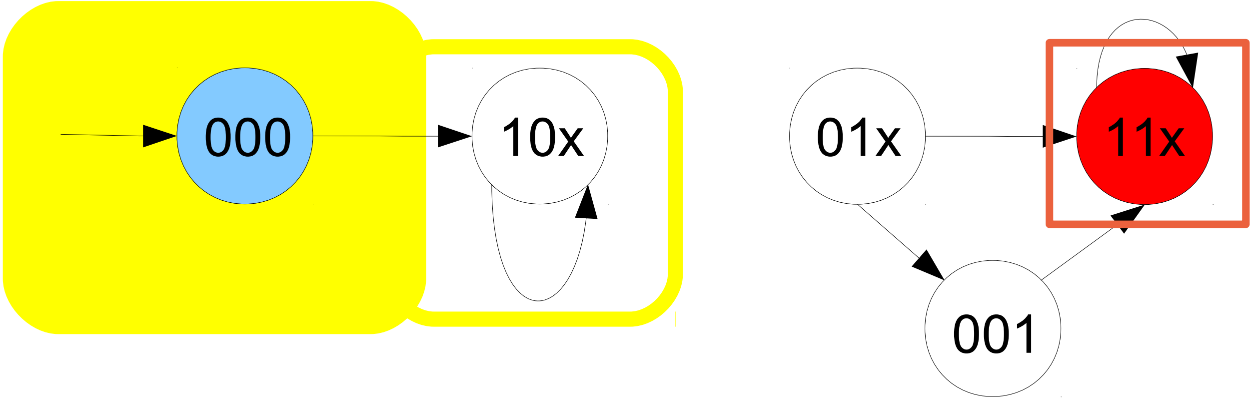

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$$F_0 = I$$
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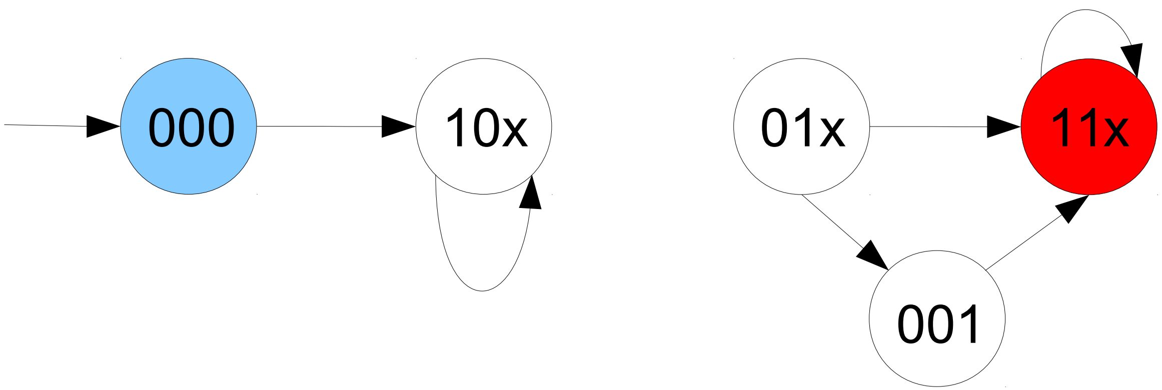
Try dropping $\neg x_1$

$$F_0 \wedge T \wedge \neg x_2 \models \neg x_2'$$

Example

Update F_1



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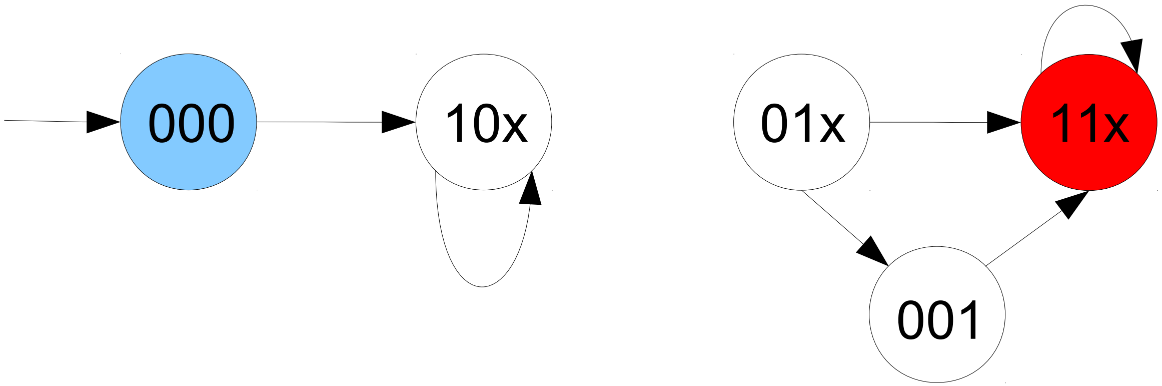
$$P = \neg x_1 \vee \neg x_2$$

$$F_0 = I$$
$$F_1 = \neg x_2$$

Example

Blocking done for F_1. Add F_2 and propagate forward



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$$F_0 = I$$
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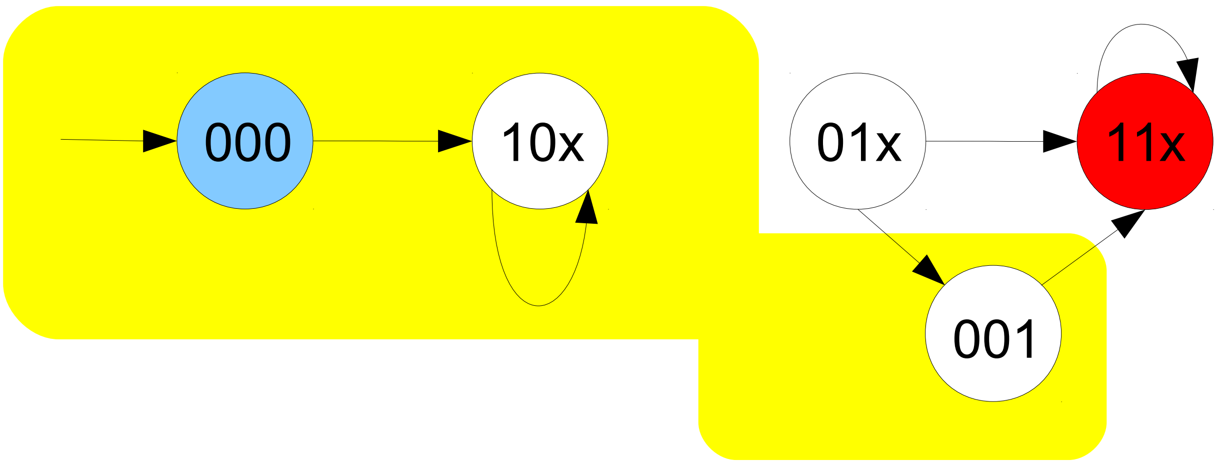
Example

No clause propagates from F_1 to F_2



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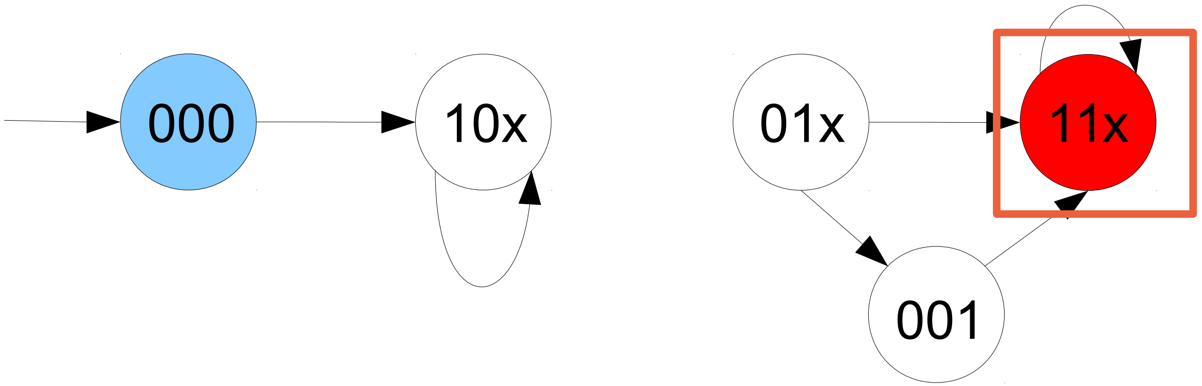
Example

