Tutorial:
Probabilistic Model Checking

Christel Baier
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Probability elsewhere

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- randomized algorithms [Rabin 1960]
  symmetry breaking, fingerprint techniques,
  random choice of waiting times or IP addresses, ...
- stochastic control theory [Bellman 1957]
  operations research
- performance modeling [Markov, Erlang, Kolm., $\sim$ 1900]
- biological systems
- resilient systems

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discrete or continuous-time Markovian models

memoryless property: future system behavior depends only on the current state, but not on the past

Probabilistic models

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	purely probabilistic	probabilistic and nondeterministic
discrete time		
continuous time		

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discrete time	discrete-time Markov chain (DTMC)	Markov decision process (MDP)
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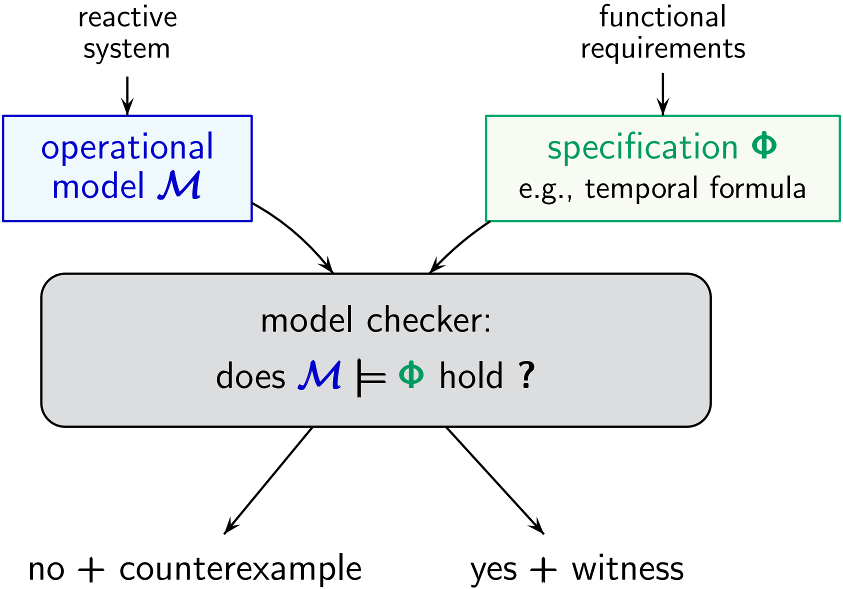
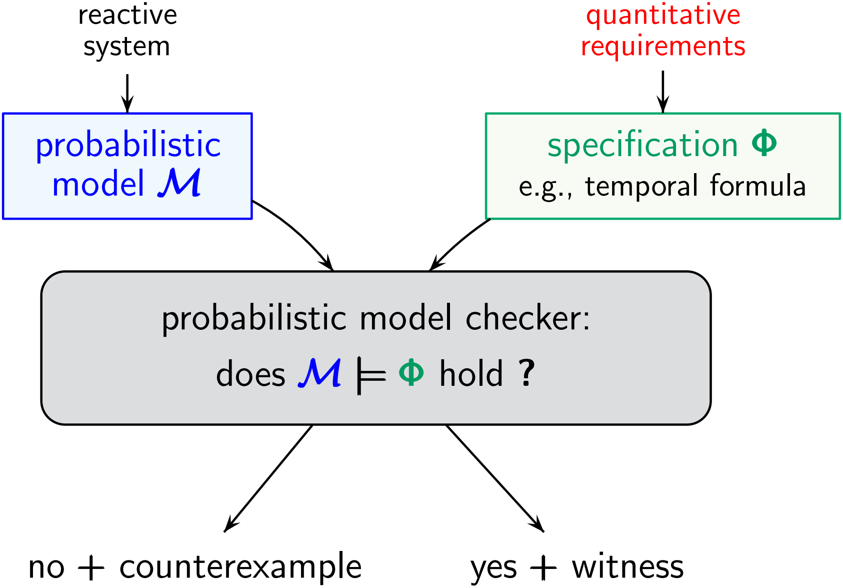
Model checking



Probabilistic model checking



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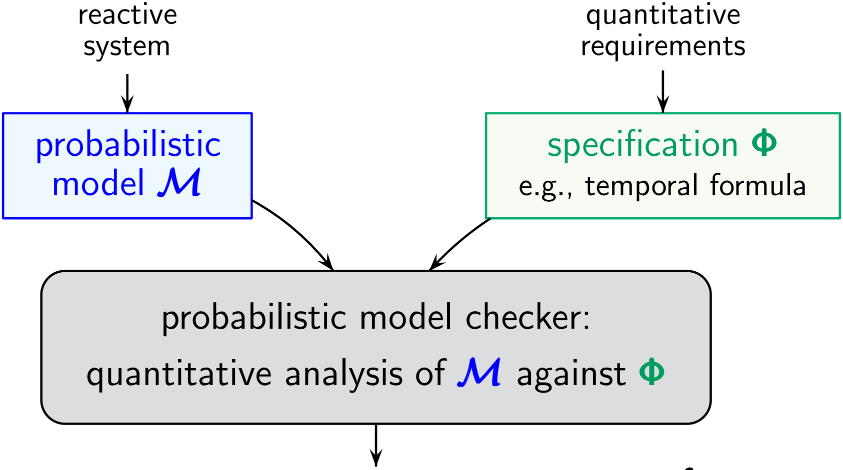

probability for "bad behaviors" is $< \mathbf{10^{-6}}$
probability for "good behaviors" is $\mathbf{1}$
expected costs for ....

Probabilistic model checking

- termination of probabilistic programs [Hart/Sharir/Pnueli'83]
- qualitative linear time properties [Vardi/Wolper'86]
  for discrete-time Markov models [Courcoubetis/Yannak.'88]

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- termination of probabilistic programs
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  for discrete-time Markov models [Courcoubetis/Yannak.'88]
- probabilistic computation tree logic [Hansson/Jonsson'94]
  for discrete-time Markov models [Bianco/de Alfaro'95]
- continuous stochastic logic [Aziz et al'96]
  for continuous-time Markov chains [Baier et al'99]
- probabilistic timed automata [Jensen'96]
  [Kwiatkowska et al'00]

  ⋮

tools: PRISM, MRMC, STORM, IscasMC, PASS,
ProbDiVinE, MARCIE, YMER, ...

Discrete-time Markov chains (DTMC)

* basic definitions
* probabilistic computation tree logic PCTL/PCTL*
* rewards, cost-utility ratios, weights
* conditional probabilities

Markov decision processes (MDP)

* basic definitions
* PCTL/PCTL* model checking
* fairness
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* rewards, quantiles
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Markov chains

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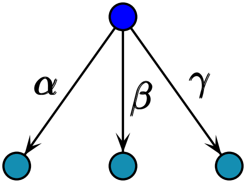
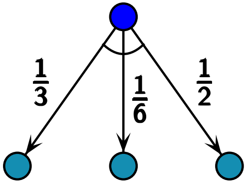


transition system

nondeterministic branching

choice between action-labeled transitions



Markov chain

probabilistic branching

discrete-time

Discrete-time Markov chain (DTMC)

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$$\text{s.t.} \quad \sum_{s' \in S} P(s, s') = 1$$

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$\uparrow$

discrete-time or time-abstract:

probability for the step $s \longrightarrow s'$

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- AP set of atomic propositions
- labeling function $L : S \to 2^{AP}$
- $\mu : S \to [0,1]$ initial distribution
- $wgt : S \to \mathbb{Z}$ where $wgt(s)$ is the reward (or weight) earned per visit of state s

Example: DTMC for communication protocol

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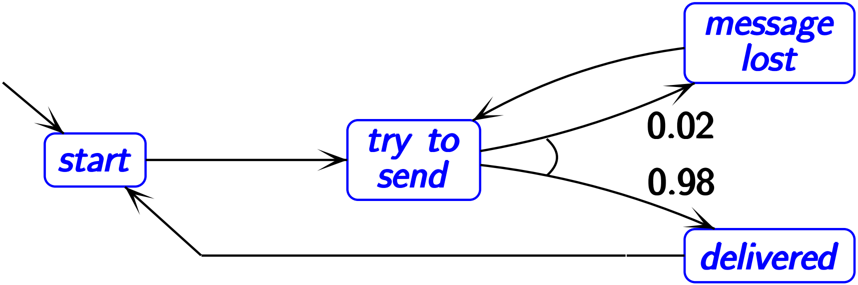

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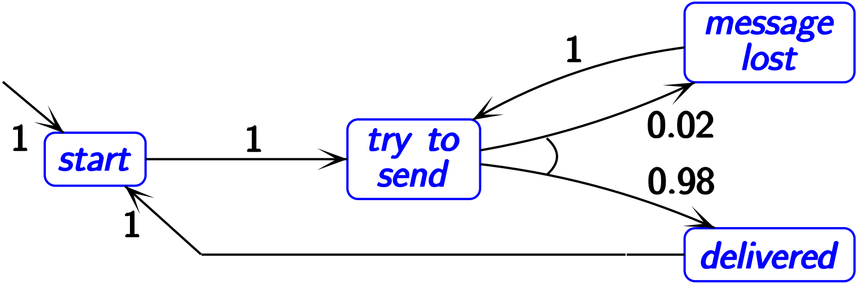

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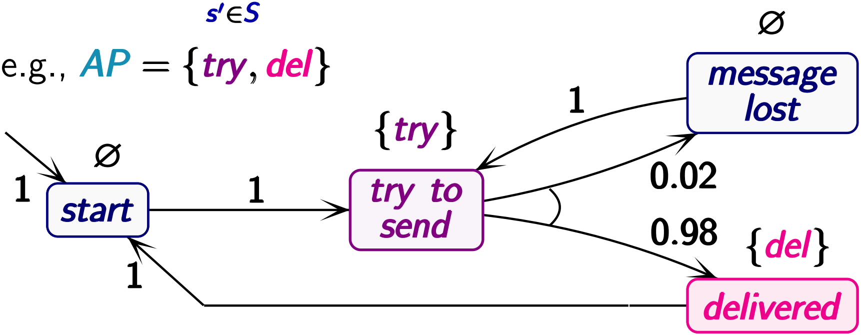
e.g., $AP = \{try, del\}$



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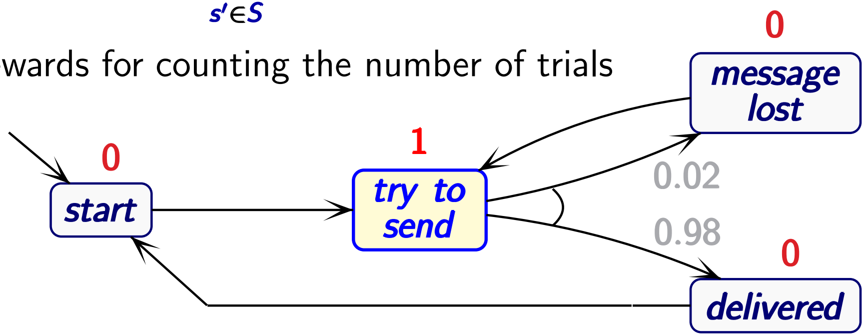
rewards for counting the number of trials



Probability measure of a Markov chain

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↑ initial distribution

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probability measure for measurable sets of paths:

consider the σ-algebra generated by cylinder sets

$\Delta(s_0\, s_1 \ldots s_n)$ = set of infinite paths $s_0\, s_1 \ldots s_n\, s_{n+1}\, s_{n+2}\, s_{n+3} \ldots$

↑ finite path

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σ-algebra on universe $\mathcal{U}$: set $\mathcal{V} \subseteq 2^{\mathcal{U}}$ s.t.

1. $\mathcal{U} \in \mathcal{V}$
2. if $T \in \mathcal{V}$ then $\mathcal{U} \setminus T \in \mathcal{V}$
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$$\Delta(s_0\, s_1 \ldots s_n) = \text{set of infinite paths} \ldots$$

here: $\mathcal{U}$ = set of infinite paths $\subseteq S^\omega$

$\mathcal{V}$ = smallest subset of $2^{\mathcal{U}}$ that contains all cylinder sets and is closed under complement and countable unions

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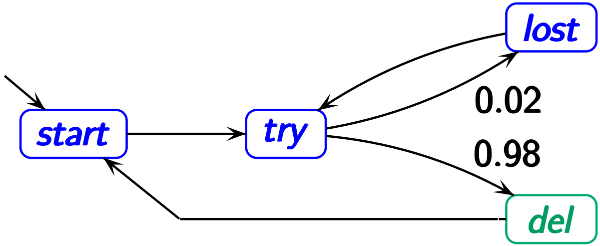
$$s_0\, s_1 \ldots s_n\, s_{n+1}\, s_{n+2}\, s_{n+3} \ldots$$

probability measure is given by:

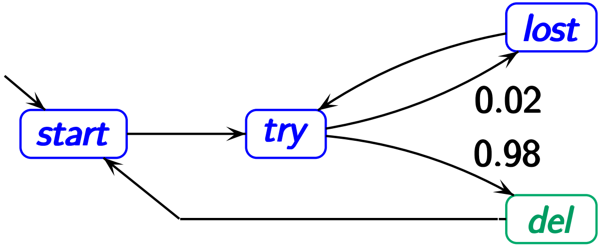
$$\Pr^{\mathcal{M}}\big(\, \Delta(s_0\, s_1 \ldots s_n)\,\big) \;=\; \mu(s_0) \cdot \prod_{1 \leqslant i \leqslant n} P(s_{i-1}, s_i)$$

Example: Markov chain



probability for delivering the message within **5** steps:

Example: Markov chain



probability for delivering the message within **5** steps:

$$= \Pr^{\mathcal{M}}(\textit{start try del}) + \Pr^{\mathcal{M}}(\textit{start try lost try del})$$

notation: $\Pr^{\mathcal{M}}(s_0\, s_1 \ldots s_n) = \Pr^{\mathcal{M}}(\Delta(s_0\, s_1 \ldots s_n))$

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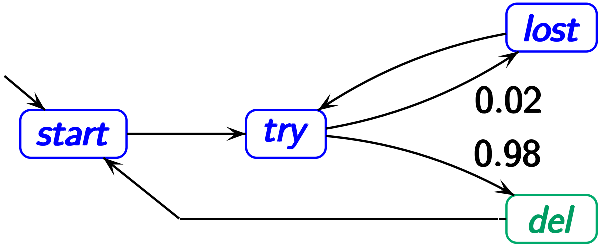

probability for delivering the message within **5** steps:

$$= \Pr^{\mathcal{M}}(\textit{start try del}) + \Pr^{\mathcal{M}}(\textit{start try lost try del})$$

$$= 0.98 + 0.02 \cdot 0.98 = 0.9996$$

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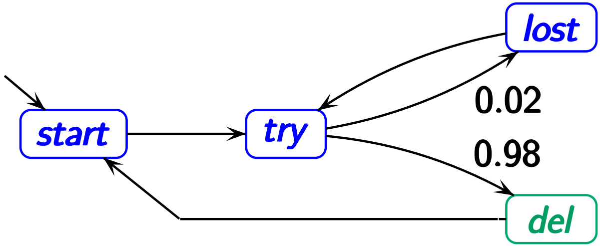

probability for eventually delivering the message:

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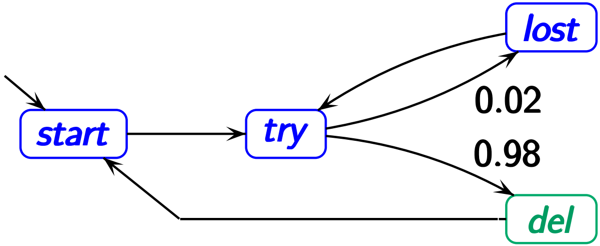

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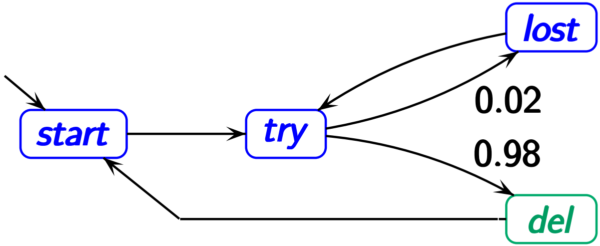

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$$= \sum_{n=0}^{\infty} 0.02^n \cdot 0.98 = 1$$

Measurability of classical properties

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A σ-algebra is a pair $(\mathcal{U}, \mathcal{V})$ where $\mathcal{U}$ is a set and $\mathcal{V} \subseteq 2^{\mathcal{U}}$ such that:

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2. if $T \in \mathcal{V}$ then $\mathcal{U} \setminus T \in \mathcal{V}$
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The elements of $\mathcal{V}$ are called events.

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DTMCs: $\mathcal{U}$ = set of infinite paths

$$\mathcal{V} = \left\{ \begin{array}{l} \sigma\text{-algebra generated by the} \\ \text{cylinder sets} \end{array} \right.$$

$$\Delta(s_0\, s_1 \ldots s_n) = \left\{ \begin{array}{l} \text{set of infinite paths } \pi \text{ of the form} \\ s_0\, s_1 \ldots s_n\, s_{n+1}\, s_{n+2}\, s_{n+3} \ldots \end{array} \right.$$

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step-bounded reachability: "visit G within n steps"

$$\Diamond^{\leqslant n} G \;=\; \bigcup_{0 \leqslant i \leqslant n} \; \bigcup_{s_0, \ldots, s_i} \Delta\big(s_0\, s_1 \ldots s_{i-1}\, s_i\big)$$

where $s_i \in G$

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$$\Pr^{\mathcal{M}}(\Diamond^{\leqslant n} G) = \sum_{0 \leqslant i \leqslant n} \ \sum_{s_0, \ldots, s_i} \Pr^{\mathcal{M}}\big(s_0\, s_1 \ldots s_{i-1}\, s_i\big)$$

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$$\Pr^{\mathcal{M}}(\Diamond G) = \sum_{i \in \mathbb{N}} \sum_{s_0, \ldots, s_i} \Pr^{\mathcal{M}}(s_0 s_1 \ldots s_{i-1} s_i)$$

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repeated reachability: "visit G infinitely often"

$$\Box\Diamond G \;=\; \bigcap_{n \in \mathbb{N}} \; \bigcup_{i \geqslant n} \; \bigcup_{s_0,\ldots,s_i} \Delta\big(s_0\, s_1 \ldots s_{i-1}\, s_i\big)$$

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where $s_i \in G$, but possibly $s_j \in G$ for some $j < i$

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persistence: "from some moment on always G"

$$\Diamond\Box G \;=\; Paths^{\mathcal{M}} \setminus \Box\Diamond\neg G$$

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persistence: "from some moment on always G"

$$\Diamond\Box G = \mathit{Paths}^{\mathcal{M}} \setminus \Box\Diamond\neg G$$

$$\Pr^{\mathcal{M}}(\Diamond\Box G) = 1 - \Pr^{\mathcal{M}}(\Box\Diamond\neg G)$$

Stochastic process of a Markov chain

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general definition of a stochastic process:

family $\left(X_t\right)_{t \in Time}$ of random variables $X_t : \mathcal{U} \to S$

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family $(X_t)_{t \in Time}$ of random variables $X_t : \mathcal{U} \to S$

- $Time$ is a time domain, e.g., $\mathbb{N}$ or $\mathbb{R}_{\geqslant 0}$
- S is a set with fixed σ-algebra
- $\mathcal{U}$ is a sample space with fixed σ-algebra

Stochastic process of a Markov chain

DTMC $\mathcal{M} = (S, P, \ldots)$

family $\left(X_t\right)_{t \in Time}$ of random variables $X_t : \mathcal{U} \to S$

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---

If $t \in \mathbb{N}$ and $\pi = s_0\, s_1\, s_2\, s_3\, \ldots\, s_t \ldots$ then $X_t(\pi) = s_t$.