Get bad cube $c = x_1 \wedge x_2$ in $F_2 \wedge \neg P$



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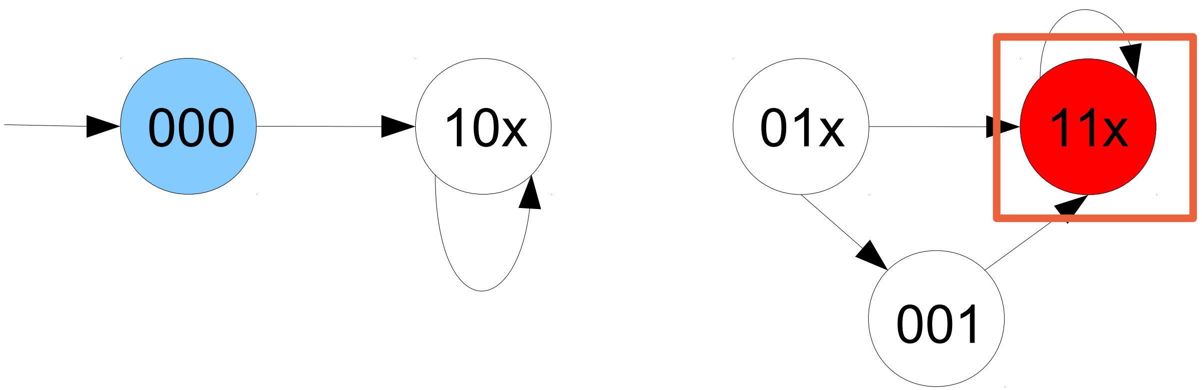
Example

Is $\neg c$ inductive relative to F_1 ? $F_1 \land T \land \neg c \models \neg c'$



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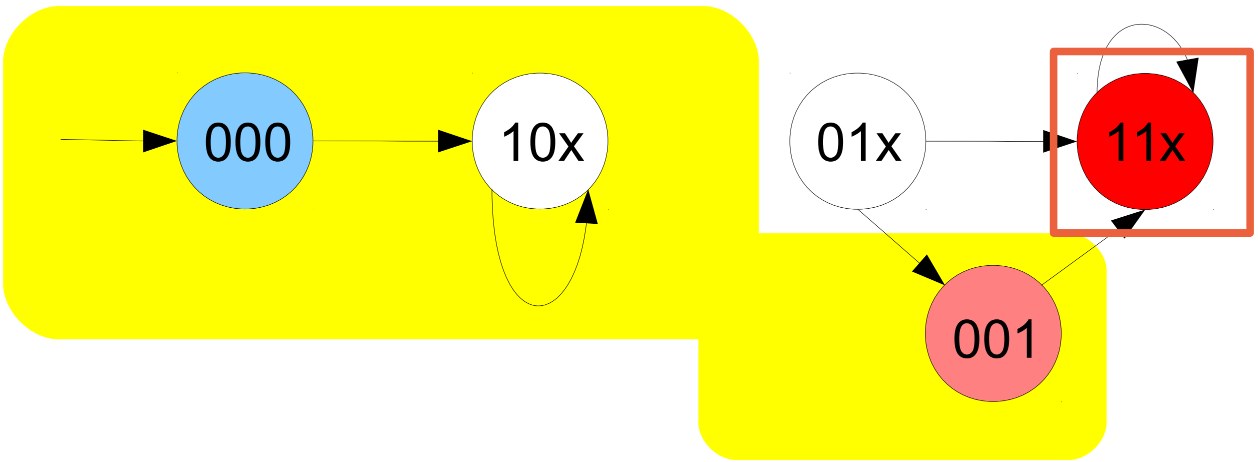
Example

No, found CTI $s = \neg x_1 \wedge \neg x_2 \wedge x_3$



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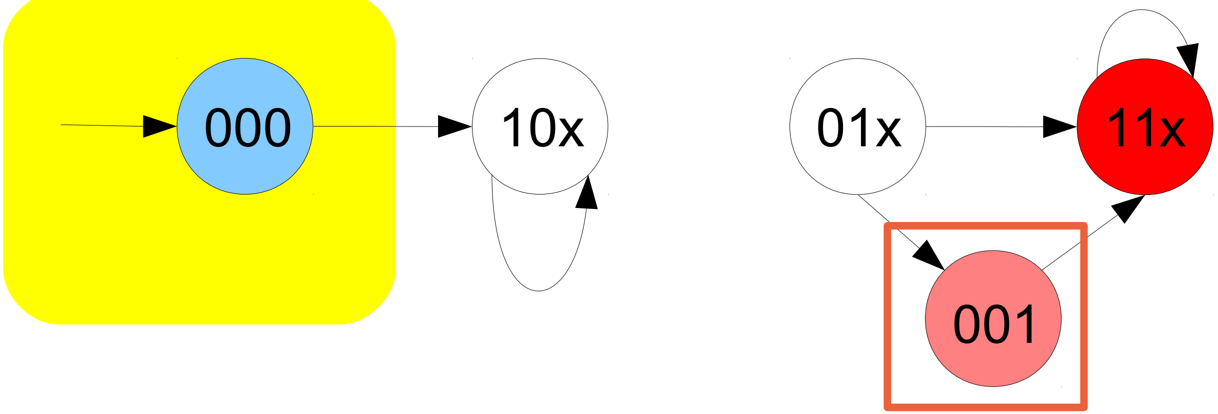
Example

Try blocking $\neg s$ at level 0: $F_0 \wedge T \wedge \neg s \models \neg s'$



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$$F_0 = I$$
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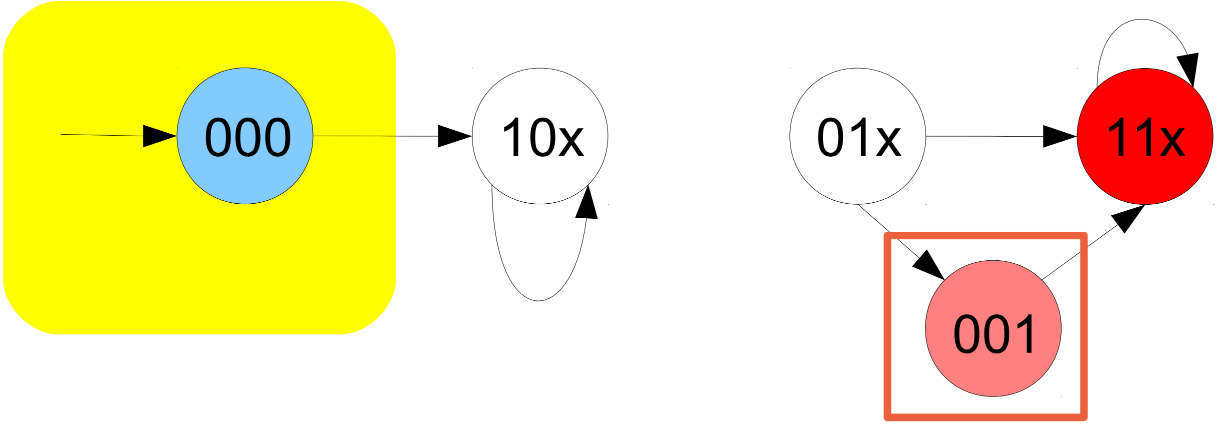
$$F_2 = \top$$

Example

Yes, generalize $\neg s = x_1 \vee x_2 \vee \neg x_3$



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$$F_0 = I$$
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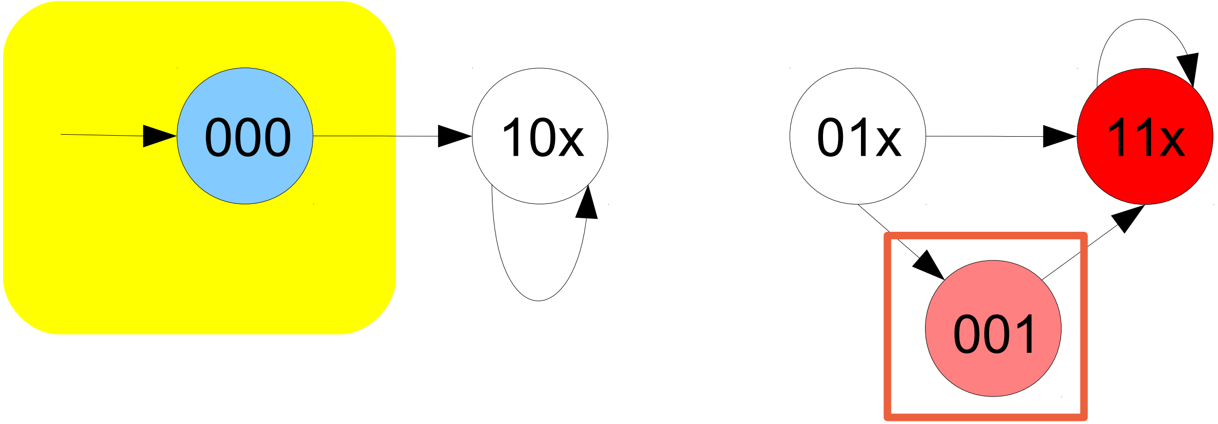
$$F_2 = \top$$