Stochastic process of a Markov chain

DTMC $\mathcal{M} = (S, P, \ldots)$

family $(X_t)_{t \in Time}$ of random variables $X_t : \mathcal{U} \to S$

- $Time$ is a time domain $\longleftarrow$ $Time = \mathbb{N}$
- S is a set $\longleftarrow$ state space
- $\mathcal{U}$ is a sample space $\longleftarrow$ set of infinite paths

---

If $t \in \mathbb{N}$ and $\pi = s_0\, s_1 \ldots s_{t-2}\, u\, s_t \ldots$ then $X_t(\pi) = s_t$.

Markov property:

$$\Pr{}^{\mathcal{M}}(X_t = s \mid X_{t-1} = u) =$$
$$\Pr{}^{\mathcal{M}}(X_t = s \mid X_{t-1} = u,\, X_{t-2} = s_{t-2},\, \ldots,\, X_0 = s_0)$$

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$$\Pr^{\mathcal{M}}\left(X_1 = s \,\middle|\, X_0 = u \right) \qquad \text{time-homogeneous}$$

Transient and long-run distribution

Transient and long-run distribution

transient: … refers to a fixed time point t

long-run: … when time tends to infinity

Transient distribution

Let $\mathcal{M} = (S, P, \mu, \ldots)$ be a DTMC, $t \in \mathbb{N}$ and $s \in S$.

transient state probability:

$$
\begin{aligned}
\mu_t(s) &= \Pr^{\mathcal{M}}\{s_0\, s_1\, s_2 \ldots \in \mathit{Paths}^{\mathcal{M}} : s_t = s\} \\
&= \Pr^{\mathcal{M}}(\, X_t = s\,)
\end{aligned}
$$

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↑ initial distribution (row vector)

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↑ t-th power of transition probability matrix

$$P^t = P^{t-1} \cdot P$$

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column vector $(0 \ldots 0, 1, 0, \ldots 0)$
representing Dirac distribution
for state s

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↑ column vector $(0 \ldots 0, 1, 0, \ldots 0)$ representing Dirac distribution for state s

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↑ transient state distribution for time point $t{-}1$

Thus: $\mu_0 = \mu$ initial distribution

$\mu_t = \mu_{t-1} \cdot P$ for $t \geqslant 1$

Long-run distributions

Let $\mathcal{M} = (S, P, \mu, \ldots)$ be a DTMC.

steady-state probability: $\widetilde{\mu}(s) = \lim_{t \to \infty} \mu_t(s)$

$\mu_t(s)$ probability for being in state s after t steps

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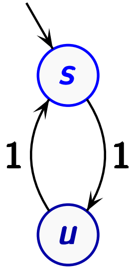

$\mu_{2t}(s) = 1$ $\mu_{2t+1}(s) = 0$

even time points   odd time points

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Let $\mathcal{M} = (S, P, \mu, \ldots)$ be a DTMC.

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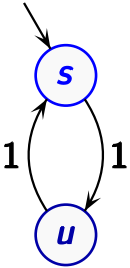
- limit may not exist or depend on the initial distribution μ



$\mu_{2t}(s) = 1$ ↑ even time points

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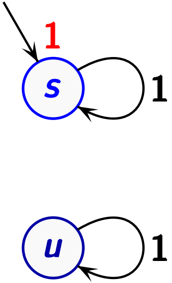

If $\mu(s) = 1$ then: $\widetilde{\mu}(s) = 1$

Long-run distributions

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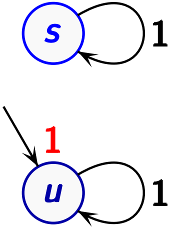

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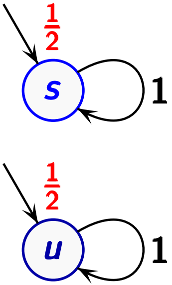

If $\mu(s) = 1$ then: $\widetilde{\mu}(s) = 1$

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If $\mu(s) = \mu(u) = \frac{1}{2}$ then:

$$\widetilde{\mu}(s) = \frac{1}{2}$$

Long-run distributions

Let $\mathcal{M} = (S, P, \mu, \ldots)$ be a DTMC.

steady-state probability: $\widetilde{\mu}(s) = \lim\limits_{t \to \infty} \mu_t(s)$

- limit may not exist or depend on the initial distribution μ
- if existing for all states s then $\widetilde{\mu} = \widetilde{\mu} \cdot P$

$\uparrow$ balance equation

Long-run distributions

Let $\mathcal{M} = (S, P, \mu, \ldots)$ be a DTMC.

steady-state probability: $\widetilde{\mu}(s) = \lim_{t \to \infty} \mu_t(s)$

long-run fraction of being in state s (Cesàro limit):

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$$\theta(s) \;=\; \lim_{T \to \infty} \frac{1}{T+1} \cdot \sum_{t=0}^{T} \mu_t(s)$$

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- Cesàro limit always exists
- if the steady-state probabilities exists: $\widetilde{\mu}(s) = \theta(s)$
- if $\mathcal{M}$ is strongly connected: θ is computable via the balance equation $\theta = \theta \cdot P$ where $\sum_{s\in S} \theta(s) = 1$

Fundamental property of finite Markov chains

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Almost surely, i.e., with probability **1**:

A bottom strongly connected component will be reached and all its states visited infinitely often.

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A bottom strongly connected component will be reached and all its states visited infinitely often.

$$\Pr^{\mathcal{M}}\Big\{ s_0\, s_1\, s_2 \ldots \in \mathit{Paths}^{\mathcal{M}} :$$

there exists $i \geqslant 0$ and a BSCC C s.t.

$$\underbrace{\forall j \geqslant i.\ s_j \in C}_{\text{eventually forever } C} \ \wedge\ \underbrace{\forall s \in C\ \overset{\infty}{\exists} j.\ s_j = s}_{\text{visit each state in } C \text{ infinitely often}} \Big\} = 1$$

Fundamental property of finite Markov chains

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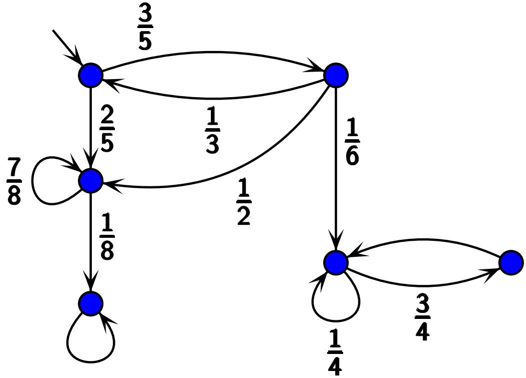

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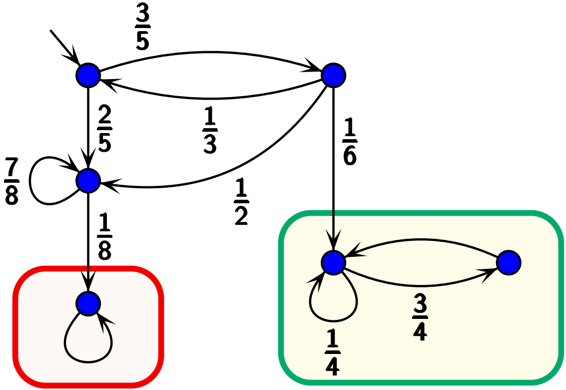

2 BSCCs

Fundamental property of finite Markov chains

> Almost surely, i.e., with probability **1**:
>
> A bottom strongly connected component will be reached and all its states visited infinitely often.

long-run distribution:

$$\theta(s) > 0 \quad \text{iff} \quad s \text{ belongs to some BSCC}$$

Fundamental property of finite Markov chains

Almost surely, i.e., with probability **1**:

A bottom strongly connected component will be reached and all its states visited infinitely often.

long-run distribution:

- $\theta(s) > 0$ iff s belongs to some BSCC
- if s is a state of BSCC B then:

$$\theta(s) = \underbrace{\Pr^{\mathcal{M}}(\Diamond B)}_{\text{probability for reaching B}} \cdot \theta^{B}(s)$$

$\theta^{B}(s)$: long-run probability for state s inside B

Tutorial: Probabilistic Model Checking

Discrete-time Markov chains (DTMC)

* basic definitions
* probabilistic computation tree logic PCTL/PCTL*
* rewards, cost-utility ratios, weights
* conditional probabilities

Markov decision processes (MDP)

* basic definitions
* PCTL/PCTL* model checking
* fairness
* conditional probabilities
* rewards, quantiles
* mean-payoff
* expected accumulated weights

Probabilistic computation tree logic

Probabilistic computation tree logic

PCTL/PCTL* [Hansson/Jonsson 1994]

- probabilistic variants of CTL/CTL*
- contains a probabilistic operator $\mathbb{P}$
  to specify lower/upper probability bounds

Probabilistic computation tree logic

PCTL/PCTL* [Hansson/Jonsson 1994]

- probabilistic variants of CTL/CTL*
- contains a probabilistic operator $\mathbb{P}$
  to specify lower/upper probability bounds
- operators for expected costs, long-run averages, ...
  will be considered later

Syntax of PCTL*

state formulas:

$$\Phi \ ::= \ \mathit{true} \mid a \mid \Phi_1 \wedge \Phi_2 \mid \neg\Phi \mid \quad \ldots$$

path formulas:

$$\varphi \ ::= \ \ldots$$

Syntax of PCTL*

state formulas:

$$\Phi \ ::= \ \mathit{true} \mid a \mid \Phi_1 \wedge \Phi_2 \mid \neg\Phi \mid \mathbb{P}_{\mathrm{I}}(\varphi)$$

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where $a \in AP$ is an atomic proposition

$\mathrm{I} \subseteq [0,1]$ is a probability interval

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qualitative properties: $\mathbb{P}_{>0}(\varphi)$ or $\mathbb{P}_{=1}(\varphi)$

quantitative properties: e.g., $\mathbb{P}_{>0.5}(\varphi)$ or $\mathbb{P}_{\leqslant 0.01}(\varphi)$

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↑ state formula

Syntax of PCTL* path formulas

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path formulas:

$$\varphi \;::=\; \Phi \;\big|\; \varphi_1 \wedge \varphi_2 \;\big|\; \neg\varphi \;\big|\; \bigcirc\varphi \;\big|\; \ldots$$

$\bigcirc \mathrel{\widehat{=}}$ next

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$\bigcirc \mathrel{\widehat{=}}$ next $\qquad$ $\mathbf{U} \mathrel{\widehat{=}}$ until

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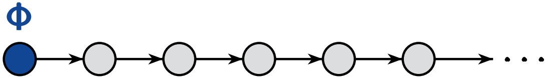
state formula



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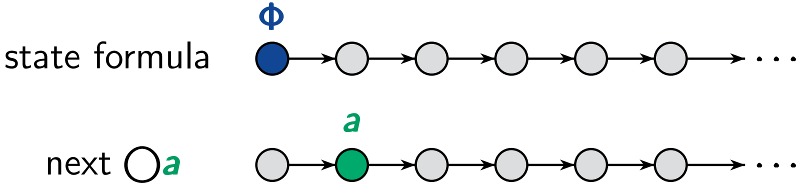

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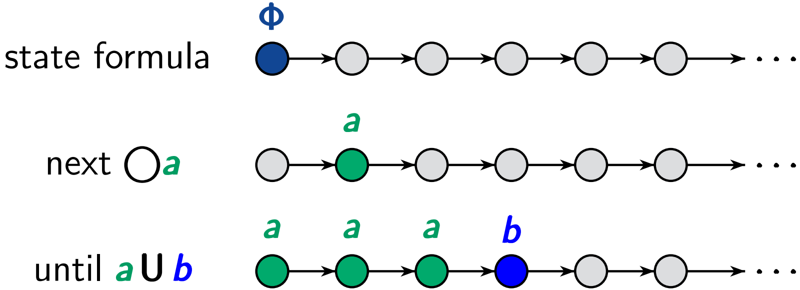

Derived path operators: eventually, always

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syntax of path formulas:

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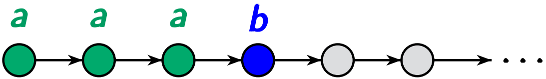
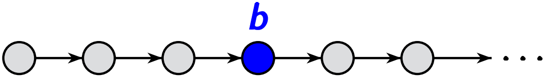
until $a \, \mathsf{U} \, b$



eventually

$\Diamond b \stackrel{\text{def}}{=} \mathit{true} \, \mathsf{U} \, b$



Derived path operators: eventually, always

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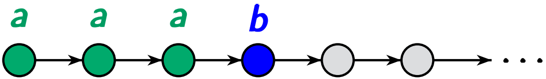

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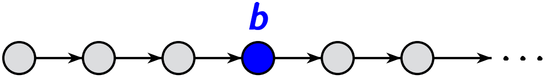
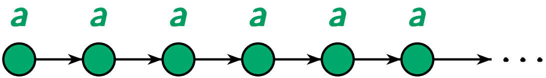


always

$\Box a \overset{\text{def}}{=} \neg\Diamond\neg a$



Semantics of PCTL*

Semantics of PCTL*

Let $\mathcal{M} = (S, P, AP, L)$ be a Markov chain.

Define by structural induction:

- a satisfaction relation $\models$ for states $s \in S$ and **PCTL*** state formulas
- a satisfaction relation $\models$ for infinite paths π in $\mathcal{M}$ and **PCTL*** path formulas

Semantics of PCTL*

$$
\begin{array}{lll}
s \models true & & \\
s \models a & \text{iff} & a \in L(s) \\
s \models \neg\Phi & \text{iff} & s \not\models \Phi \\
s \models \Phi_1 \wedge \Phi_2 & \text{iff} & s \models \Phi_1 \text{ and } s \models \Phi_2 \\
s \models \mathbb{P}_{\mathrm{I}}(\varphi) & \text{iff} & \Pr_s^{\mathcal{M}}(\varphi) \in \mathrm{I}
\end{array}
$$

Semantics of PCTL*

$s \models true$

$s \models a$ iff $a \in L(s)$

$s \models \neg\Phi$ iff $s \not\models \Phi$

$s \models \Phi_1 \wedge \Phi_2$ iff $s \models \Phi_1$ and $s \models \Phi_2$

$s \models \mathbb{P}_I(\varphi)$ iff $\Pr_s^{\mathcal{M}}(\varphi) \in I$

probability measure of the set of paths π with $\pi \models \varphi$

when s is viewed as the unique starting state

Semantics of PCTL* path formulas

let $\pi = s_0\, s_1\, s_2\, s_3 \dots$ be an infinite path in $\mathcal{M}$

Semantics of PCTL* path formulas

let $\pi = s_0\, s_1\, s_2\, s_3 \ldots$ be an infinite path in $\mathcal{M}$

$\pi \models \Phi$	iff	$s_0 \models \Phi$
$\pi \models \neg\varphi$	iff	$\pi \not\models \varphi$
$\pi \models \varphi_1 \wedge \varphi_2$	iff	$\pi \models \varphi_1$ and $\pi \models \varphi_2$
$\pi \models \bigcirc\varphi$	iff	$s_1\, s_2\, s_3 \ldots \models \varphi$
$\pi \models \varphi_1 \,\mathbf{U}\, \varphi_2$	iff	there exists $\ell \geqslant 0$ such that
		$s_\ell\, s_{\ell+1}\, s_{\ell+2} \ldots \models \varphi_2$
		$s_i\, s_{i+1}\, s_{i+2} \ldots \models \varphi_1$ for $0 \leqslant i < \ell$

Examples for PCTL*-specifications

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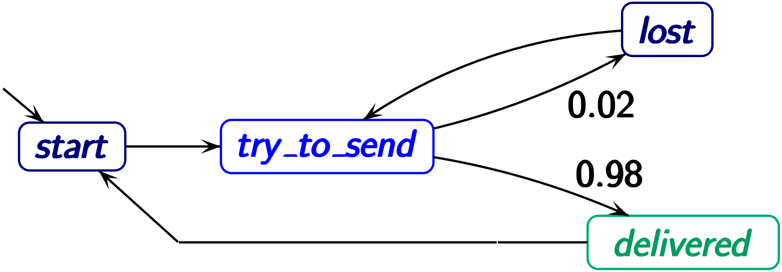
communication protocol:

$$\mathbb{P}_{=1}\big(\,\Box(\,\textit{try\_to\_send}\;\longrightarrow\;\mathbb{P}_{\geqslant 0.9}(\bigcirc \textit{delivered})\,)\,\big)$$

$$\mathbb{P}_{=1}\big(\,\Box(\,\textit{try\_to\_send}\;\longrightarrow\;\neg \textit{start}\;\mathsf{U}\;\textit{delivered}\,)\,\big)$$



Examples for PCTL*-specifications

communication protocol:

$$\mathbb{P}_{=1}\big(\ \Box(\ \mathit{try\_to\_send}\ \longrightarrow\ \mathbb{P}_{\geqslant 0.9}(\bigcirc \mathit{delivered})\ )\ \big)$$

$$\mathbb{P}_{=1}\big(\ \Box(\ \mathit{try\_to\_send}\ \longrightarrow\ \neg \mathit{start}\ \mathsf{U}\ \mathit{delivered}\ )\ \big)$$

leader election protocol for n processes:

$$\mathbb{P}_{=1}\big(\ \Diamond\ \mathit{leader\_elected}\ \big)$$

$$\mathbb{P}_{\geqslant 0.9}\big(\ \bigvee_{i \leqslant n} \bigcirc^{i}\ \mathit{leader\_elected}\ \big)$$

PCTL* model checking for DTMC

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given: Markov chain $\mathcal{M} = (S, P, AP, L, s_0)$
PCTL* state formula Φ

task: check whether $s_0 \models \Phi$

PCTL* model checking for DTMC

given: Markov chain $\mathcal{M} = (S, P, AP, L, s_0)$
PCTL* state formula Φ

task: check whether $s_0 \models \Phi$

main procedure as for CTL*:

recursively compute the satisfaction sets

$$Sat(\Psi) \;=\; \{\, s \in S : s \models \Psi \,\}$$

for all state subformulas Ψ of Φ

Recursive computation of the satisfaction sets

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$$\begin{aligned}
Sat(true) &= S \quad \text{state space of } \mathcal{M} \\
Sat(a) &= \{\, s \in S : a \in L(s) \,\} \\
Sat(\Phi_1 \wedge \Phi_2) &= Sat(\Phi_1) \cap Sat(\Phi_2) \\
Sat(\neg \Phi) &= S \setminus Sat(\Phi)
\end{aligned}$$

Recursive computation of the satisfaction sets

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Sat(\,\mathbb{P}_{\mathrm{I}}(\varphi)\,) &= \{\, s \in S : \Pr_s^{\mathcal{M}}(\varphi) \in \mathrm{I} \,\}
\end{aligned}
$$

Recursive computation of the satisfaction sets

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special case: $\varphi = \Diamond \Phi$

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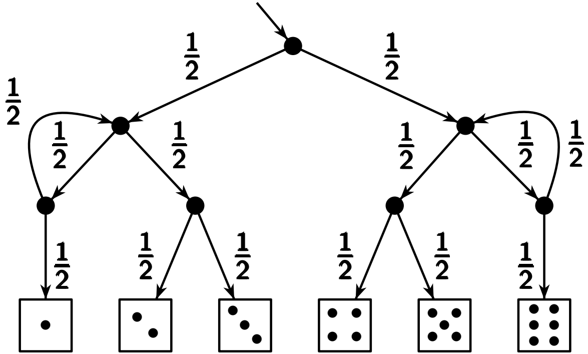
\end{aligned}$$

special case: $\varphi = \Diamond \Phi$

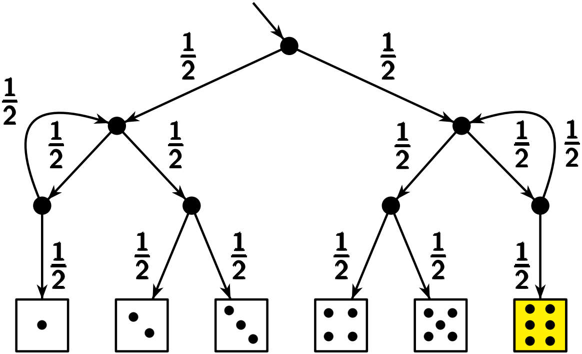
1. compute recursively $Sat(\Phi)$
2. compute $x_s = \Pr_s^{\mathcal{M}}(\Diamond \Phi)$ by solving a linear equation system

Simulating a dice by a coin

[Knuth]



Simulating a dice by a coin [KNUTH]

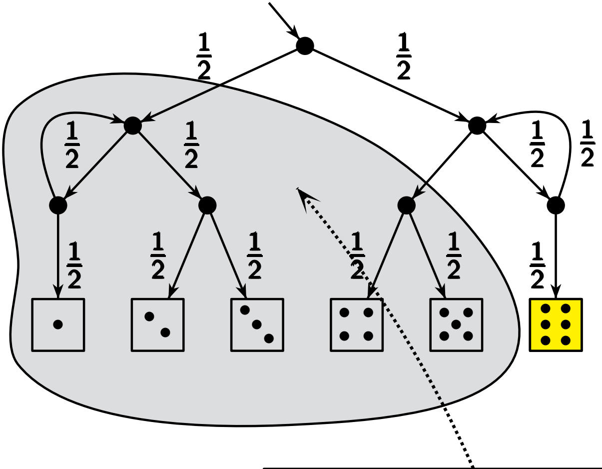


probability for the outcome **six**

$$Pr^{\mathcal{M}}(\Diamond\ \textsf{six}\ ) = \textbf{?}$$

Simulating a dice by a coin

[KNUTH]



outcome **six** unreachable

Simulating a dice by a coin

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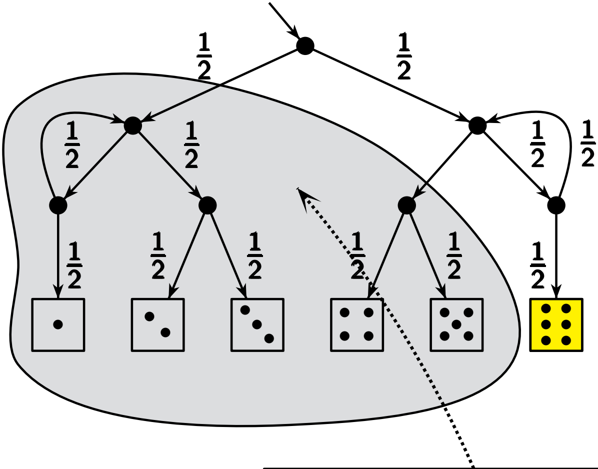
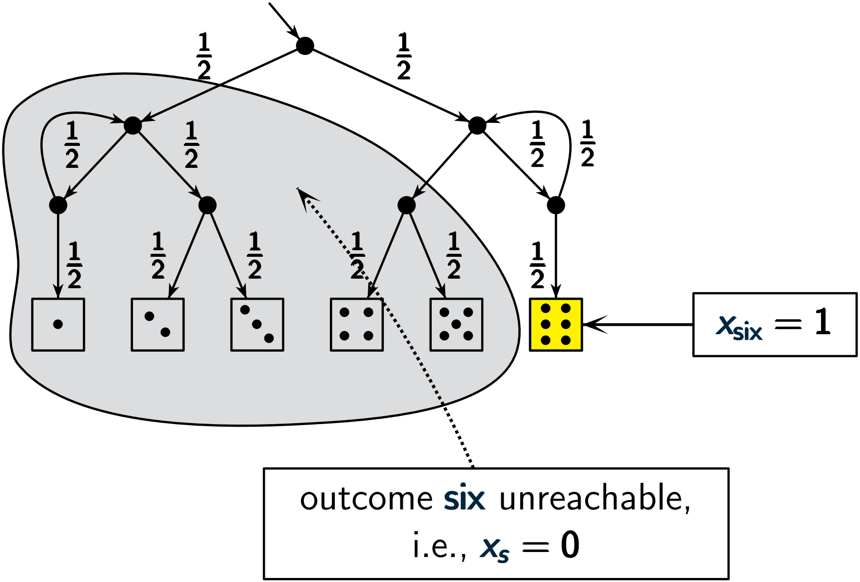


outcome **six** unreachable,
i.e., $x_s = 0$

Simulating a dice by a coin [KNUTH]



Simulating a dice by a coin

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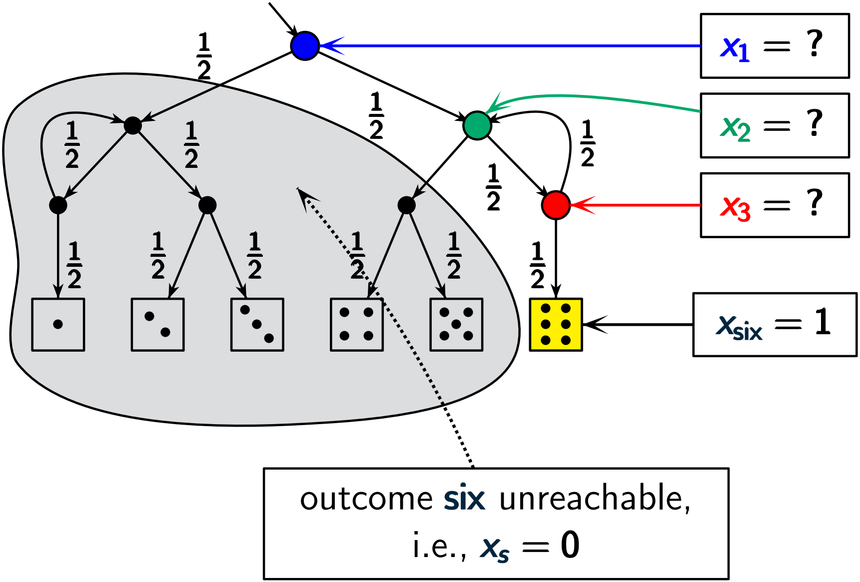

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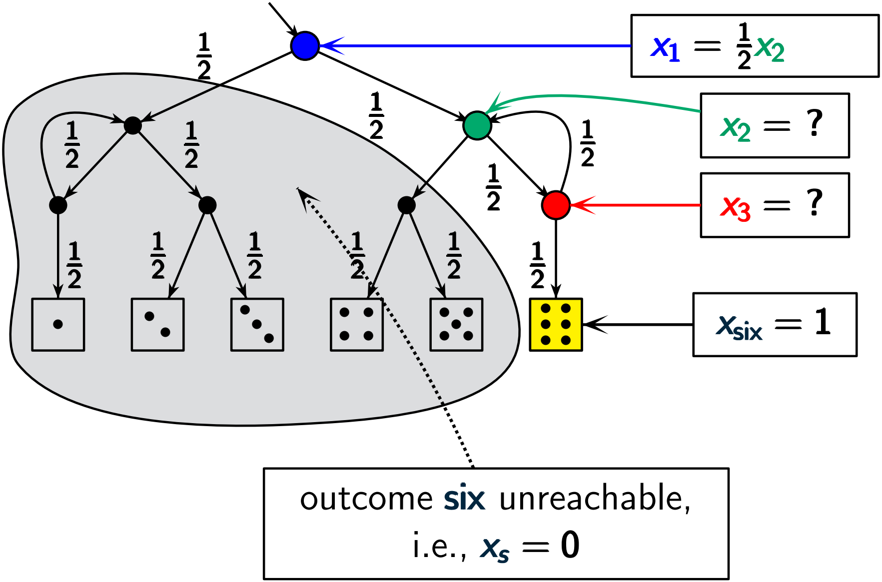

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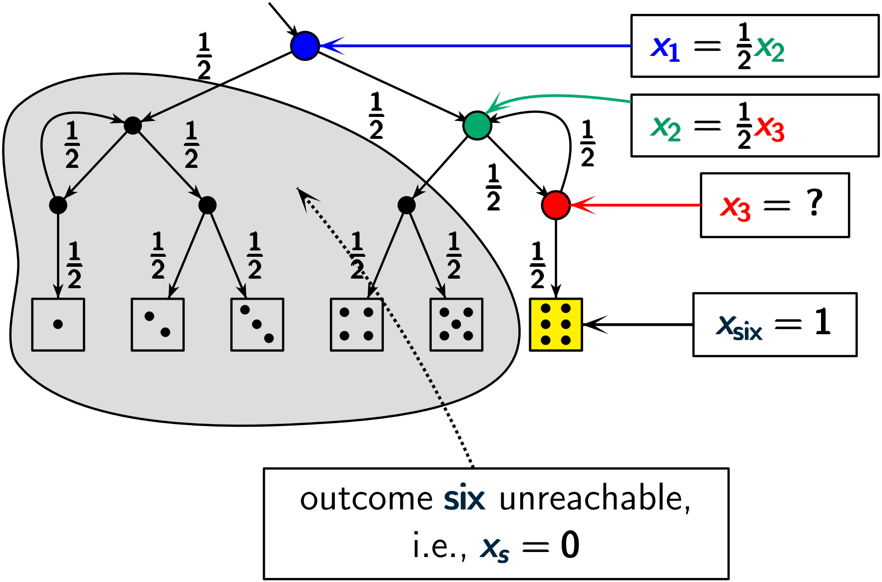

Simulating a dice by a coin

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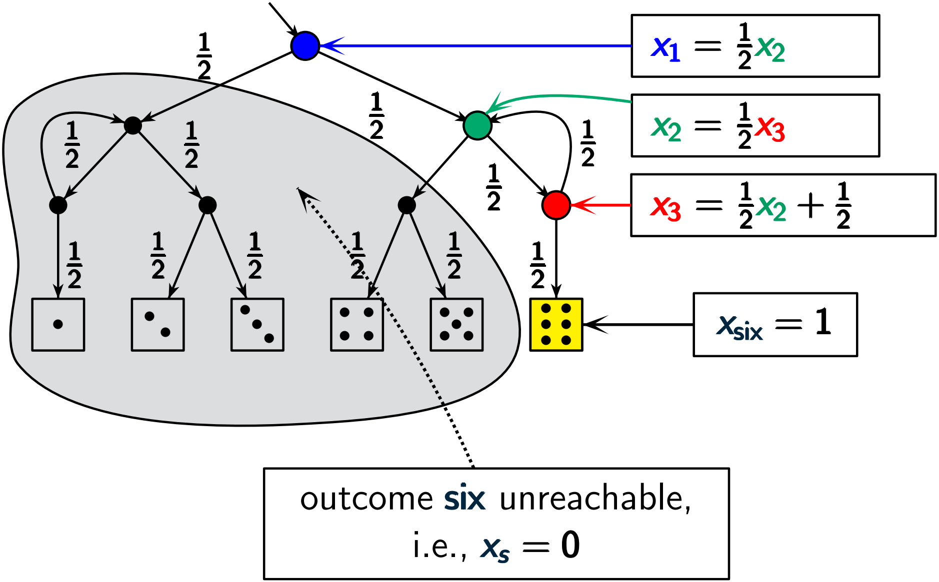
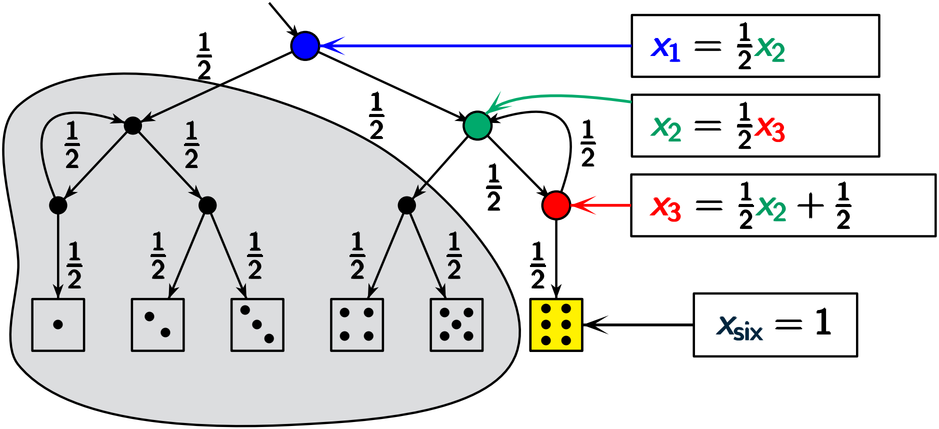


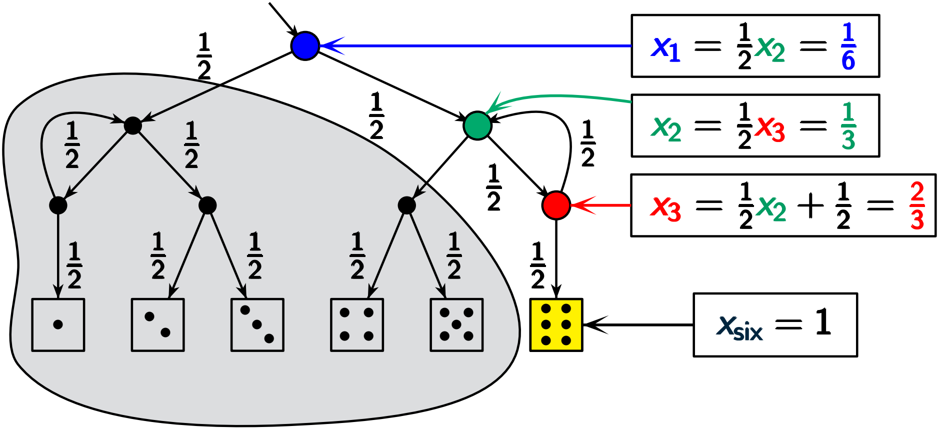
Simulating a dice by a coin

[Knuth]



$$\begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix}$$

Simulating a dice by a coin [KNUTH]



$$\Pr^{\mathcal{M}}(\Diamond \text{ six }) = x_1 = \frac{1}{6}$$

$$\begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix}$$

Computing reachability probabilities

Computing reachability probabilities

given: DTMC $\mathcal{M} = (S, P, \ldots)$ and $T \subseteq S$

task: compute $x_s = \Pr_s^{\mathcal{M}}(\Diamond T)$ for all $s \in S$

$\Diamond T$ "eventually reaching T"

Computing reachability probabilities

given: DTMC $\mathcal{M} = (S, P, \ldots)$ and $T \subseteq S$

task: compute $x_s = \Pr^{\mathcal{M}}_s(\Diamond T)$ for all $s \in S$

1. compute S^0 and S^1

$$S^0 = \{s \in S : x_s = 0\}$$
$$S^1 = \{s \in S : x_s = 1\}$$

2. ...