Try dropping x_1

$$F_0 \wedge T \wedge x_2 \vee \neg x_3 \not\models x_2' \vee \neg x_3'$$

Example

Yes, generalize $\neg s = x_1 \vee x_2 \vee \neg x_3$



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$F_0 = I$
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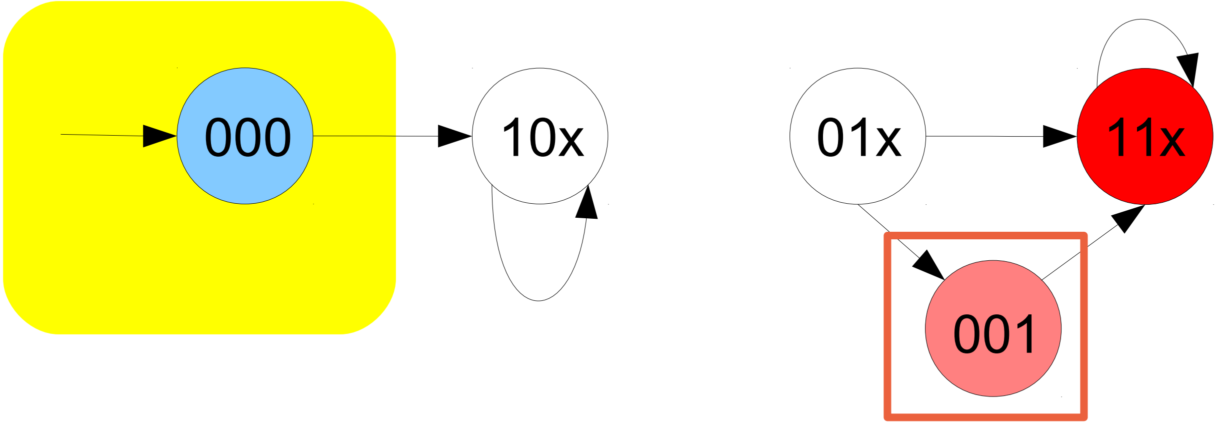
$F_2 = \top$

Try dropping x_2

$F_0 \wedge T \wedge x_1 \vee \neg x_3 \models x_1' \vee \neg x_3'$

Example

Yes, generalize $\neg s = x_1 \vee x_2 \vee \neg x_3$



$$I = \neg x_1 \wedge \neg x_2 \wedge \neg x_3$$
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$$F_0 = I$$
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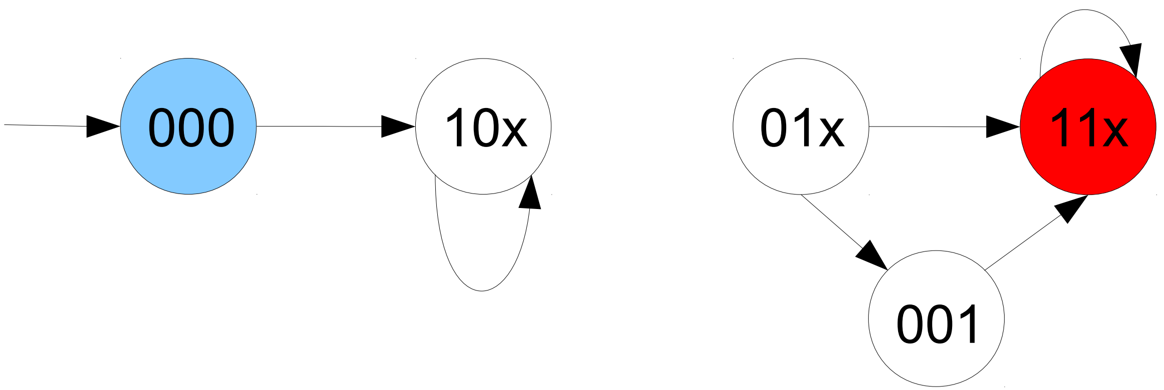
$$F_2 = \top$$

Try dropping x_3

$I \not\models x_1$

Example

Update F_1



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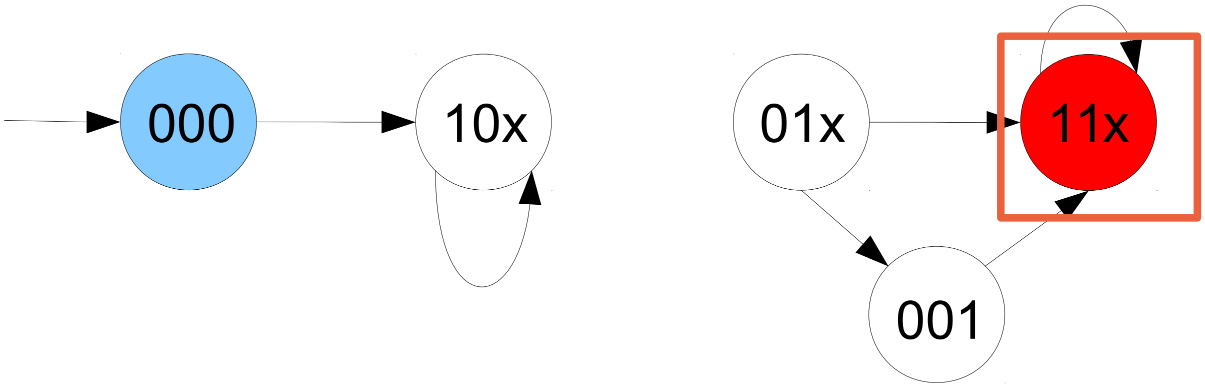
$$P = \neg x_1 \vee \neg x_2$$

$$F_0 = I$$
$$F_1 = \neg x_2 \wedge (x_1 \vee \neg x_3)$$
$$F_2 = \top$$

Example

Return to the original bad cube c



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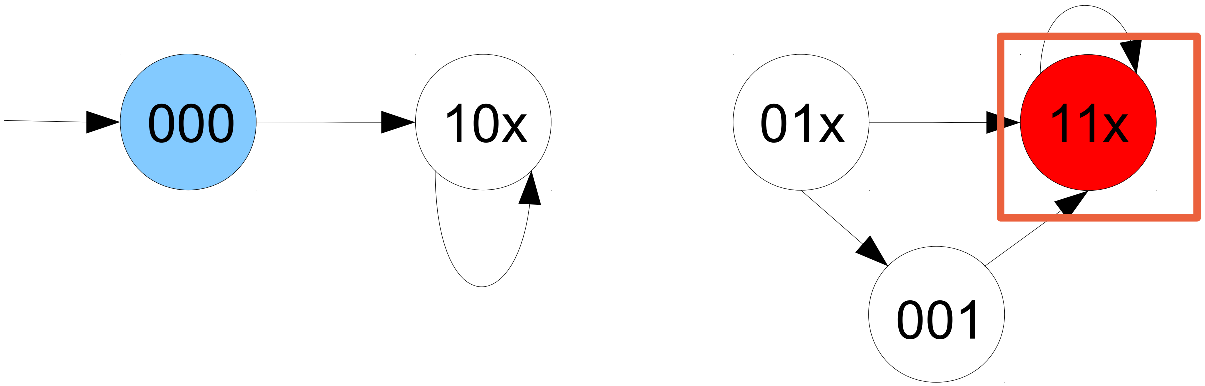
Example

Is $\neg c$ inductive relative to F_1 ? $\quad F_1 \wedge T \wedge \neg c \models \neg c'$



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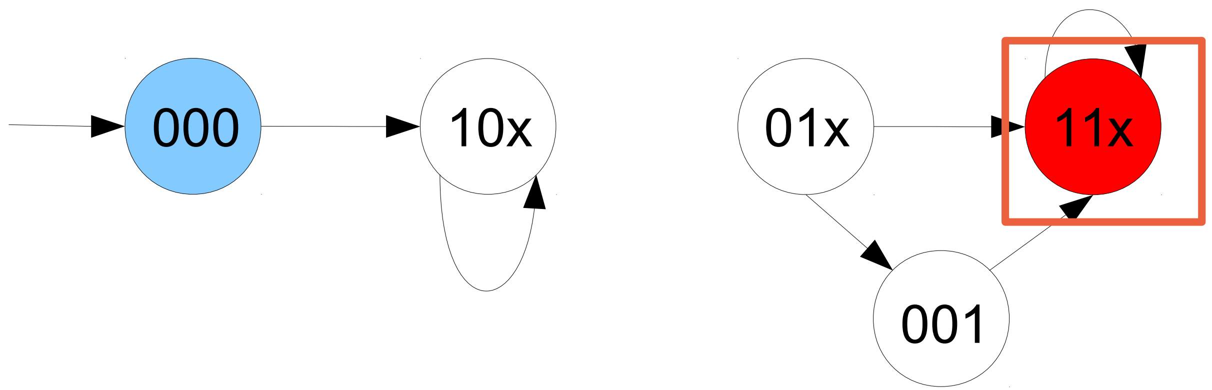
$$F_0 = I$$
$$F_1 = \neg x_2 \wedge (x_1 \vee \neg x_3)$$
$$F_2 = \top$$