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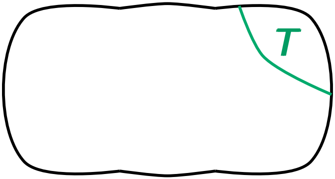

state space S

Computing reachability probabilities

given: DTMC $\mathcal{M} = (S, P, \ldots)$ and $T \subseteq S$

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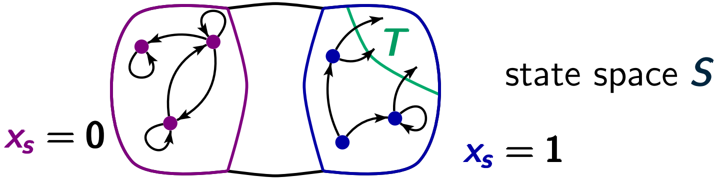

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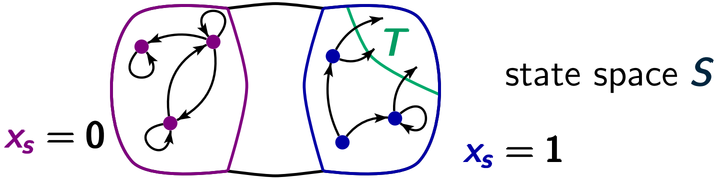

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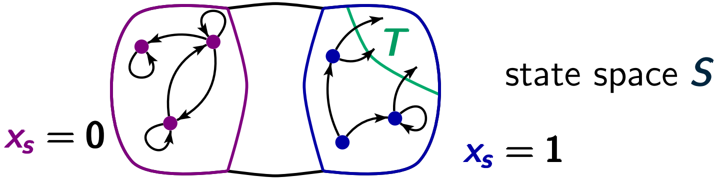

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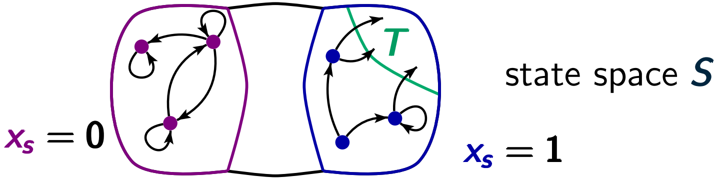

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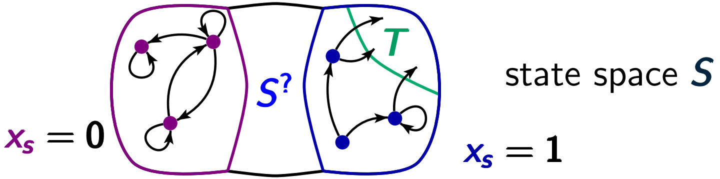
2. compute x_s for $s \in S^? = S \setminus (S^0 \cup S^1)$



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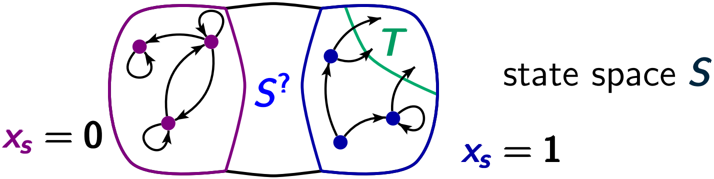
2. compute x_s for $s \in S^? = \{ s : 0 < x_s < 1 \}$



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2. compute x_s for $s \in S^? = \{s : 0 < x_s < 1\}$

↑

by solving a linear equation system

Computing reachability probabilities

task: compute $x_s = \Pr_s^{\mathcal{M}}(\Diamond T)$ for all $s \in S^?$

by solving the equation system:

$$x_s = \sum_{s' \in S^?} P(s, s') \cdot x_{s'} + P(s, S^1)$$

$$P(s, S^1) = \sum_{u \in S^1} P(s, u)$$

Computing reachability probabilities

task: compute $x_s = \Pr^{\mathcal{M}}_s(\Diamond T)$ for all $s \in S^?$

by solving the equation system:

$$x_s = \sum_{s' \in S^?} P(s,s') \cdot x_{s'} + \underbrace{P(s, S^1)}$$

↑

probability for paths of the form

$s\ \underbrace{u_1\, u_2 \ldots u_k}_{u_j \in S^1}\ t$ with $t \in T$

Computing reachability probabilities

task: compute $x_s = \Pr^{\mathcal{M}}_s(\Diamond T)$ for all $s \in S^?$

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↑

probability for paths of the form

$$\underbrace{s\ s_1\ s_2 \ldots s_m}_{s_i \in S^?}\ \underbrace{u_1\ u_2 \ldots u_k}_{u_j \in S^1}\ t \quad \text{with } t \in T \qquad m \geqslant 1$$

Computing reachability probabilities

task: compute $x_s = \Pr_s^{\mathcal{M}}(\Diamond T)$ for all $s \in S^?$

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$$x_s = \sum_{s' \in S^?} P(s,s') \cdot x_{s'} + P(s, S^1)$$

$$x = A \cdot x + b$$

Computing reachability probabilities

task: compute $x_s = \Pr^{\mathcal{M}}_s(\Diamond T)$ for all $s \in S^?$

by solving the equation system:

$$x_s \;=\; \sum_{s' \in S^?} P(s,s') \cdot x_{s'} \;+\; P(s, S^1)$$

$$x = A \cdot x + b$$

matrix $A = \big(P(s,s')\big)_{s,s' \in S^?}$

vectors $x = \big(x_s\big)_{s \in S^?}$

$b = \big(P(s,S^1)\big)_{s \in S^?}$

Computing reachability probabilities

task: compute $x_s = \Pr^{\mathcal{M}}_s(\Diamond T)$ for all $s \in S^?$

by solving the equation system:

$$x_s \;=\; \sum_{s' \in S^?} P(s,s') \cdot x_{s'} \;+\; P(s,S^1)$$

$$x = A \cdot x + b$$

iff

$$(\mathrm{I} - A) \cdot x = b$$

matrix $A = \big(P(s,s')\big)_{s,s' \in S^?}$

vectors $x = \big(x_s\big)_{s \in S^?}$

$b = \big(P(s,S^1)\big)_{s \in S^?}$

identity matrix I

Computing reachability probabilities

task: compute $x_s = \Pr^{\mathcal{M}}_s(\Diamond T)$ for all $s \in S^?$

by solving the equation system:

$$x_s = \sum_{s' \in S^?} P(s,s') \cdot x_{s'} + P(s, S^1)$$

$$x = A \cdot x + b$$
iff
$$(\mathrm{I} - A) \cdot x = b$$

linear equation system with non-singular matrix $\mathrm{I} - A$

Computing reachability probabilities

task: compute $x_s = \Pr_s^{\mathcal{M}}(\Diamond T)$ for all $s \in S^?$

by solving the equation system:

$$x_s = \sum_{s' \in S^?} P(s, s') \cdot x_{s'} + P(s, S^1)$$

$$x = A \cdot x + b$$

iff

$$(\mathrm{I} - A) \cdot x = b$$

linear equation system with non-singular matrix $\mathrm{I} - A$

$\Downarrow$

unique solution

PCTL

sublogic of PCTL* where only path formulas of the form $\bigcirc \Phi$ and $\Phi_1 \, \mathsf{U} \, \Phi_2$ are allowed

PCTL

sublogic of PCTL* where only path formulas of the form $\bigcirc \Phi$ and $\Phi_1 \, \mathsf{U} \, \Phi_2$ are allowed

state formulas:

$$\Phi \;::=\; \mathit{true} \;\big|\; a \;\big|\; \Phi_1 \wedge \Phi_2 \;\big|\; \neg\Phi \;\big|\; \mathbb{P}_{\mathrm{I}}(\varphi)$$

path formulas:

$$\varphi \;::=\; \bigcirc \Phi \;\big|\; \Phi_1 \, \mathsf{U} \, \Phi_2$$

PCTL

sublogic of PCTL* where only path formulas of the form $\bigcirc \Phi$ and $\Phi_1 \, \mathsf{U} \, \Phi_2$ are allowed

state formulas:

$$\Phi \;::=\; \mathit{true} \mid a \mid \Phi_1 \wedge \Phi_2 \mid \neg\Phi \mid \mathbb{P}_I(\varphi)$$

path formulas:

$$\varphi \;::=\; \bigcirc\Phi \mid \Phi_1 \, \mathsf{U} \, \Phi_2 \mid \Diamond\Phi \mid \Box\Phi$$

$$\mathbb{P}_I(\Diamond\Phi) \stackrel{\text{def}}{=} \mathbb{P}_I(\mathit{true} \, \mathsf{U} \, \Phi)$$

PCTL

sublogic of PCTL* where only path formulas of the form $\bigcirc\Phi$ and $\Phi_1 \,\mathsf{U}\, \Phi_2$ are allowed

state formulas:

$$\Phi \quad ::= \quad \mathit{true} \;\big|\; a \;\big|\; \Phi_1 \wedge \Phi_2 \;\big|\; \neg\Phi \;\big|\; \mathbb{P}_I(\varphi)$$

path formulas:

$$\varphi \quad ::= \quad \bigcirc\Phi \;\big|\; \Phi_1 \,\mathsf{U}\, \Phi_2 \;\big|\; \Diamond\Phi \;\big|\; \Box\Phi$$

$$\mathbb{P}_I(\Diamond\Phi) \;\stackrel{\text{def}}{=}\; \mathbb{P}_I(\mathit{true}\,\mathsf{U}\,\Phi)$$

e.g., $\mathbb{P}_{<0.4}(\Box\Phi) \;\stackrel{\text{def}}{=}\; \mathbb{P}_{>0.6}(\Diamond\neg\Phi)$

note: $\Pr^{\mathcal{M}}(s, \Box\Phi) \;=\; 1 - \Pr^{\mathcal{M}}(s, \Diamond\neg\Phi)$

PCTL model checking

given: Markov chain $\mathcal{M} = (S, P, AP, L, s_0)$
PCTL state formula Φ

task: check whether $s_0 \models \Phi$

PCTL model checking

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PCTL state formula Φ

task: check whether $s_0 \models \Phi$

recursive computation of $Sat(\Psi) = \{ s \in S : s \models \Psi \}$
for all state subformulas Ψ of Φ

in bottom-up manner, i.e.,
inner subformulas first

PCTL model checking

given: Markov chain $\mathcal{M} = (S, P, AP, L, s_0)$
PCTL state formula Φ

task: check whether $s_0 \models \Phi$

recursive computation of $Sat(\Psi) = \{ s \in S : s \models \Psi \}$
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- treatment of propositional logic fragment:

PCTL model checking

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recursive computation of $Sat(\Psi) = \{s \in S : s \models \Psi\}$
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- treatment of propositional logic fragment: $\surd$

$$
\begin{aligned}
Sat(\mathit{true}) &= S \\
Sat(a) &= \{s \in S : a \in L(s)\} \\
Sat(\neg\Psi) &= S \setminus Sat(\Psi) \\
Sat(\Psi_1 \wedge \Psi_2) &= Sat(\Psi_1) \cap Sat(\Psi_2)
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- treatment of propositional logic fragment: $\surd$
- treatment of the probability operator $\mathbb{P}_I(\varphi)$

compute $\Pr_s^{\mathcal{M}}(\varphi)$ for all states s and return

$$Sat(\mathbb{P}_I(\varphi)) = \{ s \in S : \Pr_s^{\mathcal{M}}(\varphi) \in I \}$$

PCTL model checking

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- treatment of propositional logic fragment: $\surd$
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graph algorithms + matrix/vector operations

↑

next:	matrix/vector multiplication
until:	linear equation system

PCTL model checking

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PCTL state formula Φ

task: check whether $s_0 \models \Phi$

recursive computation of $Sat(\Psi) = \{s \in S : s \models \Psi\}$
for all state subformulas Ψ of Φ

- treatment of propositional logic fragment: $\surd$
- treatment of the probability operator $\mathbb{P}_{\mathrm{I}}(\varphi)$

  graph algorithms + matrix/vector operations

time complexity: $\mathcal{O}\big(\,\mathbf{poly}(\mathcal{M}) \cdot |\Phi|\,\big)$

PCTL* model checking

given: Markov chain $\mathcal{M} = (S, P, AP, L, s_0)$
PCTL* state formula Φ

task: check whether $s_0 \models \Phi$

recursive computation of $Sat(\Psi) = \{ s \in S : s \models \Psi \}$
for all state subformulas Ψ of Φ

- treatment of propositional logic fragment: $\surd$
- treatment of the probability operator $\mathbb{P}_{\mathrm{I}}(\varphi)$

PCTL* path formula φ $\rightsquigarrow$ LTL formula φ'
$\uparrow$
path formula without probability operator

PCTL* model checking

given: Markov chain $\mathcal{M} = (S, P, AP, L, s_0)$
PCTL* state formula Φ

task: check whether $s_0 \models \Phi$

recursive computation of $Sat(\Psi) = \{s \in S : s \models \Psi\}$
for all state subformulas Ψ of Φ

- treatment of propositional logic fragment: $\surd$
- treatment of the probability operator $\mathbb{P}_I(\varphi)$

PCTL* path formula $\varphi \rightsquigarrow$ LTL formula φ'

... automata-based approach for φ' ...

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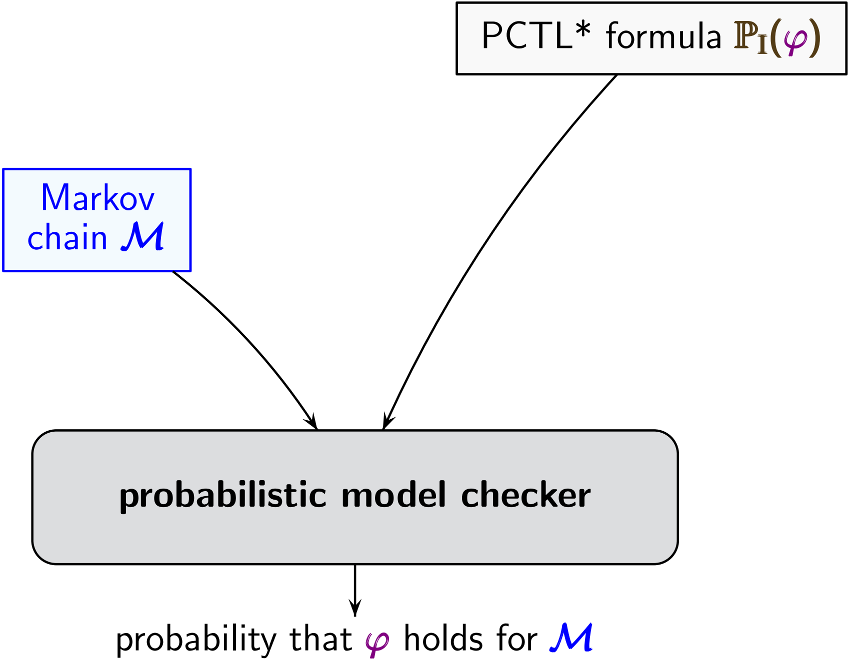
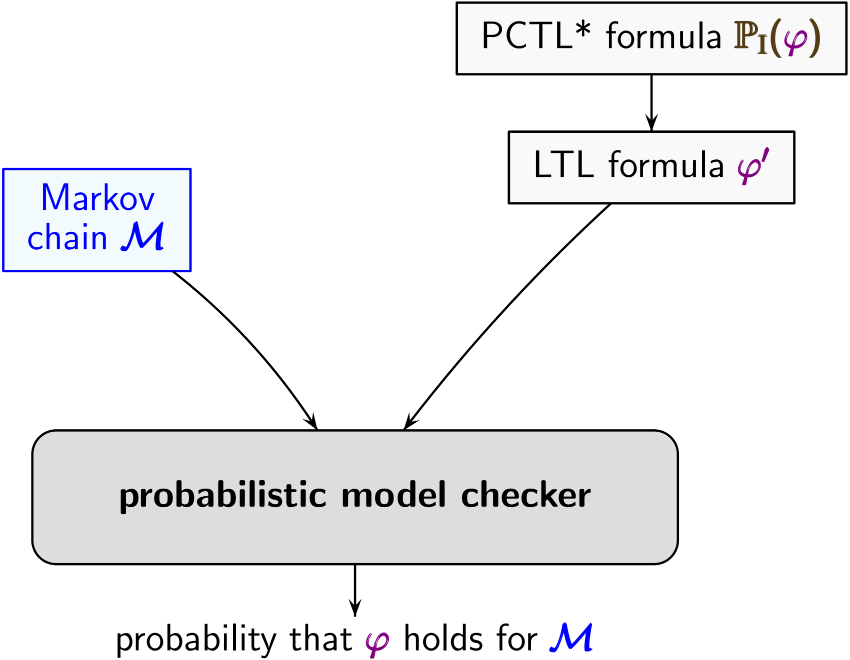
$$\Downarrow$$

$$\Diamond\big(\, a \,\mathsf{U}\, d \;\wedge\; \Box\, e\,\big)$$



PCTL* formula $\mathbb{P}_{\mathbf{I}}(\varphi)$
Markov chain $\mathcal{M}$
probabilistic model checker
probability that φ holds for $\mathcal{M}$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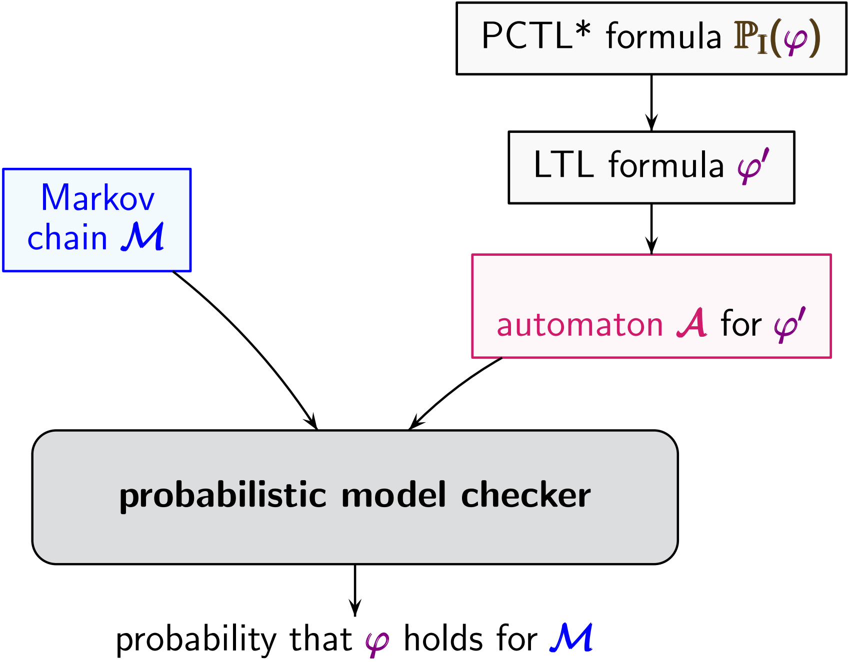

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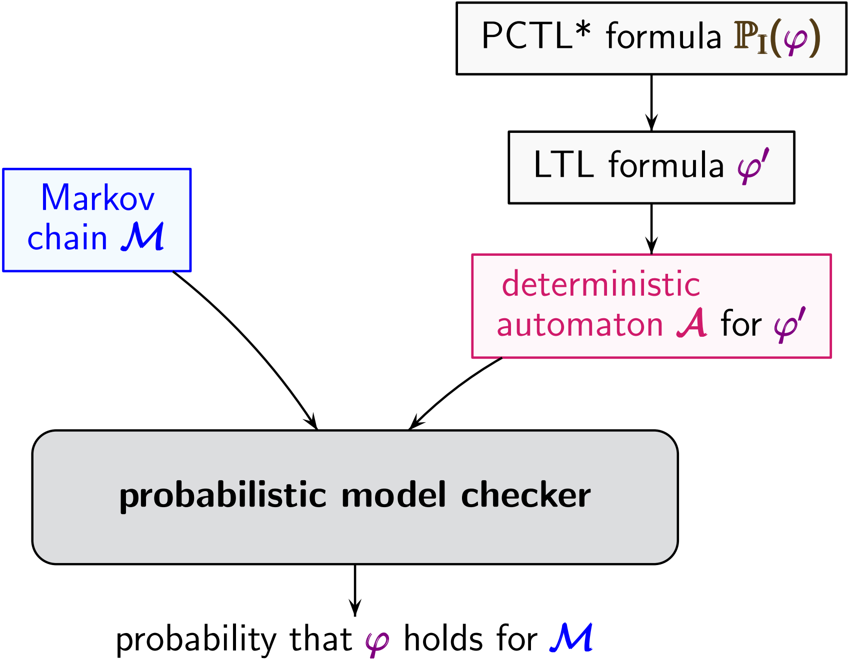
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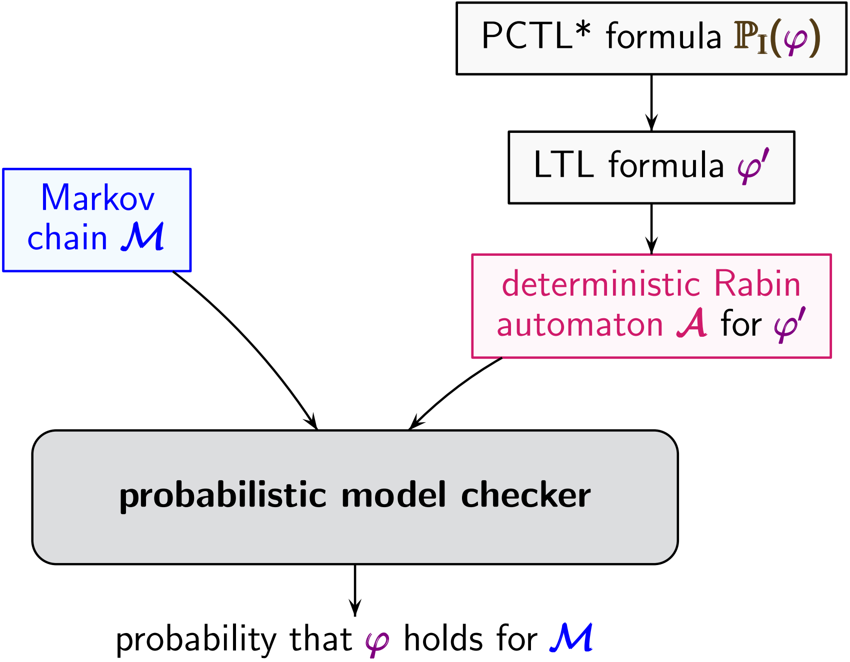
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Deterministic Rabin automata (DRA)

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A DRA is a tuple $\mathcal{A} = (Q, \Sigma, \delta, q_0, Acc)$ where

- Q finite state space
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- acceptance condition Acc is a set of pairs (L, U) with $L, U \subseteq Q$, say $Acc = \{(L_1, U_1), ..., (L_k, U_k)\}$

semantics of the acceptance condition:

$$\bigvee_{1 \leqslant i \leqslant k} (\Diamond\Box\neg L_i \ \wedge \ \Box\Diamond U_i)$$

Accepted language of a DRA

A DRA is a tuple $\mathcal{A} = (Q, \Sigma, \delta, q_0, Acc)$ where

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accepted language:

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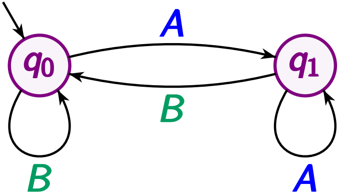
Let $\rho = q_0\, q_1\, q_2 \ldots$ be the run for some infinite word σ.