Example

Yes, generalize $\neg c = \neg x_1 \vee \neg x_2$



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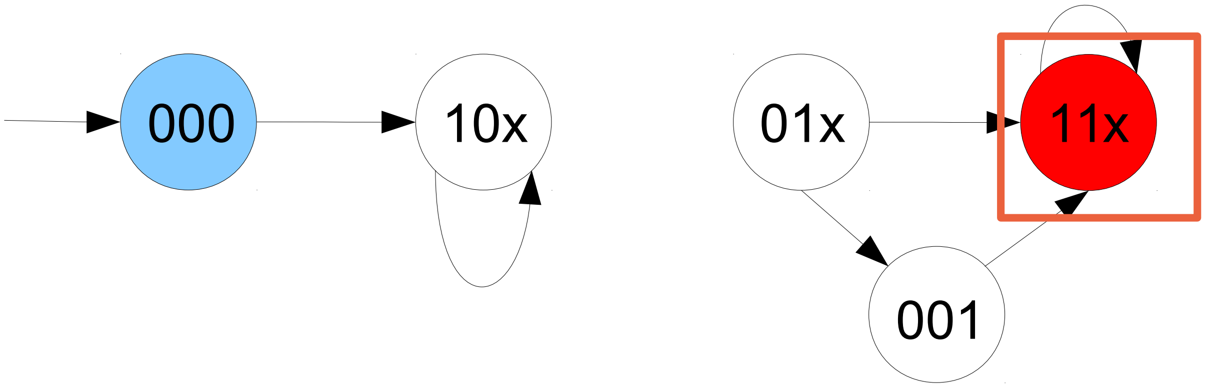
$$F_0 = I$$
$$F_1 = \neg x_2 \wedge (x_1 \vee \neg x_3)$$
$$F_2 = \top$$

Try dropping $\neg x_1$

$$F_1 \wedge T \wedge \neg x_2 \models \neg x_2'$$

Example

Update F_2 and add new frame F_3

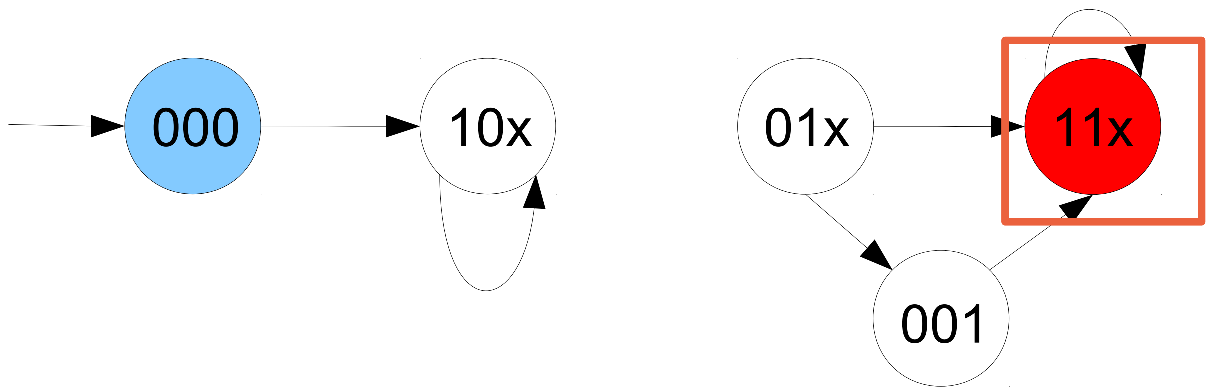


$$I = \neg x_1 \wedge \neg x_2 \wedge \neg x_3$$
$$P = \neg x_1 \vee \neg x_2$$

$$F_0 = I$$
$$F_1 = \neg x_2 \wedge (x_1 \vee \neg x_3)$$
$$F_2 = \neg x_2$$
$$F_3 = \top$$

Example

Perform forward propagation



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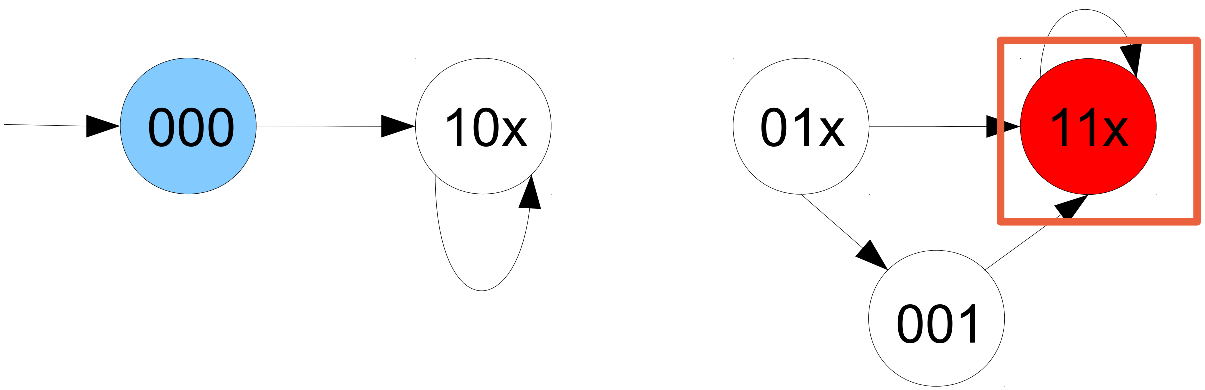
$$F_0 = I$$
$$F_1 = \neg x_2 \wedge (x_1 \vee \neg x_3)$$
$$F_2 = \neg x_2$$
$$F_3 = \top$$

From F_1 to F_2 :

$$F_1 \wedge T \wedge (x_1 \vee \neg x_3) \models (x_1' \vee \neg x_3')$$

Example

Perform forward propagation



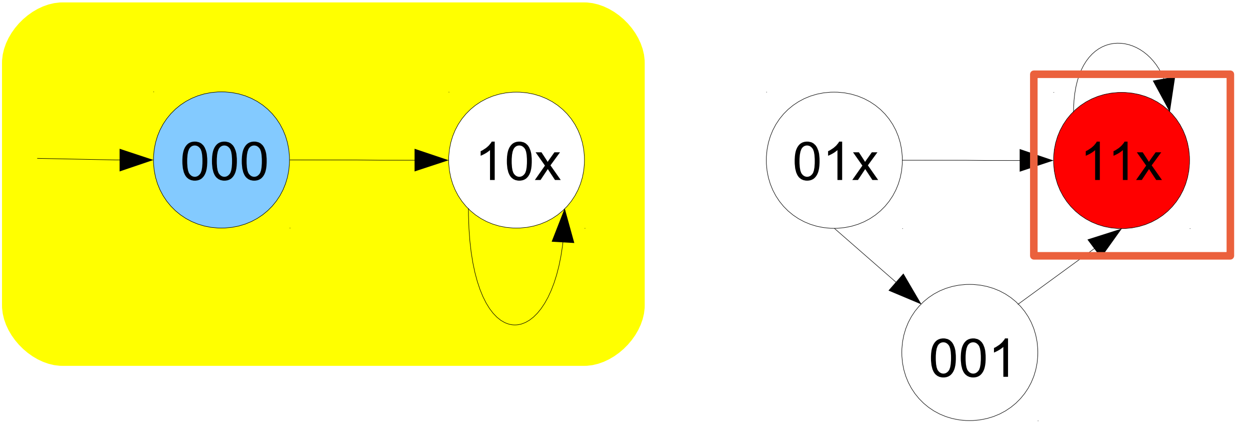
$$I = \neg x_1 \wedge \neg x_2 \wedge \neg x_3$$
$$P = \neg x_1 \vee \neg x_2$$

Found fixpoint!

$$F_0 = I$$
$$F_1 = \neg x_2 \wedge (x_1 \vee \neg x_3)$$
$$F_2 = \neg x_2 \wedge (x_1 \vee \neg x_3)$$
$$F_3 = \top$$

Example

Perform forward propagation



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$$F_0 = I$$
$$F_1 = \neg x_2 \wedge (x_1 \vee \neg x_3)$$
$$F_2 = \neg x_2 \wedge (x_1 \vee \neg x_3)$$
$$F_3 = \top$$

Inductive invariant:

$$F_1 \equiv F_2 \equiv \neg x_2 \wedge (x_1 \vee \neg x_3)$$

Outline

Introduction

IC3 for finite-state systems

SMT-based IC3 for infinite-state systems

IC3 for LTL verification

IC3 with SMT

- How to generalize from SAT to SMT?

IC3 with SMT

- How to generalize from SAT to SMT?
- Good news: replacing the SAT solver with an SMT solver is enough for partial correctness
- but what about:
  - termination?
  - efficiency?