ρ fulfills Acc iff

$$\exists i \in \{1, \ldots, k\}.\ \inf(\rho) \cap L_i = \varnothing \wedge \inf(\rho) \cap U_i \neq \varnothing$$

where $\inf(\rho) \;=\; \{q \in Q : \overset{\infty}{\exists}\, \ell \in \mathbb{N}.\ q = q_\ell\}$

Example: DRA



$$Acc = \{(\{q_0\}, \{q_1\})\}$$

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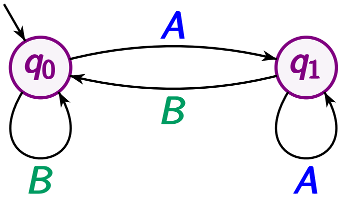

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$\Diamond\Box$ "eventually forever"

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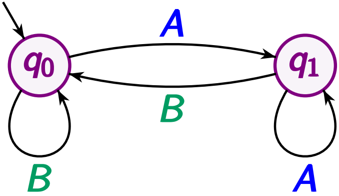

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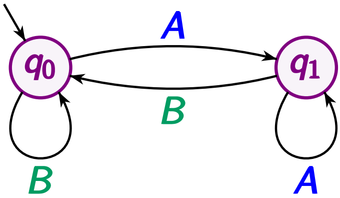

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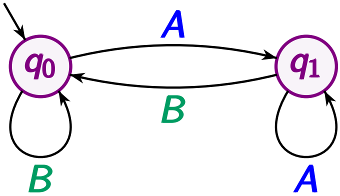

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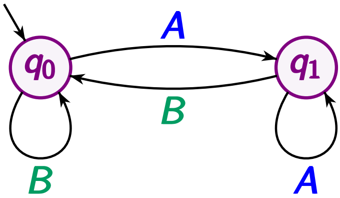

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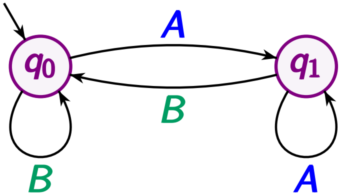

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For each LTL formula φ over AP there exists a DRA $\mathcal{A}$ with the alphabet $\Sigma = 2^{AP}$ s.t.

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LTL formula

↓

NBA

↓ determinization [SAFRA'88]

DRA

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LTL formula

↓ compositional [ESPARZA/KRETINSKY'14]

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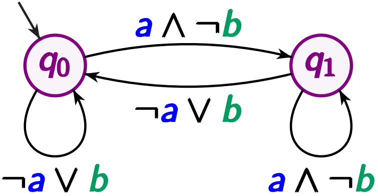

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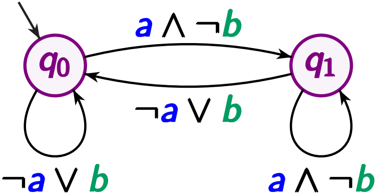

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LTL formula $\Diamond\Box(a \wedge \neg b)$

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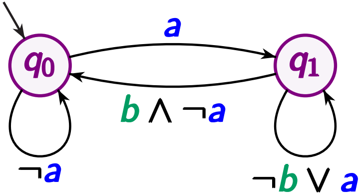

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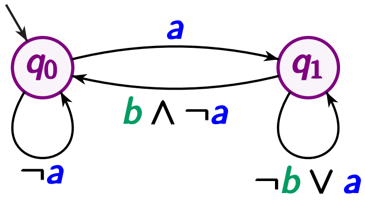

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LTL formula
$\Box(a \rightarrow \Diamond(b \wedge \neg a)) \;\wedge\; \Diamond\Box\neg a$

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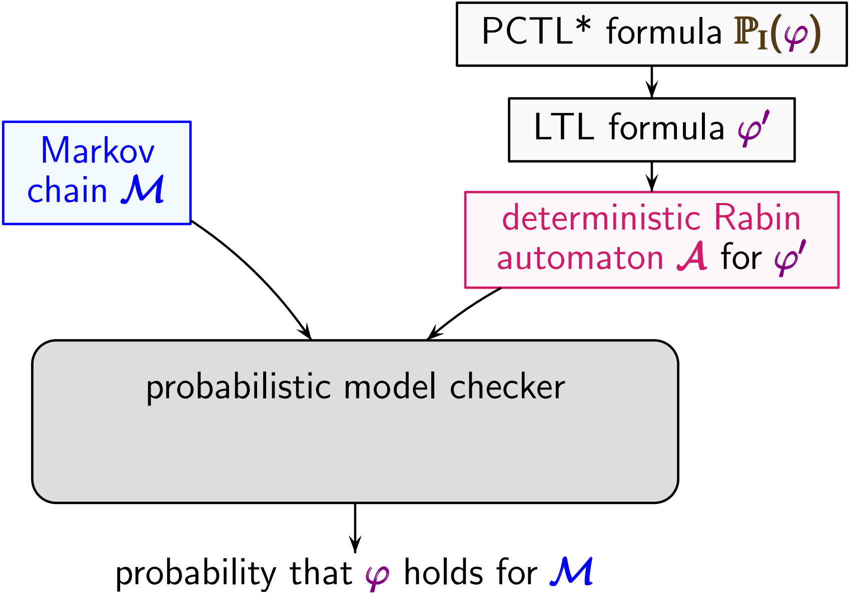

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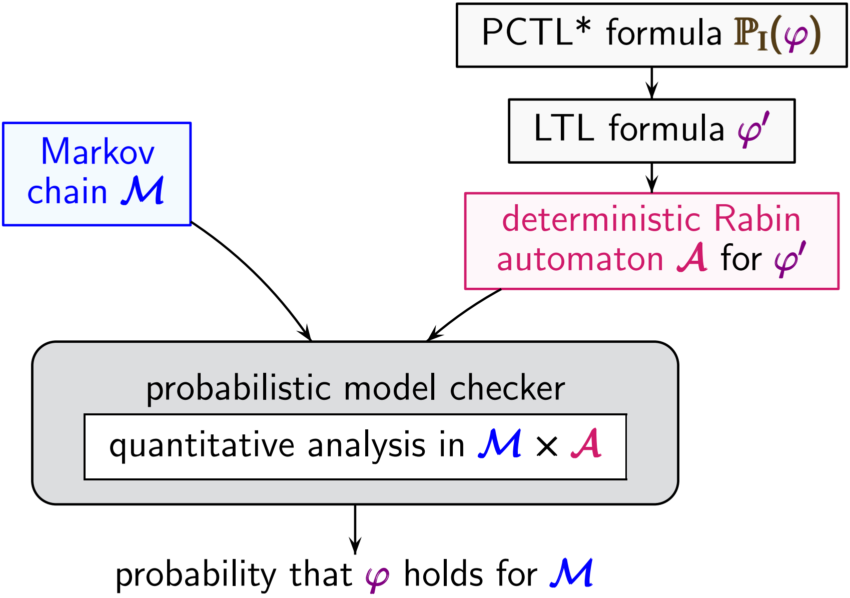

Product of a Markov chain and a DRA

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given: Markov chain $\mathcal{M} = (S, P, AP, L)$

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$$\Pr_s^{\mathcal{M}}(\mathcal{A}) \;=\; \Pr^{\mathcal{M}}\{\pi \in Paths(s) : trace(\pi) \in \mathcal{L}_\omega(\mathcal{A})\}$$

can be derived by a probabilistic reachability analysis in the product-chain $\mathcal{M} \times \mathcal{A}$

$$trace(s_0\, s_1\, s_2 \ldots) \;=\; L(s_0)\; L(s_1)\; L(s_2) \ldots \in \left(2^{AP}\right)^{\omega}$$

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path π in $\mathcal{M}$

$$s_0 \downarrow s_1 \downarrow s_2 \downarrow \vdots$$

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run for $trace(\pi)$ in $\mathcal{A}$

$$q_0 \xrightarrow{L(s_0)} q_1 \xrightarrow{L(s_1)} q_2 \xrightarrow{L(s_2)} \vdots$$

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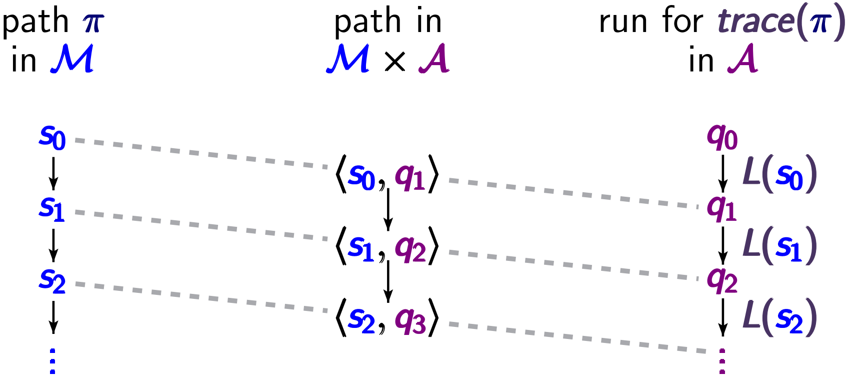

Fundamental property of the product

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↑

successor state in $\mathcal{A}$ of the initial DRA-state q_0 for the input symbol $L(s) \in 2^{AP}$

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probability measure of all paths $\pi \in Paths^{\mathcal{M}}(s)$ such that $trace(\pi) \in \mathcal{L}_\omega(\mathcal{A})$

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$$\Pr_s^{\mathcal{M}}(\mathcal{A})$$

$$= \Pr_{\langle s, q_s \rangle}^{\mathcal{M} \times \mathcal{A}}\Big(\bigvee_{1 \leqslant i \leqslant k} (\Diamond\Box\neg L_i \wedge \Box\Diamond U_i) \Big)$$

probability measure of all paths π in the product s.t. $\pi\big|_{\mathcal{A}}$ satisfies the acceptance condition of $\mathcal{A}$

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$$\begin{aligned} & \Pr_s^{\mathcal{M}}(\mathcal{A}) \\ = \; & \Pr_{\langle s, q_s \rangle}^{\mathcal{M} \times \mathcal{A}}\Big(\bigvee_{1 \leqslant i \leqslant k} (\Diamond\Box\neg L_i \wedge \Box\Diamond U_i) \Big) \\ = \; & \Pr_{\langle s, q_s \rangle}^{\mathcal{M} \times \mathcal{A}}(\Diamond \, accBSCC) \end{aligned}$$

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$$\Pr_s^{\mathcal{M}}(\mathcal{A})$$

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↑

union of accepting BSCCs in $\mathcal{M} \times \mathcal{A}$ i.e., BSCC C s.t.

$$\exists i \in \{1, \ldots, k\}. \;\; C \cap L_i = \varnothing \;\; \wedge \;\; C \cap U_i \neq \varnothing$$

Summary: PCTL* model checking

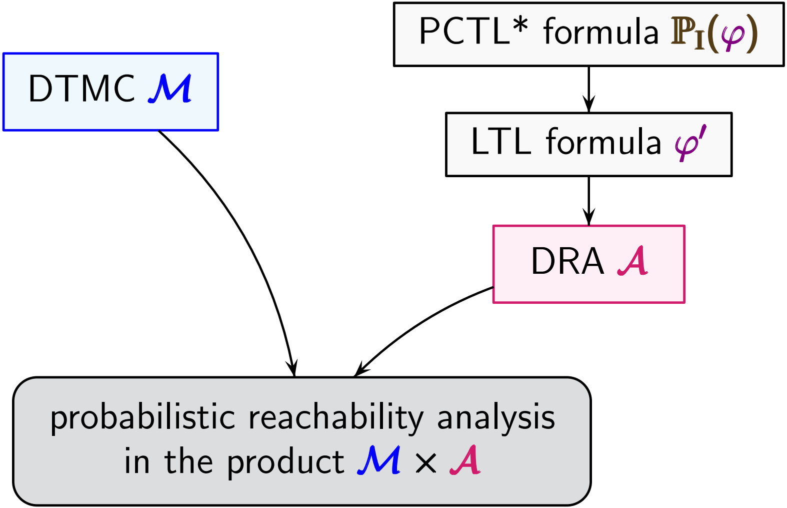
given: Markov chain $\mathcal{M} = (S, P, AP, L, s_0)$
PCTL* state formula Φ

task: check whether $\mathcal{M} \models \Phi$

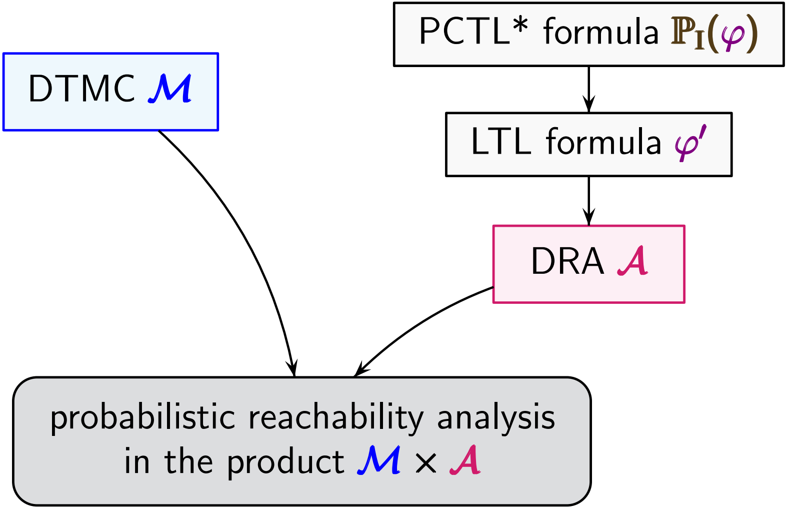
method: bottom-up treatment of state subformulas Ψ to compute

$$Sat(\Psi) = \{ s \in S : s \models \Psi \}$$

- propositional logic fragment: obvious
- probability operator $\mathbb{P}_{\mathrm{I}}(\varphi)$ via
  - construction of a DRA $\mathcal{A}$ for φ
  - probabilistic reachability analysis in $\mathcal{M} \times \mathcal{A}$



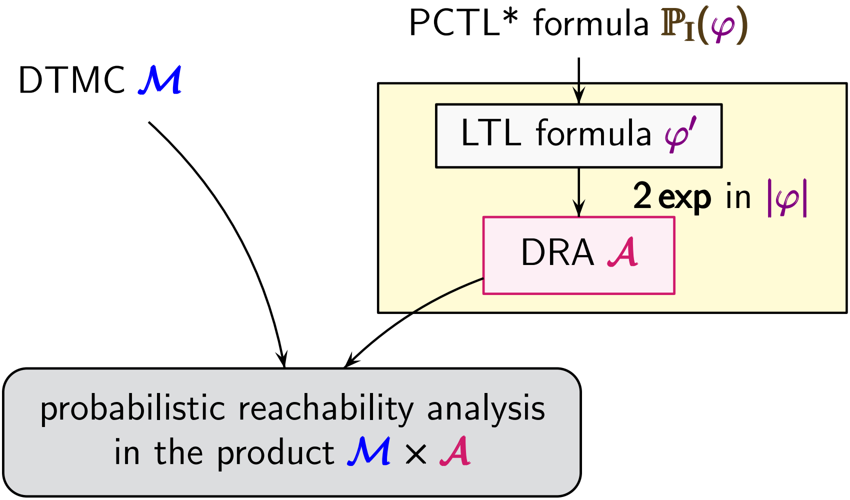
1. graph analysis to compute the accepting BSCCs of the product
2. linear equation system for the probabilities to reach an accepting BSCC



1. graph analysis to compute the accepting BSCCs of the product
2. linear equation system

time complexity:

polynomial in the sizes of $\mathcal{M}$ and $\mathcal{A}$



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Exponential-time algorithms for DTMC and LTL

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given: Markov chain $\mathcal{M}$, LTL formula φ

task: compute $\Pr^{\mathcal{M}}(\varphi)$

single exponential-time algorithms:

- iterative, automata-less approach

  [Courcoubetis/Yannakakis'88]

- using weak alternating automata

  [Bustan/Rubin/Vardi'04]

- using separated Büchi automata

  [Couvreur/Saheb/Sutre'03]

- using unambiguous Büchi automata

  [Baier/Kiefer/Klein/Klüppelholz/Müller/Worrell'16]

Tutorial: Probabilistic Model Checking

Discrete-time Markov chains (DTMC)

- basic definitions
- probabilistic computation tree logic PCTL/PCTL*
- rewards, cost-utility ratios, weights
- conditional probabilities

Markov decision processes (MDP)

- basic definitions
- PCTL/PCTL* model checking
- fairness
- conditional probabilities
- rewards, quantiles
- mean-payoff
- expected accumulated weights

Markov reward model (MRM)