IC3 with SMT

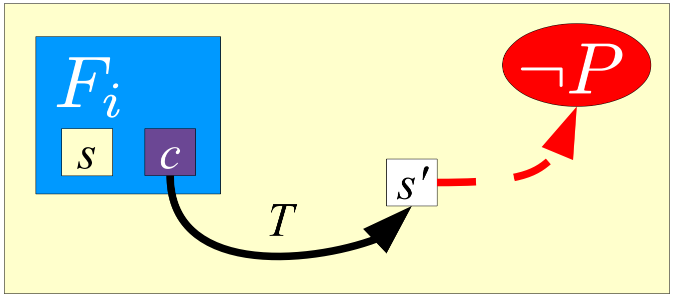
- How to generalize from SAT to SMT?

- Good news: replacing the SAT solver with an SMT solver is enough for partial correctness

- but what about:
  - termination?
    - Easy! (answer)
      - the problem is in general undecidable, so no hope here
  - efficiency?

$RelInd(F_{k-1}, T, s)$ with SMT

- When $F_i \wedge \neg s \wedge T \wedge s'$ is satisfiable:
  - *s* reaches $\neg P$ in *k-i* steps
  - *s* can be reached from F_i in 1 step
  - strengthen F_i by blocking cubes *c* in the preimage of *s*

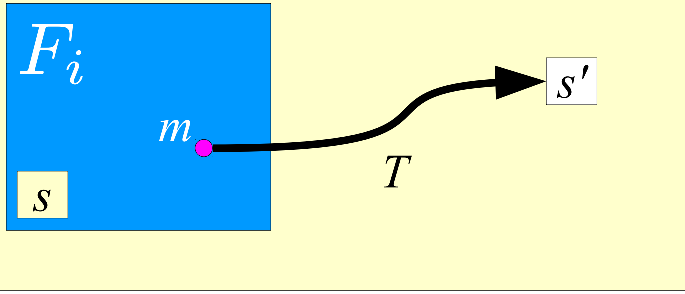


- In the Boolean case, get *c* from SAT assignment (and generalize)

- For SMT(LRA):
  - Would exclude a single point in an infinite space

Single model m from SMT solver:

$$x = 3 \wedge y = 7$$



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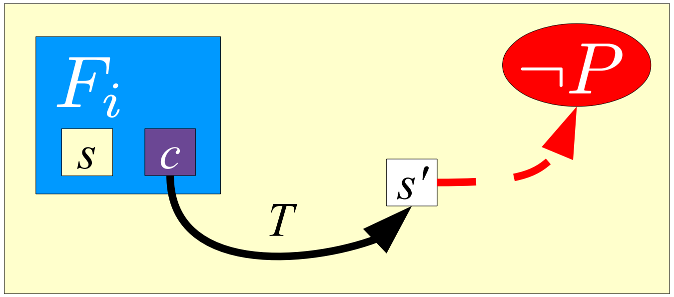
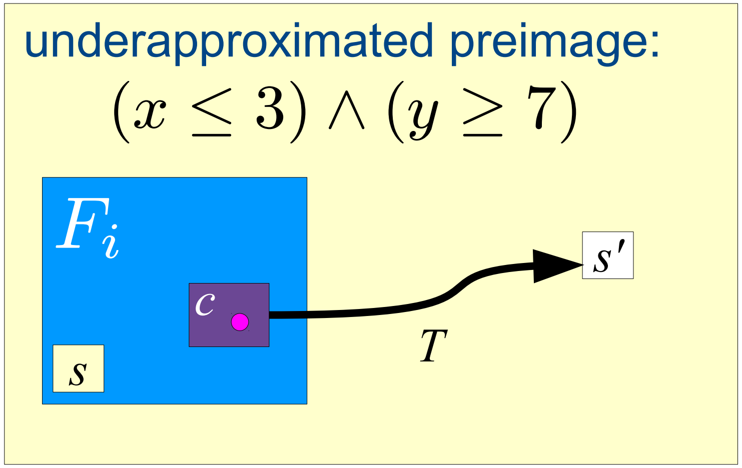


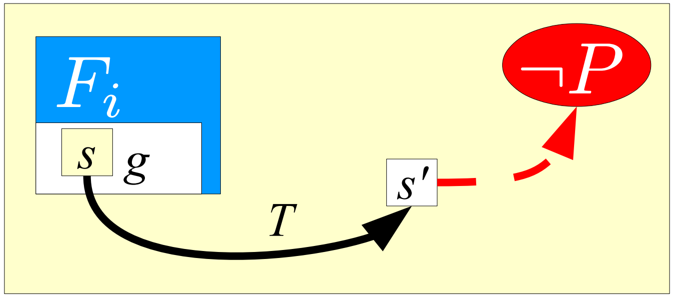
- In the Boolean case, get *c* from SAT assignment (and generalize)

- For SMT(LRA): underapproximated quantifier elimination
  - Encodes a set of predecessors
  - Cheaper than full quantifier elimination
    - But still potentially expensive
    - Not always available
      - E.g for UF+LRA



$RelInd(F_{k-1}, T, s)$ with SMT

- When $F_i \wedge \neg s \wedge T \wedge s'$ is unsatisfiable:
  - Compute a generalization *g* of *s* to block
  - Block more than a single cube at a time



- In the Boolean case, use inductive generalization algorithms
- For SMT, Boolean algorithms plus theory-specific "ad hoc" techniques
  - Based on Farkas' lemma for LRA [HB SAT'12]
  - [WK DATE'13] for BV
  - [KJN FORMATS'12] for timed automata

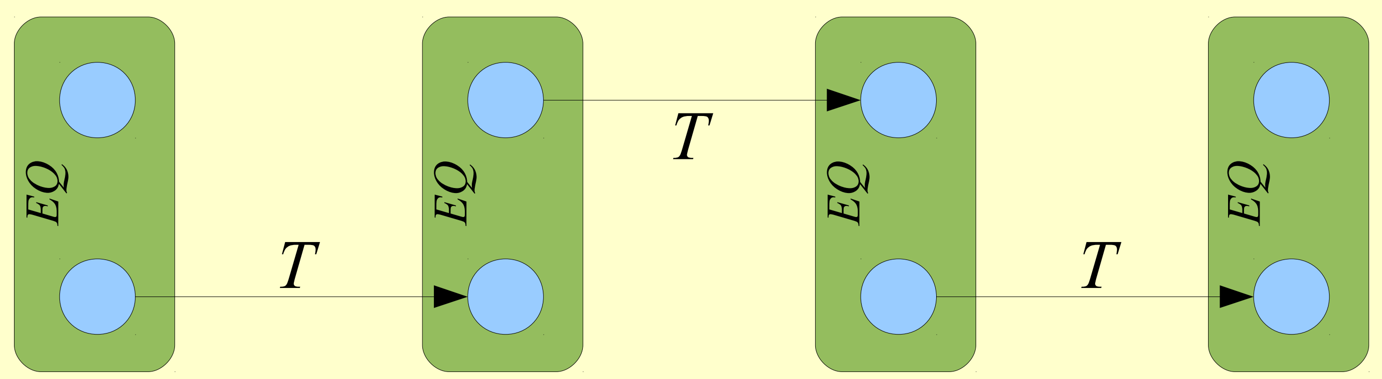
Implicit Predicate Abstraction [Tonetta FM'09]

- Abstract version of k-induction, avoiding explicit computation of the abstract transition relation
  - By embedding the abstraction in the SMT encoding

- Given a set of predicates $\mathbb{P}$ and an unrolling depth k, the abstract path $\widehat{\mathrm{Path}}_{k,\mathbb{P}}$ is

$$\bigwedge_{1 \leq h < k} (T(Y^{h-1}, X^h) \wedge \bigwedge_{p \in \mathbb{P}} (p(X^h) \leftrightarrow p(Y^h)) \wedge T(Y^{k-1}, X^k)$$

$EQ \stackrel{\mathrm{def}}{=} \bigwedge_{p \in \mathbb{P}} (p(Y) \leftrightarrow p(X))$



IC3 with Implicit Abstraction

- Integrate the idea of Implicit Abstraction within IC3
- Use abstract transition relation $T(X, Y')$ instead of $T(X, X')$
- Learn clauses only over predicates $\mathbb{P}$
- Use abstract relative induction check:

$$\begin{aligned}\mathrm{AbsRelInd}(F, T, s, \mathbb{P}) := & F(X) \wedge s(X) \wedge T(X, Y') \wedge \\ & \bigwedge_{p \in \mathbb{P}} (p(X') \leftrightarrow p(Y')) \wedge \neg s(X')\end{aligned}$$

IC3 with Implicit Abstraction

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- If UNSAT ⇨inductive strengthening as in the Boolean case
  - No theory-specific technique needed
  - Theory reasoning confined within the SMT solver

IC3 with Implicit Abstraction

- Integrate the idea of Implicit Abstraction within IC3
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$$\mathrm{AbsRelInd}(F, T, s, \mathbb{P}) := F(X) \wedge s(X) \wedge T(X, Y') \wedge \bigwedge_{p \in \mathbb{P}} (p(X') \leftrightarrow p(Y')) \wedge \neg s(X')$$