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Markov chain $\mathcal{M} = (S, P, AP, L, rew)$ with a reward function for the states:

$$rew : S \to \mathbb{N}$$

idea: reward $rew(s)$ will be earned when leaving s

analogously: rewards for edges $rew : S \times S \to \mathbb{N}$

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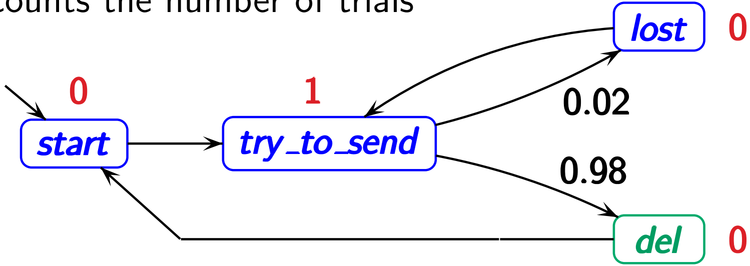
formalization by accumulated rewards of finite paths

$$rew(s_0\, s_1 \ldots s_n) \quad = \quad \sum_{0 \leqslant i < n} rew(s_i)$$

analogously: rewards for edges $rew : S \times S \to \mathbb{N}$

Example: Markov reward model

communication protocol with reward function that counts the number of trials



accumulated reward of finite paths, e.g.,

$$rew(start\ try\ lost\ try\ del) = 2$$

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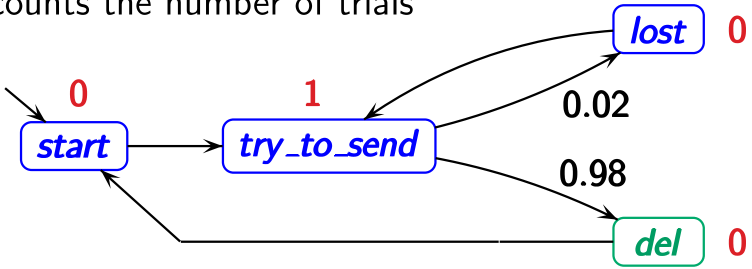

measures of interest, e.g.,

$\Pr^{\mathcal{M}}(\lozenge^{\leqslant 3} \mathit{del})$ probability to deliver a message within at most three trials

reachability with reward bound $\leqslant 3$

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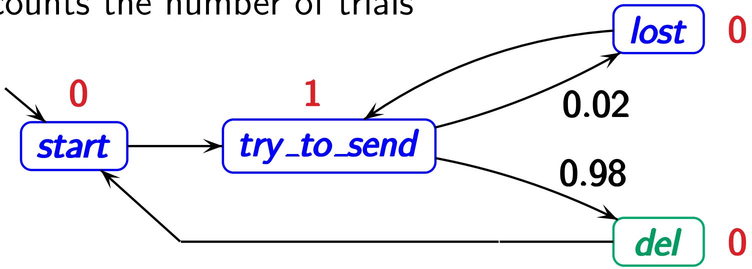

measures of interest, e.g.,

$\Pr^{\mathcal{M}}(\Diamond^{\leqslant 3} del)$ probability to deliver a message within at most three trials

$\mathbb{E}(\oplus\, del)$ expected number of trials until delivered

Reward-based extension of PCTL

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probability operator for reward-bounded path formulas:

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$s \models \mathbb{E}_{\leqslant r}(\lozenge \Phi)$ iff $\begin{cases} \text{expected accumulated reward on} \\ \text{paths from } s \text{ to a } \Phi\text{-state is} \leqslant r \end{cases}$

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example: communication protocol

$\mathbb{P}_{\geqslant 0.9}(\Diamond^{\leqslant 3} del)$ probability for delivering the message within at most three trials is at least 0.9

$\mathbb{E}_{\leqslant 5}(\Diamond\!\!\!\!+ del)$ average number of trials is less or equal 5

Model checking reward-based properties

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treatment of $\mathbb{P}_I(\Phi_1 \, \mathsf{U}^{\leqslant r} \, \Phi_2)$ where $r \in \mathbb{N}$

compute $\Pr_s^{\mathcal{M}}(\Phi_1 \, \mathsf{U}^{\leqslant i} \, \Phi_2)$ iteratively
for increasing reward bound $i = 0, 1, 2, \ldots, r$

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$$x_{s,i} \;=\; \sum_{s' \in S} P(s, s') \cdot x_{s', i - \mathit{rew}(s)}$$

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if $s \models \Phi_2$ then: $x_{s,i} = 1$

in all other cases: $x_{s,i} = 0$

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compute the expected accumulated rewards
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$$
\begin{aligned}
x_s &= rew(s) + \sum_{s' \in S} P(s, s') \cdot x_{s'} && \text{if } s \not\models \Phi \\
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also applicable for rational-valued weight fct.

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time complexity:

expected rewards: polynomial in $\mathit{size}(\mathcal{M})$

reward-bounded until: polynomial in $\mathit{size}(\mathcal{M})$ and r

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unit rewards: polynomial in $size(\mathcal{M})$ and $\log r$
repeated squaring

general case: polynomial in $size(\mathcal{M})$ and r
"pseudo-polynomial"

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NP-hard [Laroussinie/Sproston'05]

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NP-hardness [Laroussinie/Sproston'05]

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The threshold problem for Markov chains is NP-hard:

given: Markov chain $\mathcal{M} = (S, P, s_{init}, rew)$,
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Polynomial reduction from counting variant of SUBSUM:

given: $x_1, \ldots, x_n, y, k \in \mathbb{N}$

task: check whether there are at least k subsets N
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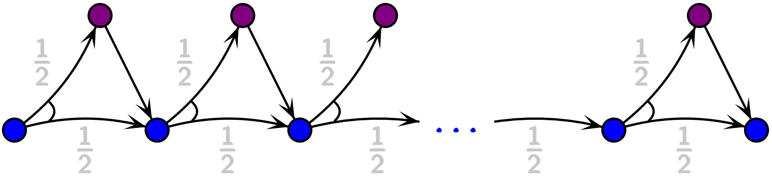
Markov chain: $2n+1$ states



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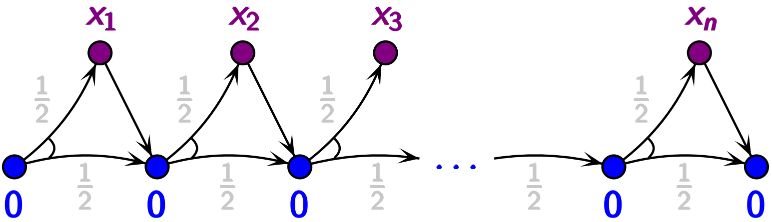

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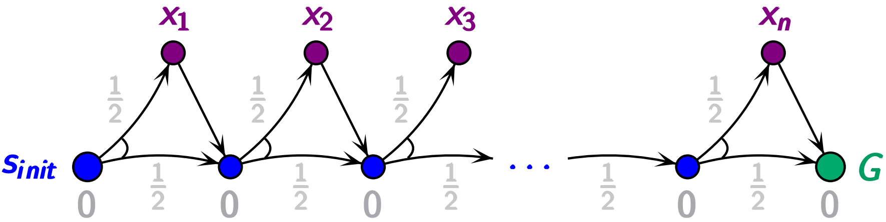

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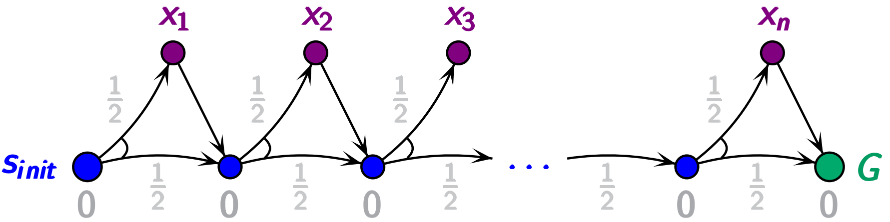
$\mathrm{Pr}_{s_{init}}(\Diamond^{\leqslant y} G) \geqslant \frac{k}{2^n}$ iff there are at least k subsets ....



Mean-payoff (a.k.a. long-rung average)

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given: a weighted graph without trap states

mean-payoff functions $\overline{\mathrm{MP}}, \underline{\mathrm{MP}} : \mathit{InfPaths} \to \mathbb{R}$:

$$\overline{\mathrm{MP}}(s_0\, s_1\, s_2 \ldots) = \limsup_{n\to\infty} \frac{1}{n+1} \cdot \sum_{i=0}^{n} wgt(s_i)$$

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if $\mathit{wgt}(s) = +1$, $\mathit{wgt}(t) = -1$ then there exists $n_1, n_2, \ldots$ and $k_1, k_2, \ldots \in \mathbb{N}$ s.t. for $\pi = s^{n_1}\, t^{k_1}\, s^{n_2}\, t^{k_2} \ldots$:

$$\underline{\mathrm{MP}}(\pi) \;<\; 0 \;<\; \overline{\mathrm{MP}}(\pi)$$

Expected mean-payoff in finite MC

fundamental results:

in finite MC: $\mathbb{E}_s(\underline{\mathrm{MP}}) = \mathbb{E}_s(\overline{\mathrm{MP}})$

notation: $\mathbb{E}_s(\mathrm{MP})$ rather than $\mathbb{E}_s(\underline{\mathrm{MP}})$ resp. $\mathbb{E}_s(\overline{\mathrm{MP}})$

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BSCC: bottom strongly connected component

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Almost all paths eventually enter a BSCC and visit all its states infinitely often ...

... with the same long-run frequencies ...

BSCC: bottom strongly connected component

Long-run frequencies in finite MC

steady-state probabilities in BSCC B of a finite MC:

$$\theta^B(s) \;=\; \lim_{n\to\infty} \frac{1}{n} \cdot \sum_{i=1}^{n} \Pr_t(\bigcirc^i s) \quad \text{for each } t \in B$$

$\bigcirc^i s \;\triangleq\;$ "after i steps in state s"

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computable by a linear equation system:

$$\theta^B(s) \;=\; \sum_{t\in B} \theta^B(t) \cdot P(t,s)$$

"balance equations"

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unique solution of the linear equation system

$$x = x \cdot P|_B$$

$$\sum_{s\in B} x_s = 1$$

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for almost all paths $\pi = s_0\, s_1\, s_2 \ldots$ with $\pi \models \lozenge B$:

$$\theta^B(s) \;=\; \underbrace{\lim_{n\to\infty} \frac{1}{n+1} \cdot \mathit{freq}(s, s_0\, s_1 \ldots s_n)}_{\text{long-run frequency of state } s \text{ in path } \pi}$$

... limit exists for almost all paths ...

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if $\pi \models \Diamond B$ where B is a BSCC then almost surely

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$$\mathrm{MP}(\pi) \;=\; \sum_{s\in B} \theta^B(s) \cdot \mathit{wgt}(s) \;\stackrel{\text{def}}{=}\; \mathrm{MP}(B)$$

expected mean-payoff: $\displaystyle\sum_{B} \Pr_{s_0}(\Diamond B) \cdot \mathrm{MP}(B)$

Long-run ratios in finite MC

MC with two reward functions $cost, util : S \to \mathbb{N}$

Examples:

- energy-utility ratio
- number of SLA violations per day
- recovery time per failure

Long-run ratios in finite MC

MC with two reward functions $cost, util : S \to \mathbb{N}$

long-run cost-utility ratio $lrrat : InfPaths \to \mathbb{R}$

$$lrrat(s_0\, s_1\, s_2 \ldots) \;=\; \lim_{n\to\infty} \frac{cost(s_0\, s_1 \ldots s_n)}{util(s_0\, s_1 \ldots s_n)}$$

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does the limit exist for almost all paths ?

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- number of SLA violations per day
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MC with two reward functions $\mathit{cost}, \mathit{util} : S \to \mathbb{N}$

long-run cost-utility ratio $\mathit{lrrat} : \mathit{InfPaths} \to \mathbb{R}$

$$\mathit{lrrat}(s_0\, s_1\, s_2 \ldots) = \lim_{n\to\infty} \frac{\mathit{cost}(s_0\, s_1 \ldots s_n)}{\mathit{util}(s_0\, s_1 \ldots s_n)}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n+1} \cdot \sum_{i=0}^{n} \mathit{cost}(s_i)}{\frac{1}{n+1} \cdot \sum_{i=0}^{n} \mathit{util}(s_i)}$$

Long-run ratios in finite MC

MC with two reward functions $cost, util : S \to \mathbb{N}$

long-run cost-utility ratio $lrrat : InfPaths \to \mathbb{R}$

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$$= \lim_{n\to\infty} \frac{\frac{1}{n+1} \cdot \sum_{i=0}^{n} cost(s_i)}{\frac{1}{n+1} \cdot \sum_{i=0}^{n} util(s_i)}$$

$$= \frac{\mathrm{MP}[cost](s_0\, s_1\, s_2 \ldots)}{\mathrm{MP}[util](s_0\, s_1\, s_2 \ldots)}$$

Long-run ratios in finite MC

MC with two reward functions $cost, util : S \to \mathbb{N}$

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$$= \lim_{n\to\infty} \frac{\frac{1}{n+1} \cdot \sum_{i=0}^{n} cost(s_i)}{\frac{1}{n+1} \cdot \sum_{i=0}^{n} util(s_i)}$$

$$= \frac{\mathrm{MP}[cost](s_0\, s_1\, s_2 \ldots)}{\mathrm{MP}[util](s_0\, s_1\, s_2 \ldots)}$$

in particular:
limit exists for
almost all paths

Long-run ratios in finite MC

MC with two reward functions $cost, util : S \to \mathbb{N}$

long-run cost-utility ratio $lrrat : InfPaths \to \mathbb{R}$

$$lrrat(s_0\, s_1\, s_2 \ldots) \; = \; \lim_{n \to \infty} \frac{cost(s_0\, s_1 \ldots s_n)}{util(s_0\, s_1 \ldots s_n)}$$

if $\pi \models \Diamond B$ where B is a BSCC then almost surely

$$lrrat(\pi) \; = \; \frac{\mathrm{MP}[cost](B)}{\mathrm{MP}[util](B)}$$

$$MP[wgt](B) = \sum_{s \in B} \theta^B(s) \cdot wgt(s)$$ mean-payoff for weight function

Long-run ratios in finite MC

MC with two reward functions $\mathit{cost}, \mathit{util} : S \to \mathbb{N}$

long-run cost-utility ratio $\mathit{lrrat} : \mathit{InfPaths} \to \mathbb{R}$

$$\mathit{lrrat}(s_0\, s_1\, s_2 \ldots) \;=\; \lim_{n\to\infty} \frac{\mathit{cost}(s_0\, s_1 \ldots s_n)}{\mathit{util}(s_0\, s_1 \ldots s_n)}$$

if $\pi \models \Diamond B$ where B is a BSCC then almost surely

$$\mathit{lrrat}(\pi) \;=\; \underbrace{\frac{\mathrm{MP}[\mathit{cost}](B)}{\mathrm{MP}[\mathit{util}](B)}}_{\text{only depends on } B} \;\stackrel{\text{def}}{=}\; \mathit{lrrat}(B)$$

Long-run ratios in finite MC

MC with two reward functions $cost, util : S \to \mathbb{N}$

long-run cost-utility ratio $lrrat : InfPaths \to \mathbb{R}$

$$lrrat(s_0\, s_1\, s_2 \ldots) \;=\; \lim_{n\to\infty} \frac{cost(s_0\, s_1 \ldots s_n)}{util(s_0\, s_1 \ldots s_n)}$$

if $\pi \models \Diamond B$ where B is a BSCC then almost surely

$$lrrat(\pi) \;=\; \frac{\mathrm{MP}[cost](B)}{\mathrm{MP}[util](B)} \;\stackrel{\text{def}}{=}\; lrrat(B)$$

expected long-run ratio: $\displaystyle\sum_{B} \Pr^{\mathcal{M}}(\Diamond B) \cdot lrrat(B)$

Best threshold for long-run ratios

given: MC with reward functions $\mathit{cost}, \mathit{util} : S \to \mathbb{N}$
rational probability bound p

compute $r_{opt} \;=\; \inf\left\{\, r \in \mathbb{R} : \Pr^{\mathcal{M}}(\, \mathit{lrrat} \leqslant r \,) > p \,\right\}$

↑

random variable for the
long-run cost-utility ratio
(as before)

Best threshold for long-run ratios

given: MC with reward functions $cost, util : S \to \mathbb{N}$
rational probability bound p

compute $r_{opt} = \inf \{ r \in \mathbb{R} : \Pr^{\mathcal{M}}(lrrat \leqslant r) > p \}$

$$r_{opt} = \inf \{ r \in \mathbb{R} : \Pr^{\mathcal{M}}(\Box\Diamond(\frac{cost}{util} \leqslant r)) > p \}$$

if $\pi = s_0\, s_1\, s_2 \ldots$ is an infinite path then

$$\pi \models \Box\Diamond(\frac{cost}{util} \leqslant r) \quad \text{iff} \quad \overset{\infty}{\exists}\, n \text{ s.t. } \frac{cost(s_0\, s_1 \ldots s_n)}{util(s_0\, s_1 \ldots s_n)} \leqslant r$$

Best threshold for long-run ratios

given: MC with reward functions $cost, util : S \to \mathbb{N}$
rational probability bound p

compute $r_{opt} = \inf \{ r \in \mathbb{R} : \Pr^{\mathcal{M}}(\mathit{lrrat} \leqslant r) > p \}$

$$r_{opt} = \inf \{ r \in \mathbb{R} : \Pr^{\mathcal{M}}(\Box\Diamond(\tfrac{cost}{util} \leqslant r)) > p \}$$

$$= \inf \{ r \in \mathbb{R} : \Pr^{\mathcal{M}}(\Diamond\Box(\tfrac{cost}{util} \leqslant r)) > p \}$$

$\pi \models \Box\Diamond(\frac{cost}{util} \leqslant r)$ iff $\overset{\infty}{\exists}\, n$ s.t. $\frac{cost(s_0\, s_1 \ldots s_n)}{util(s_0\, s_1 \ldots s_n)} \leqslant r$