- If SAT $\Rightarrow$ abstract predecessor c from the SMT model μ
  - $c \stackrel{\text{def}}{=} \{p(X) \mid p \in \mathbb{P} \wedge \mu \models p(X)\} \cup \{\neg p(X) \mid \mu \not\models p(X)\}$
  - No quantifier elimination needed

Example

- $T \stackrel{\text{def}}{=} (2x_1' - 3x_1 \leq 4x_2' + 2x_2 + 3) \wedge (3x_1 - 2x_2' = 0)$
- $\mathbb{P} \stackrel{\text{def}}{=} \{(x_1 - x_2 \geq 4), (x_1 < 3)\}$
- $s \stackrel{\text{def}}{=} \neg(x_1 - x_2 \geq 4) \wedge (x_1 < 3)$
- $RelInd(\emptyset, T, s)$ is SAT
- Compute a predecessor with $\exists_{\text{approx}} x_1', x_2'.(\neg s \wedge T \wedge s')$
  $(\frac{5}{2} \leq 3x_1 + x_2) \wedge \neg(x_1 - x_2 \geq 4) \wedge (x_1 < 3) \wedge \neg(-\frac{2}{3} \leq x_1)$

Example

- $T \stackrel{\text{def}}{=} (2x'_1 - 3x_1 \leq 4x'_2 + 2x_2 + 3) \wedge (3x_1 - 2x'_2 = 0)$
- $\mathbb{P} \stackrel{\text{def}}{=} \{(x_1 - x_2 \geq 4), (x_1 < 3)\}$
- $s \stackrel{\text{def}}{=} \neg(x_1 - x_2 \geq 4) \wedge (x_1 < 3)$
- $RelInd(\emptyset, T, s)$ is SAT
- Compute a predecessor with $\exists_{\text{approx}} x'_1, x'_2.(\neg s \wedge T \wedge s')$
  $(\frac{5}{2} \leq 3x_1 + x_2) \wedge \neg(x_1 - x_2 \geq 4) \wedge (x_1 < 3) \wedge \neg(-\frac{2}{3} \leq x_1)$
- $$\begin{aligned} \mathrm{AbsRelInd}(\emptyset, T, s, \mathbb{P}) := & \; T[X' \mapsto Y'] \wedge \\ & \neg s \wedge s' \wedge \\ & (x'_1 - x'_2 \geq 4) \leftrightarrow (y'_1 - y'_2 \geq 4) \wedge (x'_1 < 3) \leftrightarrow (y'_1 < 3) \end{aligned}$$
- Compute predecessor from SMT model $\mu \stackrel{\text{def}}{=} \{x_1 \mapsto 0, x_2 \mapsto 1\}$
  $\neg(x_1 - x_2 \geq 4) \wedge (x_1 < 3)$

Example

- $T \stackrel{\text{def}}{=} (2x_1' - 3x_1 \leq 4x_2' + 2x_2 + 3) \wedge (3x_1 - 2x_2' = 0)$
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- $s \stackrel{\text{def}}{=} \neg(x_1 - x_2 \geq 4) \wedge (x_1 < 3)$
- $RelInd(\emptyset, T, s)$ is SAT
- Compute a predecessor with $\exists_{\text{approx}} x_1', x_2'.(\neg s \wedge T \wedge s')$
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- Compute predecessor from SMT model $\mu \stackrel{\text{def}}{=} \{x_1 \mapsto 0, x_2 \mapsto 1\}$
  $\neg(x_1 - x_2 \geq 4) \wedge (x_1 < 3)$

Abstraction Refinement

- Abstract predecessors are overapproximations
  - Spurious counterexamples can be generated
- We can apply standard abstraction refinement techniques
  - Use sequence interpolants to discover new predicates
  - Sequence of abstract states $s_0 \rightarrow s_1 \rightarrow \ldots \rightarrow s_n$
  - SMT check on $s_0^0 \wedge T_{\text{concrete}}^0 \wedge s_1^1 \wedge \ldots \wedge T_{\text{concrete}}^{k-1} \wedge s_k^k$
  - If unsat, compute sequence of interpolants for

$$[s_0^0 \wedge T_{\text{concrete}}^0 \wedge \ldots \wedge T_{\text{concrete}}^{i-1}], [s_i^i \wedge \ldots \wedge T_{\text{concrete}}^{k-1} \wedge s_k^k]$$

  - Add all the predicates in the interpolants to $\mathbb{P}$

Incrementality

- Abstraction refinement is *fully incremental*
- No restart from scratch
- Can keep all the clauses of $F_1, \ldots, F_k$
  - Refinements monotonically strengthen T

    $T_{\text{new}} \stackrel{\text{def}}{=} T_{\text{old}} \wedge \bigwedge_{p \in \mathbb{P}_{\text{new}}} (p(X) \leftrightarrow p(Y)) \wedge (p(X') \leftrightarrow p(Y'))$
  - All IC3 invariants on $F_1, \ldots, F_k$ are preserved

    $F_{i+1} \subseteq F_i$ (so $F_i \models F_{i+1}$) ✔

    for all $i < k, F_i \models P$ ✔

    $\mathrm{F}_i \wedge T_{\text{new}} \models F'_{i+1}$ ✔
- Abstract counterexample check can use incremental SMT

Example

- System *S* with 2 state vars *c* and *d*
  - Init: $(d = 1) \wedge (c \geq d)$
  - Trans: $(c' = c + d) \wedge (d' = d + 1)$
  - Property: $(d > 2) \implies (c > d)$
- Predicates $\mathbb{P}$

$$(d = 1) \quad (c \geq d)$$
$$(d > 2) \quad (c > d)$$

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- System *S* with 2 state vars *c* and *d*
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- Predicates $\mathbb{P}$

$$(d = 1) \quad (c \geq d)$$
$$(d > 2) \quad (c > d)$$

- Check base case: $\text{Init} \models \text{Property}$ ✔

Example

- System *S* with 2 state vars *c* and *d*
  - Init: $(d = 1) \wedge (c \geq d)$
  - Trans: $(c' = c + d) \wedge (d' = d + 1)$
  - Property: $(d > 2) \implies (c > d)$
- Predicates $\mathbb{P}$

$$(d = 1) \quad (c \geq d)$$
$$(d > 2) \quad (c > d)$$

- Trace: $F_0 := \mathrm{Init}$
  $F_1 := \top$
- Get bad cube
  - SMT check $F_1 \wedge \neg \mathrm{Prop}$
  - SAT with model $\mu := \{c = 0, d = 2\}$
  - Evaluate predicates wrt. μ
    - Return $c := \{\neg(d = 1), \neg(c \geq d), (d > 2), \neg(c > d)\}$

Example

- System *S* with 2 state vars *c* and *d*
  - Init: $(d = 1) \wedge (c \geq d)$
  - Trans: $(c' = c + d) \wedge (d' = d + 1)$
  - Property: $(d > 2) \implies (c > d)$
- Rec. block *c*
  - Check

$$AbsRelInd(F_0, T, c, \mathbb{P}) := \text{Init} \wedge$$

$$\boxed{\begin{array}{l} ({\color{red}y_c} = c + d) \wedge ({\color{red}y_d} = d + 1) \wedge \\ ((d' = 1) \leftrightarrow ({\color{red}y_d} = 1)) \wedge ((c' \geq d') \leftrightarrow ({\color{red}y_c} \geq {\color{red}y_d})) \wedge \\ ((d' > 2) \leftrightarrow ({\color{red}y_d} > 2)) \wedge ((c' > d') \leftrightarrow ({\color{red}y_c} > {\color{red}y_d})) \wedge \end{array}}$$

$$\neg c \wedge c'$$

- Predicates $\mathbb{P}$

$$\begin{array}{ll} (d = 1) & (c \geq d) \\ (d > 2) & (c > d) \end{array}$$

- Trace: $F_0 := \text{Init}$
  $F_1 := \top$

Example

- System *S* with 2 state vars *c* and *d*
  - Init: $(d = 1) \wedge (c \geq d)$
  - Trans: $(c' = c + d) \wedge (d' = d + 1)$
  - Property: $(d > 2) \implies (c > d)$
- Rec. block *c*
  - Check
  $AbsRelInd(F_0, T, c, \mathbb{P})$ ✔
  - Unsat core: $\{(d' > 2)\}$
  - Update $F_1 := F_1 \wedge \neg(d > 2)$
- Predicates $\mathbb{P}$
  $(d = 1) \quad (c \geq d)$
  $(d > 2) \quad (c > d)$
- Trace: $F_0 := \text{Init}$
  $F_1 := \top$

Example

- System *S* with 2 state vars *c* and *d*
  - Init: $(d = 1) \wedge (c \geq d)$
  - Trans: $(c' = c + d) \wedge (d' = d + 1)$
  - Property: $(d > 2) \implies (c > d)$
- Forward propagation

- Predicates $\mathbb{P}$

  $$(d = 1) \quad (c \geq d)$$
  $$(d > 2) \quad (c > d)$$

- Trace: $F_0 := \mathrm{Init}$

  $$F_1 := \neg(d > 2)$$
  $$F_2 := \top$$

Example

- System S with 2 state vars c and d
  - Init: $(d = 1) \wedge (c \geq d)$
  - Trans: $(c' = c + d) \wedge (d' = d + 1)$
  - Property: $(d > 2) \implies (c > d)$
- Get bad cube at 2
  - $c := \{\neg(d = 1), \neg(c \geq d),$
    $(d > 2), \neg(c > d)\}$
- Predicates $\mathbb{P}$
  $(d = 1) \quad (c \geq d)$
  $(d > 2) \quad (c > d)$
- Trace: $F_0 := \mathrm{Init}$
  $F_1 := \neg(d > 2)$
  $F_2 := \top$

Example

- System S with 2 state vars c and d
  - Init: $(d = 1) \wedge (c \geq d)$
  - Trans: $(c' = c + d) \wedge (d' = d + 1)$
  - Property: $(d > 2) \implies (c > d)$
- Rec. block c
  - $\ldots$
  - Update $F_1 := F_1 \wedge (c \geq d)$
  - $\ldots$
  - Update $F_2 := F_2 \wedge (c > d) \vee \neg(d > 2)$
- Predicates $\mathbb{P}$

$$\begin{array}{cc} (d = 1) & (c \geq d) \\ (d > 2) & (c > d) \end{array}$$

- Trace: $F_0 := \mathrm{Init}$

$$F_1 := \neg(d > 2)$$
$$F_2 := \top$$

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  - Init: $(d = 1) \wedge (c \geq d)$
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$$(d = 1) \quad (c \geq d)$$
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- Trace: $F_0 := \mathrm{Init}$

$$F_1 := \neg(d > 2) \wedge (c \geq d) \wedge F_2$$
$$F_2 := (c > d) \vee \neg(d > 2)$$
$$F_3 := \top$$

Example

- System *S* with 2 state vars *c* and *d*
  - Init: $(d = 1) \wedge (c \geq d)$
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$$(d = 1) \quad (c \geq d)$$
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- Trace: $F_0 := \mathrm{Init}$

$$F_1 := \neg(d > 2) \wedge (c \geq d) \wedge F_2$$
$$F_2 := (c > d) \vee \neg(d > 2)$$
$$F_3 := \top$$

- Get bad cube at 3
  - $c := \{\neg(d = 1), \neg(c \geq d), (d > 2), \neg(c > d)\}$

Example

- System S with 2 state vars c and d
  - Init: $(d = 1) \wedge (c \geq d)$
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$$F_1 := \neg(d > 2) \wedge (c \geq d) \wedge F_2$$
$$F_2 := (c > d) \vee \neg(d > 2)$$
$$F_3 := \top$$

- Rec block c
  - Check

$$AbsRelInd(F_2, T, c, \mathbb{P})$$ ✘

  - SMT model

$$\mu := \{c = 0, d = 2, c' = 0, d' = 3, y_c = 2, y_d = 3\}$$

  - (Abstract) predecessor

$$s := \{\neg(d > 2), \neg(c > d), \neg(d = 1), \neg(c \geq d)\}$$

Example

- System S with 2 state vars c and d
  - Init: $(d = 1) \wedge (c \geq d)$
  - Trans: $(c' = c + d) \wedge (d' = d + 1)$
  - Property: $(d > 2) \implies (c > d)$
- Rec block s (at level 2)
  - $\ldots$
  - Reached level 0, abstract cex:

$$q := \neg(d > 2), \neg(c > d), (d = 1), (c \geq d)$$
$$p := \neg(d > 2), \neg(c > d), \neg(d = 1), (c \geq d)$$
$$s := \neg(d > 2), \neg(c > d), \neg(d = 1), \neg(c \geq d)$$
$$c := \neg(d = 1), \neg(c \geq d), (d > 2), \neg(c > d)$$

- Predicates $\mathbb{P}$

$$(d = 1) \quad (c \geq d)$$
$$(d > 2) \quad (c > d)$$

- Trace: $F_0 := \mathrm{Init}$

$$F_1 := \neg(d > 2) \wedge (c \geq d) \wedge F_2$$
$$F_2 := (c > d) \vee \neg(d > 2)$$
$$F_3 := \top$$

Example

- System *S* with 2 state vars *c* and *d*
  - Init: $(d = 1) \wedge (c \geq d)$
  - Trans: $(c' = c + d) \wedge (d' = d + 1)$
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- Check abstract counterexample
  - SMT check

    $I_0 \wedge q_0 \wedge T_{0 \mapsto 1} \wedge p_1 \wedge T_{1 \mapsto 2} \wedge s_2 \wedge T_{2 \mapsto 3} \wedge c_3$

    UNSAT

- Predicates $\mathbb{P}$

  $(d = 1) \quad (c \geq d)$

  $(d > 2) \quad (c > d)$

- Trace: $F_0 := \text{Init}$

  F_1

  F_2

  F_3

Example

- System *S* with 2 state vars *c* and *d*
  - Init: $(d = 1) \wedge (c \geq d)$
  - Trans: $(c' = c + d) \wedge (d' = d + 1)$
  - Property: $(d > 2) \implies (c > d)$
- Check abstract counterexample
  - Compute sequence interpolant

$$\underbrace{I_0 \wedge q_0 \wedge T_{0\mapsto 1}}_{A_1} \wedge \underbrace{p_1 \wedge T_{1\mapsto 2} \wedge s_2 \wedge T_{2\mapsto 3} \wedge c_3}_{B_1}$$

$$\varphi_1 := (d_1 \geq 2)$$

- Predicates $\mathbb{P}$

$$(d = 1) \quad (c \geq d)$$
$$(d > 2) \quad (c > d)$$

- Trace: $F_0 := \mathrm{Init}$

F_1

F_2

F_3

Example

- System S with 2 state vars c and d
  - Init: $(d = 1) \wedge (c \geq d)$
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  $(d = 1) \quad (c \geq d)$

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- Trace: $F_0 := \mathrm{Init}$

  F_1

  F_2

  F_3
- Check abstract counterexample
  - Compute sequence interpolant

$$\underbrace{I_0 \wedge q_0 \wedge T_{0\mapsto 1} \wedge p_1 \wedge T_{1\mapsto 2}}_{A_2} \wedge \underbrace{s_2 \wedge T_{2\mapsto 3} \wedge c_3}_{B_2}$$

$$\varphi_1 := (d_1 \geq 2)$$

$$\varphi_2 := (d_2 \geq 3)$$

Example

- System *S* with 2 state vars *c* and *d*
  - Init: $(d = 1) \wedge (c \geq d)$
  - Trans: $(c' = c + d) \wedge (d' = d + 1)$
  - Property: $(d > 2) \implies (c > d)$

- Predicates $\mathbb{P}$

$$(d = 1) \quad (c \geq d)$$
$$(d > 2) \quad (c > d)$$
$$\boxed{(d \geq 2) \quad (d \geq 3)}$$

- Trace: $F_0 := \text{Init}$

$$F_1$$
$$F_2$$
$$F_3$$

- Check abstract counterexample
  - Compute sequence interpolant

$$\underbrace{I_0 \wedge q_0 \wedge T_{0 \mapsto 1} \wedge p_1 \wedge T_{1 \mapsto 2} \wedge s_2 \wedge T_{2 \mapsto 3}}_{A_3} \wedge \underbrace{c_3}_{B_3}$$

$$\varphi_1 := (d_1 \geq 2)$$
$$\varphi_2 := (d_2 \geq 3)$$
$$\varphi_3 := \bot$$

Update predicates $\mathbb{P}$

Example

- System S with 2 state vars c and d
  - Init: $(d = 1) \wedge (c \geq d)$
  - Trans: $(c' = c + d) \wedge (d' = d + 1)$
  - Property: $(d > 2) \implies (c > d)$
- Update abstract trans
- Resume IC3 from level 3
  - ...