Best threshold for long-run ratios

given: MC with reward functions $\mathit{cost}, \mathit{util} : S \to \mathbb{N}$
rational probability bound p

compute $\quad r_{opt} \quad = \quad \inf \{ r \in \mathbb{R} : \Pr^{\mathcal{M}}(\mathit{lrrat} \leqslant r) > p \}$

$$\begin{aligned} r_{opt} &= \inf \{ r \in \mathbb{R} : \Pr^{\mathcal{M}}(\Box\Diamond(\tfrac{\mathit{cost}}{\mathit{util}} \leqslant r)) > p \} \\ &= \inf \{ r \in \mathbb{R} : \Pr^{\mathcal{M}}(\Diamond\Box(\tfrac{\mathit{cost}}{\mathit{util}} \leqslant r)) > p \} \\ &= \min \{ r \in \mathbb{Q} : \Pr^{\mathcal{M}}(\Diamond C_r) > p \} \end{aligned}$$

where C_r = union of all BSCCs B with $\mathit{lrrat}(B) \leqslant r$

Best threshold for long-run ratios

given: MC with reward functions $cost, util : S \to \mathbb{N}$
rational probability bound p

compute $$r_{opt} = \inf\{\, r \in \mathbb{R} : \Pr^{\mathcal{M}}(\, lrrat \leqslant r\,) > p \,\}$$
$$= \min\{\, r \in \mathbb{Q} : \Pr^{\mathcal{M}}(\Diamond C_r) > p \,\}$$

where C_r = union of all BSCCs B with $lrrat(B) \leqslant r$

↑ expected long-run ratio of B

Best threshold for long-run ratios

given: MC with reward functions $cost, util : S \to \mathbb{N}$
rational probability bound p

compute $$r_{opt} = \inf \{ r \in \mathbb{R} : \Pr^{\mathcal{M}}(lrrat \leqslant r) > p \}$$
$$= \min \{ r \in \mathbb{Q} : \Pr^{\mathcal{M}}(\Diamond C_r) > p \}$$

where C_r = union of all BSCCs B with $lrrat(B) \leqslant r$

1. compute the BSCCs $B_1, \ldots, B_k$ and $r_i = lrrat(B_i)$

Best threshold for long-run ratios

given: MC with reward functions $cost, util : S \to \mathbb{N}$
rational probability bound p

compute $r_{opt} = \inf \{ r \in \mathbb{R} : \Pr^{\mathcal{M}}(lrrat \leqslant r) > p \}$
$= \min \{ r \in \mathbb{Q} : \Pr^{\mathcal{M}}(\Diamond C_r) > p \}$

where C_r = union of all BSCCs B with $lrrat(B) \leqslant r$

1. compute the BSCCs $B_1, \ldots, B_k$ and $r_i = lrrat(B_i)$

   w.l.o.g. $r_1 < r_2 < \ldots < r_k$

Best threshold for long-run ratios

given: MC with reward functions $cost, util : S \to \mathbb{N}$
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compute $r_{opt} = \inf \{ r \in \mathbb{R} : \Pr^{\mathcal{M}}(lrrat \leqslant r) > p \}$
$= \min \{ r \in \mathbb{Q} : \Pr^{\mathcal{M}}(\Diamond C_r) > p \}$

where C_r = union of all BSCCs B with $lrrat(B) \leqslant r$

1. compute the BSCCs $B_1, \ldots, B_k$ and $r_i = lrrat(B_i)$
   w.l.o.g. $r_1 < r_2 < \ldots < r_k$
2. determine the minimal $i \in \{1, \ldots, k\}$ such that
   $\Pr^{\mathcal{M}}(\Diamond B_1) + \ldots + \Pr^{\mathcal{M}}(\Diamond B_i) > p$ and return r_i

Cost-utility ratios: invariances

Given an MC with two positive reward functions $\mathit{cost}, \mathit{util} : S \to \mathbb{N}$, consider their ratio:

$$\mathit{ratio} \;=\; \frac{\mathit{cost}}{\mathit{util}} : \mathit{FinPaths} \to \mathbb{Q}$$

$$\mathit{ratio}(\pi) \;=\; \frac{\mathit{cost}(\pi)}{\mathit{util}(\pi)} \quad \text{for all finite paths } \pi$$

decision problems: given an ω-regular property φ and probability bound $q \in [0,1[$, ratio threshold $r \in \mathbb{Q}$:

- does $\Pr^{\mathcal{M}}(\Box(\mathit{ratio} \leqslant r) \wedge \varphi) > q$ hold ?
- does $\Pr^{\mathcal{M}}(\Box(\mathit{ratio} \leqslant r) \wedge \varphi) = 1$ hold ?

Cost-utility ratio via weight functions

Given an MC with two positive reward functions $cost, util : S \to \mathbb{N}$, consider their ratio:

$$ratio = \frac{cost}{util} : FinPaths \to \mathbb{Q}$$

$$ratio(\pi) = \frac{cost(\pi)}{util(\pi)} \quad \text{for all finite paths } \pi$$

replace ratio by weight constraints:

$$\Box(\, ratio \leqslant r \,) \quad \equiv \quad \Box(\, wgt \leqslant 0 \,)$$

Cost-utility ratio via weight functions

Given an MC with two positive reward functions $cost, util : S \to \mathbb{N}$, consider their ratio:

$$ratio = \frac{cost}{util} : FinPaths \to \mathbb{Q}$$

$$ratio \leqslant r \quad \text{iff} \quad wgt \leqslant 0$$

$$\text{where} \quad wgt = cost - r \cdot util$$

$$\Box(ratio \leqslant r) \equiv \Box(wgt \leqslant 0)$$

Cost-utility ratio via weight functions

Given an MC with two positive reward functions $cost, util : S \to \mathbb{N}$, consider their ratio:

$$ratio = \frac{cost}{util} : FinPaths \to \mathbb{Q}$$

$$ratio \leqslant r \quad \text{iff} \quad wgt \leqslant 0$$

$$\text{where} \quad wgt = cost - r \cdot util \in \mathbb{Q}$$

$$\Box(ratio \leqslant r) \equiv \Box(wgt \leqslant 0)$$

Cost-utility ratio via weight functions

Given an MC with two positive reward functions $cost, util : S \to \mathbb{N}$, consider their ratio:

$$ratio = \frac{cost}{util} : FinPaths \to \mathbb{Q}$$

$$ratio \leqslant r \quad \text{iff} \quad wgt > 0$$

where $wgt = (cost - r \cdot util) \cdot const \in \mathbb{Z}$

↑ integer-valued weight function

$$\Box(\, ratio \leqslant r \,) \quad \equiv \quad \Box(\, wgt > 0 \,)$$

Weight invariances for MC

Given an MC with a weight function $\boldsymbol{wgt} : \boldsymbol{S} \to \mathbb{Z}$.

Weight invariances for MC

Given an MC with a weight function $wgt : S \to \mathbb{Z}$.

almost-sure problem:

does $\Pr^{\mathcal{M}}_{s_0}\big(\Box(wgt > 0) \wedge \varphi\big) = 1$ hold ?

positive problem:

does $\Pr^{\mathcal{M}}_{s_0}\big(\Box(wgt > 0) \wedge \varphi\big) > 0$ hold ?

quantitative problems, e.g.:

does $\Pr^{\mathcal{M}}_{s_0}\big(\Box(wgt > 0) \wedge \varphi\big) > \frac{1}{2}$ hold ?

Weight invariances for MC

Given an MC with a weight function $wgt : S \to \mathbb{Z}$.

almost-sure problem:

does $\Pr^{\mathcal{M}}_{s_0}\big(\Box(wgt > 0) \wedge \varphi\big) = 1$ hold ?

simple

positive problem:

does $\Pr^{\mathcal{M}}_{s_0}\big(\Box(wgt > 0) \wedge \varphi\big) > 0$ hold ?

quantitative problems, e.g.:

does $\Pr^{\mathcal{M}}_{s_0}\big(\Box(wgt > 0) \wedge \varphi\big) > \frac{1}{2}$ hold ?

difficult

Almost-sure weight invariances

$$\Pr_{s}^{\mathcal{M}}\big(\Box(wgt > 0) \wedge \varphi\big) = 1$$

Almost-sure weight invariances

$$\Pr_s^{\mathcal{M}}\big(\Box(wgt > 0) \wedge \varphi\big) = 1$$

iff $\Pr_s^{\mathcal{M}}\big(\Box(wgt > 0)\big) = 1$ and $\Pr_s^{\mathcal{M}}\big(\varphi\big) = 1$

Almost-sure weight invariances

$$\Pr_s^{\mathcal{M}}\big(\Box(wgt > 0) \wedge \varphi\big) = 1$$

iff $\Pr_s^{\mathcal{M}}\big(\Box(wgt > 0)\big) = 1$ and $\Pr_s^{\mathcal{M}}\big(\varphi\big) = 1$

iff $s \not\models \exists\Diamond(wgt \leqslant 0)$ and $\Pr_s^{\mathcal{M}}\big(\varphi\big) = 1$

Almost-sure weight invariances

$$\Pr_s^{\mathcal{M}}\big(\Box(wgt > 0) \wedge \varphi\big) = 1$$

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iff $s \not\models \exists\Diamond(wgt \leqslant 0)$ and $\Pr_s^{\mathcal{M}}(\varphi) = 1$

↑

solvable by standard
shortest-path algorithms

Almost-sure weight invariances

$$\Pr_s^{\mathcal{M}}\big(\Box(wgt > 0) \wedge \varphi\big) = 1$$

iff $\Pr_s^{\mathcal{M}}\big(\Box(wgt > 0)\big) = 1$ and $\Pr_s^{\mathcal{M}}(\varphi) = 1$

iff $s \not\models \exists\Diamond(wgt \leqslant 0)$ and $\Pr_s^{\mathcal{M}}(\varphi) = 1$

solvable by standard shortest-path algorithms

standard methods for ω-regular path properties

polynomially time-bounded for reachability or Büchi properties

Almost-sure weight invariances

$$\Pr^{\mathcal{M}}_{s}\big(\Box(wgt > 0) \wedge \varphi\big) = 1$$

iff $\Pr^{\mathcal{M}}_{s}\big(\Box(wgt > 0)\big) = 1$ and $\Pr^{\mathcal{M}}_{s}(\varphi) = 1$

iff $s \not\models \exists\Diamond(wgt \leqslant 0)$ and $\Pr^{\mathcal{M}}_{s}(\varphi) = 1$

Best threshold computable by shortest-path algorithms:

$$\textbf{sup}\ \{\ r \in \mathbb{Z}\ :\ \Pr^{\mathcal{M}}_{s}\big(\Box(wgt > r) \wedge \varphi\big) = 1\ \}$$

$1 +$ length of a shortest path starting in state s,
provided that φ holds almost surely and there are no negative cycles

Weight invariances for MC

Given an MC with a weight function $wgt : S \to \mathbb{Z}$.

almost-sure problem:

does $\Pr^{\mathcal{M}}_{s_0}\big(\Box(wgt > 0) \wedge \varphi\big) = 1$ hold ?

simple

positive problem:

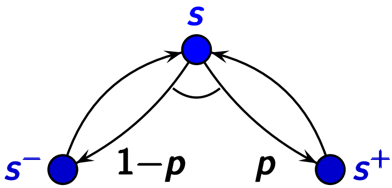
does $\Pr^{\mathcal{M}}_{s_0}\big(\Box(wgt > 0) \wedge \varphi\big) > 0$ hold ?

quantitative problems, e.g.:

does $\Pr^{\mathcal{M}}_{s_0}\big(\Box(wgt > 0) \wedge \varphi\big) > \frac{1}{2}$ hold ?

difficult

Markov chain with weight function



$wgt(s) = +1$

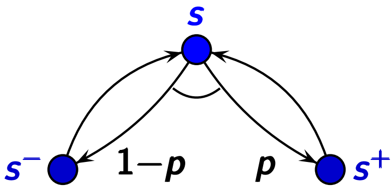
$wgt(s^-) = -2$

$wgt(s^+) = 0$

probability parameter

$0 < p < 1$

Markov chain with weight function



$wgt(s) = +1$

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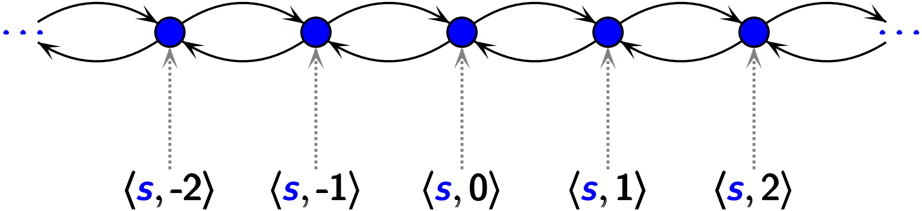
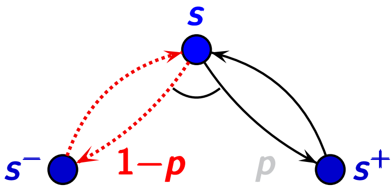
$wgt(s^+) = 0$

random walk:



Markov chain with weight function



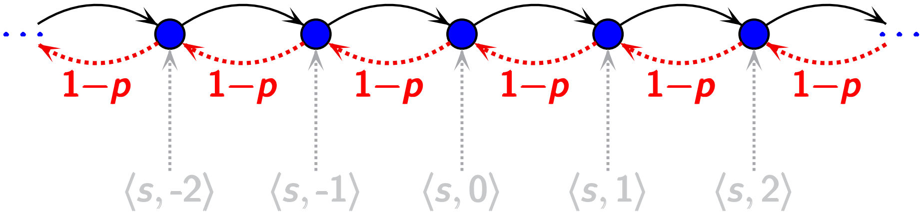
$wgt(s) = +1$

$wgt(s^-) = -2$

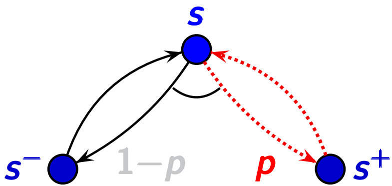
$wgt(s^+) = 0$

weight $-\mathbf{1}$ for the cycle $s\ s^-\ s$

random walk:



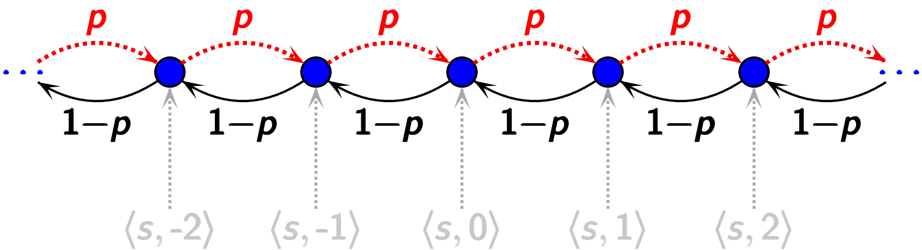
Markov chain with weight function



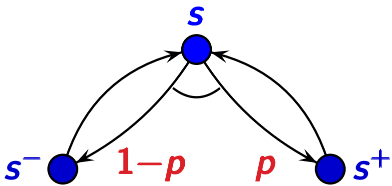
$$wgt(s) = +1$$
$$wgt(s^-) = -2$$
$$wgt(s^+) = 0$$

weight **+1** for the cycle $s\ s^+\ s$

random walk:



Markov chain with weight function



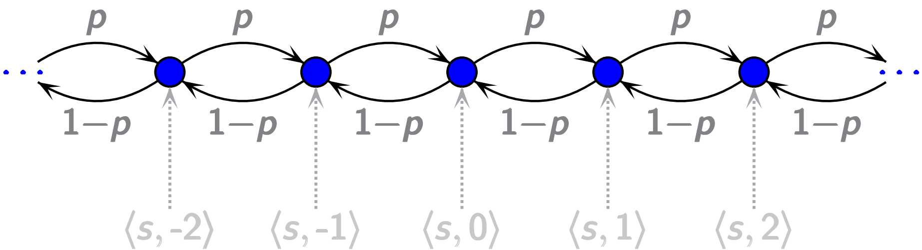
$wgt(s) = +1$

$wgt(s^-) = -2$

$wgt(s^+) = 0$

$$\Pr_s(\Box(wgt > 0)) > 0 \quad \text{iff} \quad p > \frac{1}{2}$$

random walk:



Weight invariance problem: positive case

The problem "does $\Pr_s(\Box(wgt > r) \wedge \varphi) > 0$ hold ?"

- depends on the concrete transition probabilities

where φ is a ω-regular property and $0 \leqslant q < 1$

Weight invariance problem: positive case

The problem "does $\mathrm{Pr}_s(\Box(wgt > r) \wedge \varphi) > 0$ hold ?"

- depends on the concrete transition probabilities
- is solvable in polynomial time

  BSCC-analysis and variants of shortest-paths algorithms, assuming φ is a Rabin or Streett or reachability condition

  [Brázdil/Kiefer/Kučera/Novotný/Katoen'14]
  [Krähmann/Schubert/Baier/Dubslaff'15]

where φ is a ω-regular property and $0 \leqslant q < 1$

Weight invariance problem: positive case

The problem "does $\mathbf{Pr}_s(\Box(wgt > r) \wedge \varphi) > 0$ hold ?"

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- is solvable in polynomial time

  BSCC-analysis and variants of shortest-paths algorithms, assuming φ is a Rabin or Streett condition

check whether there exists a good BSCC B s.t.

1. $\mathrm{MP}(B) > 0$ or $\mathrm{MP}(B) = 0$ & no negative cycle in B
2. there is a path π from s to B s.t. π and its prefixes have sufficiently high weight

Weight invariance problem: quantitative case

The problem "does $\Pr_s(\Box(wgt > r) \land \varphi) > 0$ hold ?"

- depends on the concrete transition probabilities
- is solvable in polynomial time

  BSCC-analysis and variants of shortest-paths algorithms, assuming φ is a Rabin or Streett condition

The problem "does $\Pr_s(\Box(wgt > 0) \land \varphi) > q$ hold ?"

- is reducible to the threshold problem for probabilistic pushdown automata (exponential blowup)

Weight invariance problem: quantitative case

The problem "does $\mathrm{Pr}_s(\Box(wgt > r) \wedge \varphi) > 0$ hold ?"

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- is solvable in polynomial time

  BSCC-analysis and variants of shortest-paths algorithms, assuming φ is a Rabin or Streett condition

The problem "does $\mathrm{Pr}_s(\Box(wgt > 0) \wedge \varphi) > q$ hold ?"

- is reducible to the threshold problem for probabilistic pushdown automata (exponential blowup)
- is PosSLP-hard, even for unit weights and $\varphi = true$

  [Etessami/Yannak.'09], [Brázdil/Brozek/Etes./Kučera/Wojt.'10]

Weight invariance problem: almost-sure case

The problem "does $\mathrm{Pr}_s(\Box(wgt > r) \wedge \varphi) = 1$ hold ?"

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Weight invariance problem: almost-sure case

The problem "does $\Pr_s(\Box(wgt > r) \wedge \varphi) = 1$ hold ?"

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$$\Pr_s(\Box(wgt > r) \wedge \varphi) = 1$$

iff $\Pr_s(\Box(wgt > r)) = 1$ and $\Pr_s(\varphi) = 1$

Weight invariance problem: almost-sure case

The problem "does $\Pr_s(\Box(wgt > r) \wedge \varphi) = 1$ hold ?"

- independent from the concrete transition probabilities
- is solvable in polynomial time

$$\Pr_s(\Box(wgt > r) \wedge \varphi) = 1$$

iff $\Pr_s(\Box(wgt > r)) = 1$ and $\boxed{\Pr_s(\varphi) = 1}$

↑

standard algorithm

polynomial-time for

reachability, Rabin or Streett

Weight invariance problem: almost-sure case

The problem "does $\mathbf{Pr}_s(\Box(wgt > r) \wedge \varphi) = 1$ hold ?"

- independent from the concrete transition probabilities
- is solvable in polynomial time

$$\mathbf{Pr}_s(\Box(wgt > r) \wedge \varphi) = 1$$

iff $\boxed{\mathbf{Pr}_s(\Box(wgt > r)) = 1}$ and $\mathbf{Pr}_s(\varphi) = 1$

↑

shortest-path algorithm

check chether the weight of a shortest path from s is at least r+1

Weight-bounded reachability in MC

Weight-bounded reachability in MC

given: weighted MC $\mathcal{M}$, weight bound $r \in \mathbb{Z}$
and a distinguished states s, $goal$

decision problems:

positive prob: does $\Pr_s(\Diamond^{\leqslant r} goal) > 0$ hold ?

almost-sure: does $\Pr_s(\Diamond^{\leqslant r} goal) = 1$ hold ?

quantitative: does $\Pr_s(\Diamond^{\leqslant r} goal) > \frac{1}{2}$ hold ?