- Predicates $\mathbb{P}$

$$\begin{array}{ll} (d = 1) & (c \geq d) \\ (d > 2) & (c > d) \\ (d \geq 2) & (d \geq 3) \end{array}$$

- Trace: $F_0 := \mathrm{Init}$

$$F_1 := \neg(d > 2) \wedge (c \geq d) \wedge F_2$$

$$F_2 := (c > d) \vee \neg(d > 2)$$

$$F_3 := \top$$

Example

- System S with 2 state vars c and d
  - Init: $(d = 1) \wedge (c \geq d)$
  - Trans: $(c' = c + d) \wedge (d' = d + 1)$
  - Property: $(d > 2) \implies (c > d)$
- Update abstract trans
- Resume IC3 from level 3
  - $\ldots$

- Predicates $\mathbb{P}$

$$\begin{array}{ll} (d = 1) & (c \geq d) \\ (d > 2) & (c > d) \\ (d \geq 2) & (d \geq 3) \end{array}$$

- Trace: $F_0 := \mathrm{Init}$

$$F_1 := \neg(d > 2) \wedge (c \geq d) \wedge F_2$$

$$F_2 := (c \geq d) \vee \neg(d \geq 2) \wedge F_3$$

$$F_3 := (d = 1) \vee (d \geq 2) \wedge \neg(c \geq d) \wedge F_4$$

$$F_4 := (c > d) \vee \neg(d > 2)$$

Example

- System *S* with 2 state vars *c* and *d*
  - Init: $(d = 1) \wedge (c \geq d)$
  - Trans: $(c' = c + d) \wedge (d' = d + 1)$
  - Property: $(d > 2) \implies (c > d)$
- Update abstract trans
- Resume IC3 from level 3
  - $\ldots$
  - Forward propagation

- Predicates $\mathbb{P}$

$$(d = 1) \quad (c \geq d)$$
$$(d > 2) \quad (c > d)$$
$$(d \geq 2) \quad (d \geq 3)$$

- Trace: $F_0 := \text{Init}$

$$F_1 := \neg(d > 2) \wedge (c \geq d) \wedge F_2$$
$$F_2 := (c \geq d) \vee \neg(d \geq 2) \wedge F_3$$
$$F_3 := (d = 1) \vee (d \geq 2) \wedge$$
$$\neg(c \geq d) \wedge F_4$$
$$F_4 := (c > d) \vee \neg(d > 2)$$

Example

- System S with 2 state vars c and d
  - Init: $(d = 1) \wedge (c \geq d)$
  - Trans: $(c' = c + d) \wedge (d' = d + 1)$
  - Property: $(d > 2) \implies (c > d)$
- Predicates $\mathbb{P}$

$$\begin{array}{ll} (d = 1) & (c \geq d) \\ (d > 2) & (c > d) \\ (d \geq 2) & (d \geq 3) \end{array}$$

- Trace: $F_0 := \mathrm{Init}$

$$F_1 := \neg(d > 2) \wedge (c \geq d) \wedge F_2$$

$$F_2 := (c \geq d) \vee \neg(d \geq 2) \wedge F_3$$

$$F_3 := (d = 1) \vee (d \geq 2) \wedge \neg(c \geq d) \wedge F_4$$

$$F_4 := (c > d) \vee \neg(d > 2)$$

- Update abstract trans
- Resume IC3 from level 3
  - $\ldots$
  - Forward propagation

$$F_2 \wedge \widehat{T}_{\mathbb{P}} \models (c' \geq d') \vee \neg(d' \geq 2)$$

Example

- System *S* with 2 state vars *c* and *d*
  - Init: $(d = 1) \wedge (c \geq d)$
  - Trans: $(c' = c + d) \wedge (d' = d + 1)$
  - Property: $(d > 2) \implies (c > d)$
- Update abstract trans
- Resume IC3 from level 3
  - $\ldots$
  - Forward propagation

SAFE ✔

- Predicates $\mathbb{P}$

$$(d = 1) \quad (c \geq d)$$
$$(d > 2) \quad (c > d)$$
$$(d \geq 2) \quad (d \geq 3)$$

- Trace: $F_0 := \mathrm{Init}$

$$F_1 := \neg(d > 2) \wedge (c \geq d) \wedge F_2$$
$$\boxed{F_2 := F_3}$$
$$F_3 := (c \geq d) \vee \neg(d \geq 2) \wedge$$
$$(d = 1) \vee (d \geq 2) \wedge$$
$$\neg(c \geq d) \wedge F_4$$
$$F_4 := (c > d) \vee \neg(d > 2)$$

Implementing IC3-IA

- Get the code at:
  http://es-static.fbk.eu/people/griggio/vtsa2015/
  - Open source (GPLv3) implementation on top of MathSAT
    http://mathsat.fbk.eu/
    - Incremental interface
    - Assumptions and unsat core
    - Interpolation

- Simple (~1700 lines of C++, including parser and statistics, according to David A. Wheeler's 'SLOCCount') yet competitive
  - Input in VMT format (a simple extension of SMT-LIB)

    https://nuxmv.fbk.eu/index.php?n=Languages.VMT

- Let's analyse it!

Outline

Introduction

IC3 for finite-state systems

SMT-based IC3 for infinite-state systems

IC3 for LTL verification

Linear Temporal Logic

- Syntax
  - A (quantifier-free) first-order formula φ
  - $\mathbf{X}\varphi$ (ne**X**t φ)
  - $\varphi\mathbf{U}\psi$ (φ **Until** ψ)
  - $\mathbf{F}\varphi$ (**F**inally φ)
  - $\mathbf{G}\varphi$ (**G**lobally φ)

- Semantics
  - Given an infinite path $\pi := s_0, s_1, \ldots, s_i, \ldots$
    - $\pi \models \mathbf{X}\varphi \text{ iff } s_1, \ldots \models \varphi$
    - $\pi \models \varphi\mathbf{U}\psi \text{ iff } \exists j > 0. s_j, \ldots \models \psi \text{ and } \forall 0 \leq k < j. s_k, \ldots \models \varphi$
    - $\pi \models \mathbf{F}\varphi \text{ iff } \exists j. s_j, \ldots \models \varphi$
    - $\pi \models \mathbf{G}\varphi \text{ iff } \forall j. s_j, \ldots \models \varphi$
  - A system *S* satisfies an LTL formula φ ($S \models \varphi$) iff all inifinite paths of *S* satisfy φ

LTL verification

- Automata-based approach:
  - Given an LTL property φ, build a transition system $S_{\neg\varphi}$ with a fairness condition $f_{\neg\varphi}$, such that

$$\boxed{S \models \varphi \quad \text{iff} \quad S \times S_{\neg\varphi} \models \mathbf{FG}\neg f_{\neg\varphi}}$$

- Finite-state case:
  - **lasso-shaped counterexamples**, with $f_{\neg\varphi}$ at least once in the loop
  - **liveness to safety** transformation: absence of lasso-shaped counterexamples as an invariant property
    - Duplicate the state variables $X_{\text{copy}} = \{x_c | x \in X\}$
    - Non-deterministically save the current state
    - Remember when $f_{\neg\varphi}$ in extra state var triggered
    - Invariant: $\mathbf{G}\neg(X = X_{\text{copy}} \wedge \text{triggered})$

Liveness to Safety for Inifinite States

- Unsound for infinite-state systems
  - Not all counterexamples are lasso-shaped

$$I(S) \stackrel{\text{def}}{=} (x = 0) \qquad T(S) \stackrel{\text{def}}{=} (x' = x + 1) \qquad \varphi \stackrel{\text{def}}{=} \mathbf{FG}(x < 5)$$

- Liveness to safety with Implicit Abstraction
  - Apply the l2s transformation to the abstract system
    - Save the values of the predicates instead of the concrete state
  - Do it on-the-fly, **tightly integrating** l2s with IC3
  - Sound but incomplete
    - When abstract loop found, simulate in the concrete and refine
    - Might still diverge during refinement
      - Intrinsic limitation of **state** predicate abstraction

K-liveness

- Simple but effective technique for LTL verification of finite-state systems
- Key insight: $M \times M_{\neg\varphi} \models \mathbf{FG} \neg f_{\neg\varphi}$ iff exists *k* such that $f_{\neg\varphi}$ **is visited at most *k* times**
  - Again, a safety property

- K-liveness: increase k incrementally, within IC3
  - Liveness checking as a sequence of safety checks
  - Exploits the highly incremental nature of IC3
  - Sound also for infinite-state systems
    - What about completeness?

K-liveness for hybrid automata

- K-liveness is incomplete for infinite-state systems
  - Even if $M \times M_{\neg\varphi} \models \mathbf{FG}\neg f_{\neg\varphi}$, there might be **no concrete *k*** bound for the number of violations of $\neg f_{\neg\varphi}$

$$I(S) \stackrel{\text{def}}{=} (x = n) \qquad T(S) \stackrel{\text{def}}{=} (x' = x + 1) \qquad \varphi \stackrel{\text{def}}{=} \mathbf{FG}(x > n)$$

- K-zeno: extension of K-liveness for hybrid automata
  - Key idea: exploit progress of time to make k-liveness converge
  - By extending the original model with a **"symbolic fairness monitor"** Z^{φ}_{β} that forces time progress
  - Under certain conditions, restores completeness of k-liveness
    - If $M \times M_{\neg\varphi} \models \mathbf{FG}\neg f_{\neg\varphi}$, then exists k such that $M \times M_{\neg\varphi} \times Z^{\varphi}_{\beta}$ visits f_Z at most k times
    - (clearly, safety check can still diverge)

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DISCLAIMER: again, this is definitely incomplete. Apologies to missing authors/works

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Thank You