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given: weighted MC $\mathcal{M}$, weight bound $r \in \mathbb{Z}$
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decision problems:

positive prob: does $\Pr_s(\Diamond^{\leqslant r} goal) > 0$ hold ?

solvable in poly-time using shortest-path algorithms

almost-sure: does $\Pr_s(\Diamond^{\leqslant r} goal) = 1$ hold ?

solvable in poly-time using shortest-path algorithms;
a bit tricky if goal is not a trap

quantitative: does $\Pr_s(\Diamond^{\leqslant r} goal) > \frac{1}{2}$ hold ?

solvable in poly-space using algorithms for prob PDA

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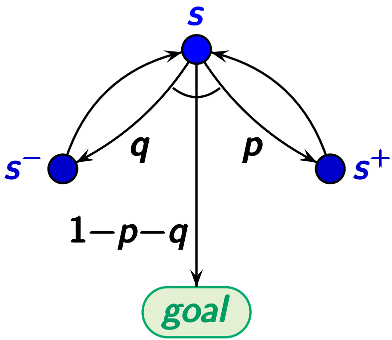
solvable in poly-time using shortest-path algorithms;
a bit tricky if goal is not a trap

quantitative: does $\Pr_s(\Diamond^{\leqslant r} goal) > \frac{1}{2}$ hold ?

solvable in poly-space using algorithms for prob PDA

Is there an algorithm to compute $\Pr_s(\Diamond^{\leqslant r} goal)$?

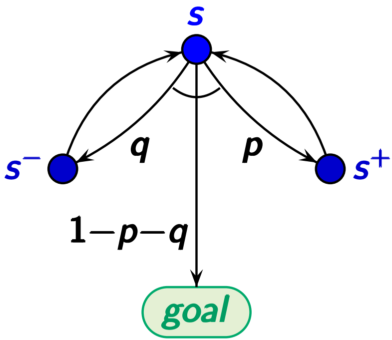
Weight-bounded reachability in MC [Kiefer'17]



$wgt(s) = 0$
$wgt(s^-) = -1$
$wgt(s^+) = +1$

probability parameters
p and q with $0 < p, q < 1$
and $p + q < 1$

Weight-bounded reachability in MC [Kiefer'17]



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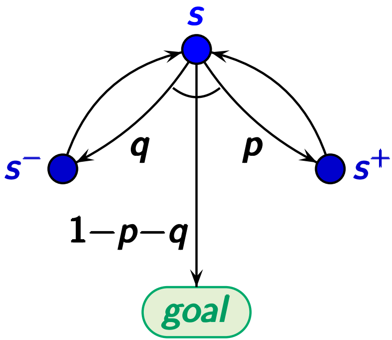
$$\Pr_s(\Diamond^{=0} goal) = (1-p-q) \cdot \sum_{n=0}^{\infty} \binom{2n}{n} \cdot p^n \cdot q^n$$

Weight-bounded reachability in MC [KIEFER'17]



$$wgt(s) = 0$$
$$wgt(s^-) = -1$$
$$wgt(s^+) = +1$$

$$\Pr_s(\Diamond^{=0} goal) = (1-p-q) \cdot \sum_{n=0}^{\infty} \binom{2n}{n} \cdot p^n \cdot q^n$$

$$= \frac{1-p-q}{\sqrt{1-4 \cdot p \cdot q}}$$ ... irrational

Best threshold for ratio invariances

Given a Markov chain $\mathcal{M}$ with two reward functions $rew_1, rew_2 : S \to \mathbb{N}$ with $rew_2 > \mathbf{0}$, consider their ratio

$$ratio : FinPaths \to \mathbb{Q}, \quad ratio(\pi) = \frac{rew_1(\pi)}{rew_2(\pi)}$$

examples:

- energy-utility ratio
- cost of repair mechanisms per failure
- SLA violations per day

Best threshold for ratio invariances

Given a Markov chain $\mathcal{M}$ with two reward functions $rew_1, rew_2 : S \to \mathbb{N}$ with $rew_2 > 0$, consider their ratio:

$$ratio : FinPaths \to \mathbb{Q}, \quad ratio(\pi) \;=\; \frac{rew_1(\pi)}{rew_2(\pi)}$$

$$\Pr_s(\,\Box(ratio > r)\,)$$

examples:

- energy-utility ratio
- cost of repair mechanisms per failure
- SLA violations per day

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best threshold for qualitative ratio invariances:

$$\sup \{ \ r \in \mathbb{Q} : \ \Pr_s(\Box(ratio > r)) > 0 \ \}$$

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... are computable in polynomial time ...

[Krähmann/Schubert/Baier/Dubslaff'15]

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Positive ratio quantiles

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$$ratio = \frac{rew_1}{rew_2} : FinPaths \to \mathbb{Q} \quad \text{where } rew_2 > 0$$

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reduction to positive weight invariances:

$$ratio > r \quad \text{iff} \quad rew_1 - r \cdot rew_2 > 0$$

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reduction to positive weight invariances:

$$ratio > r \quad \text{iff} \quad \underbrace{rew_1 - r \cdot rew_2}_{\text{weight function}} > 0$$

If $r \in \mathbb{Q}$ then pick some $c \in \mathbb{N}$ such that $(rew_1 - r \cdot rew_2) \cdot c$ is an integer weight function.

Positive ratio quantiles

$$\mathbf{sup} \left\{ r \in \mathbb{Q} : \Pr_s(\Box(ratio > r)) > 0 \right\}$$

- inner decision problem for fixed r is solvable in polynomial time
- quantile can be approximated using a binary search

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for all finite paths π:

$$0 \leqslant ratio(\pi) \leqslant \frac{\max rew_1}{\min rew_2}$$

$ratio = \frac{rew_1}{rew_2} : FinPaths \to \mathbb{Q}$ where $rew_2 > 0$

Positive ratio quantiles

$$\sup \{ \ r \in \mathbb{Q} : \ \Pr_s(\Box(ratio > r)) > 0 \ \}$$

- inner decision problem for fixed r is solvable in polynomial time
- quantile can be approximated using a binary search and is one of the values
  - expected long-run ratio of a BSCC

If B is a BSCC then the expected long-run ratio is:

$$\frac{\mathrm{MP}_B[rew_1]}{\mathrm{MP}_B[rew_2]} \quad \text{where} \quad \mathrm{MP}_B[rew] = \left\{ \begin{array}{l} \text{mean-payoff} \\ \text{of } rew \text{ in } B \end{array} \right.$$

Positive ratio quantiles

$$\textbf{sup} \left\{ \, r \in \mathbb{Q} : \; \Pr_s(\Box(\mathit{ratio} > r)) > 0 \, \right\}$$

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  - $\mathit{ratio}(\pi)$ for a simple path π from s
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finitely many values

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finitely many values

- computation using the continued-fraction method

Positive ratio quantiles

$$\sup \left\{ \, r \in \mathbb{Q} : \; \Pr_s(\Box(ratio > r)) > 0 \, \right\} \; = \; \frac{c}{d}$$

where $c, d \in \mathbb{N}$ with $d > 0$

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  - expected long-run ratio of a BSCC
  - $ratio(\pi)$ for a simple path π from s
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Positive ratio quantiles

$$\mathbf{sup}\ \{\ r \in \mathbb{Q} :\ \Pr_s(\Box(\mathit{ratio} > r)) > 0\ \}\ =\ \frac{c}{d}$$

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- quantile can be approximated using a binary search and is one of the values ... and therefore rational
  - expected long-run ratio c_B/d_B of BSCC B
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$$\left| \frac{c}{d} - p \right| < \varepsilon$$

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$$\left| \frac{a}{b} - p \right| < \varepsilon \quad \text{iff} \quad \frac{a}{b} = \frac{c}{d}$$

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$$p = p_1 + \cfrac{1}{p_2 + \cfrac{1}{p_3 + \cfrac{1}{p_4 + \cfrac{1}{\cdots}}}}$$

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   [Grötschel/Lovász/Schrijver'87]

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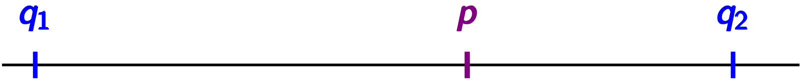

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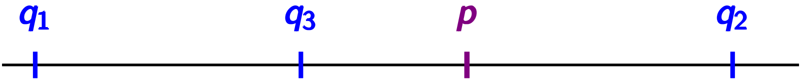

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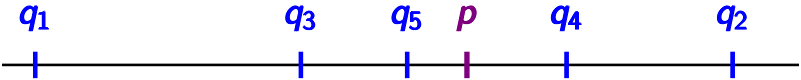

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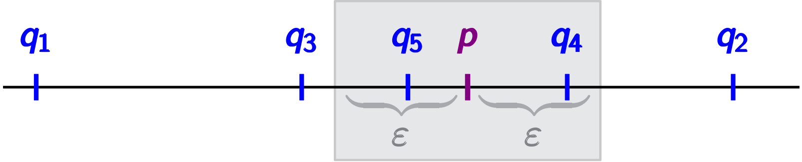

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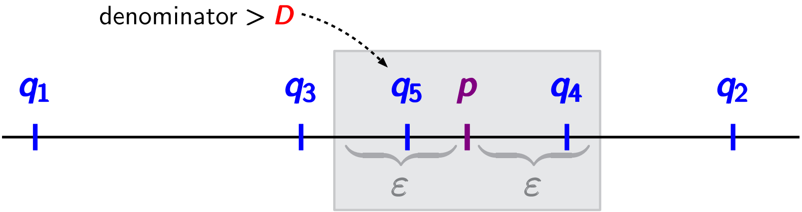

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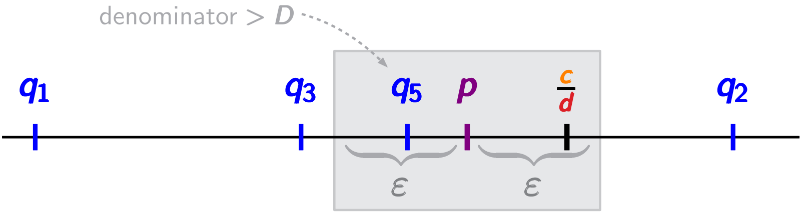

Polynomially computable ratio quantiles in MC

qualitative quantiles for ratio invariances:

$$\textbf{sup}\,\{\, r \in \mathbb{Q} \,:\, \Pr_s\big(\Box(\textit{ratio} > r) \wedge \varphi\,\big) \,>\, \mathbf{0}\,\}$$

$$\textbf{sup}\,\{\, r \in \mathbb{Q} \,:\, \Pr_s\big(\Box(\textit{ratio} > r) \wedge \varphi\,\big) \,=\, \mathbf{1}\,\}$$

where φ is a reachability, Rabin or Streett condition

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where $\mathit{wgt}_r = \mathit{cost} - r \cdot \mathit{util}$

... binary search for maximal r and shortest-path algorithms ...

where φ is a reachability, Rabin or Streett condition

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qualitative and quantitative quantiles for long-run ratios:

$$\sup \{ r \in \mathbb{Q} : \Pr_s(\Box\Diamond(ratio > r) \wedge \varphi) = 1 \}$$

$$\sup \{ r \in \mathbb{Q} : \Pr_s(\Box\Diamond(ratio > r) \wedge \varphi) > q \}$$

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$$= \min \{ r \in \mathbb{Q} : \Pr_s(\Diamond C_r) > q \}$$

where C_r = union of "good" BSCCs B with $lrrat(B) \geqslant r$

Tutorial: Probabilistic Model Checking

Discrete-time Markov chains (DTMC)

* basic definitions
* probabilistic computation tree logic PCTL/PCTL*
* rewards, cost-utility ratios, weights
* conditional probabilities

Markov decision processes (MDP)

* basic definitions
* PCTL/PCTL* model checking
* fairness
* conditional probabilities
* rewards, quantiles
* mean-payoff
* expected accumulated weights

Conditional probabilities

Conditional probabilities

- useful for various multi-objective properties

  e.g. analyze the gained utility for a given energy budget

$$\Pr_s\big(\Diamond_{\geqslant u}\, goal \mid \Diamond^{\leqslant e}\, goal\big) \quad \text{or}$$

$$\mathrm{ExpUtil}_s\big(\oplus\, goal \mid \Diamond^{\leqslant e}\, goal\big)$$

$\Diamond_{\geqslant u}\, goal$ "gained utility for reaching the goal is at least u"

$\Diamond^{\leqslant e}\, goal$ "consumed energy until reaching the goal is at most e"

Conditional probabilities

- useful for various multi-objective properties

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- useful for failure diagnosis

  e.g. study the impact of failures and cost of repair mechanisms in resilient systems

$$\Pr_s\big(\Diamond goal \mid \Diamond failure\big) \quad \text{or}$$

$$\mathrm{ExpCost}_s\big(\oplus\, goal \mid \Diamond failure\big)$$

Conditional probabilities

for Markov chains:

$$\Pr_s^{\mathcal{M}}(\varphi \mid \psi) \;=\; \frac{\Pr_s^{\mathcal{M}}(\varphi \wedge \psi)}{\Pr_s^{\mathcal{M}}(\psi)}$$

provided $\Pr_s^{\mathcal{M}}(\psi) > 0$

Conditional probabilities

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- discrete MCs and PCTL [Andrés/Rossum'08] [Ji/Wu/Chen'13]
- continuous-time MCs and CSL [Gao/Xu/Zhan/Zhang'13]

PCTL: probabilistic computation tree logic

CSL: continuous stochastic logic

Conditional probabilities

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transformation-based approach for LTL conditions

MC $\mathcal{M} \rightsquigarrow$ MC $\mathcal{M}_\psi$: [Baier/Klein/Klüppelholz/Märcker'14]

$$\Pr_s^{\mathcal{M}}(\varphi \mid \psi) \;=\; \Pr_s^{\mathcal{M}_\psi}(\varphi)$$

Transformation-based approach for MC

given: Markov chain $\mathcal{M} = (S, P)$ and $\psi = \Diamond G$

define Markov chain $\mathcal{M}_\psi$ s.t. for all LTL formulas φ

$$\Pr_s^{\mathcal{M}}(\varphi \mid \Diamond G) \;=\; \Pr_s^{\mathcal{M}_\psi}(\varphi)$$

LTL: linear temporal logic

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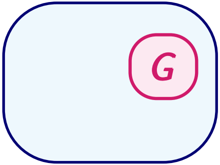
MC $\mathcal{M}$



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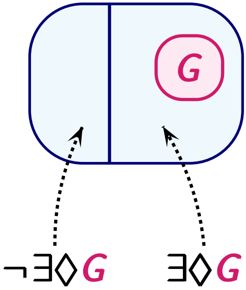

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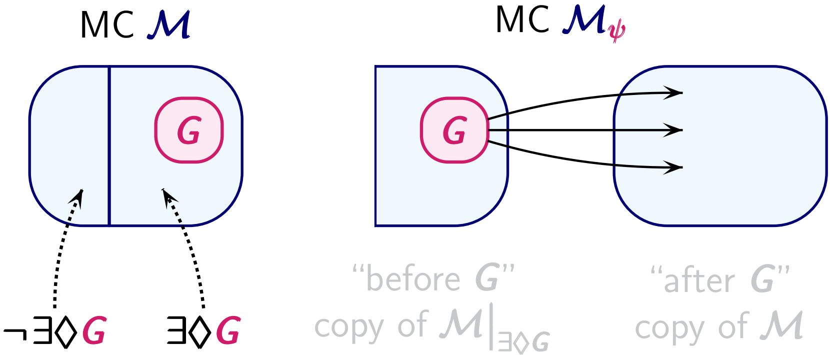
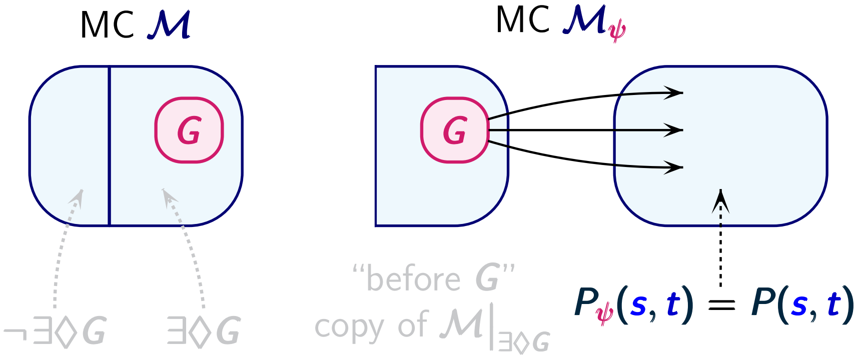


Transformation-based approach for MC

given: Markov chain $\mathcal{M} = (S, P)$ and $\psi = \Diamond G$

define Markov chain $\mathcal{M}_\psi$ s.t. for all LTL formulas φ

$$\Pr_s^{\mathcal{M}}(\varphi \mid \Diamond G) = \Pr_s^{\mathcal{M}_\psi}(\varphi)$$



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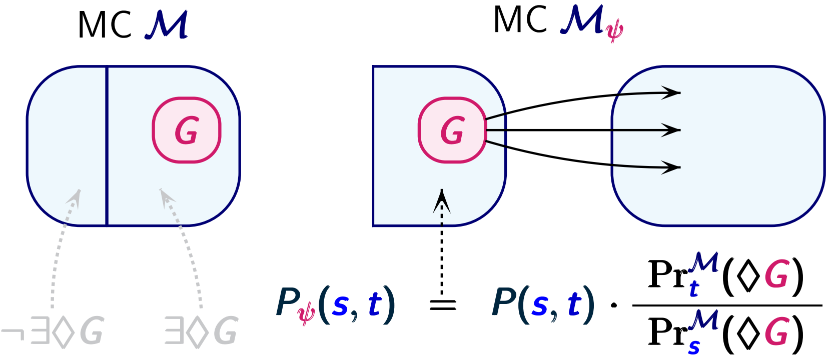

$$P_\psi(s,t) = P(s,t) \cdot \frac{\Pr_t^{\mathcal{M}}(\Diamond G)}{\Pr_s^{\mathcal{M}}(\Diamond G)}$$

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... can be generalized for other temporal conditions ψ

either by adapting the definition of $\mathcal{M}_\psi$ or
by using an ω-automaton for LTL conditions

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same method applicable for conditional expectations

$$\mathbb{E}_s^{\mathcal{M}}(f \mid \psi) = \mathbb{E}_s^{\mathcal{M}_\psi}(f')$$

e.g.: $\mathbb{E}_s^{\mathcal{M}}($ "energy until reaching the goal" $\mid \Diamond goal$ $)$

Tutorial: Probabilistic Model Checking

Discrete-time Markov chains (DTMC)

* basic definitions
* probabilistic computation tree logic PCTL/PCTL*
* rewards, cost-utility ratios, weights
* conditional probabilities

Markov decision processes (MDP)

* basic definitions
* PCTL/PCTL* model checking
* fairness
* conditional probabilities
* rewards, quantiles
* mean-payoff
* expected accumulated weights