Introduction to Permission-Based Program Logics

Part II – Concurrent Programs

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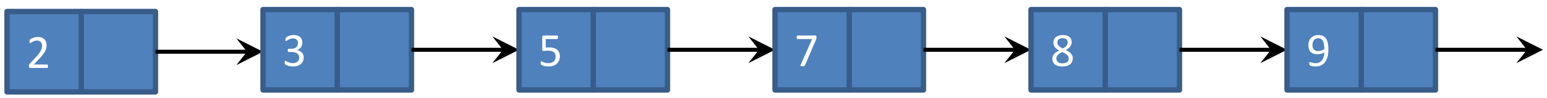
Example: Lock-Coupling List



- There is one lock per node; threads acquire locks in a hand over hand fashion.
- If a node is locked, we can insert a node just after it.
- If two adjacent nodes are locked, we can remove the second.

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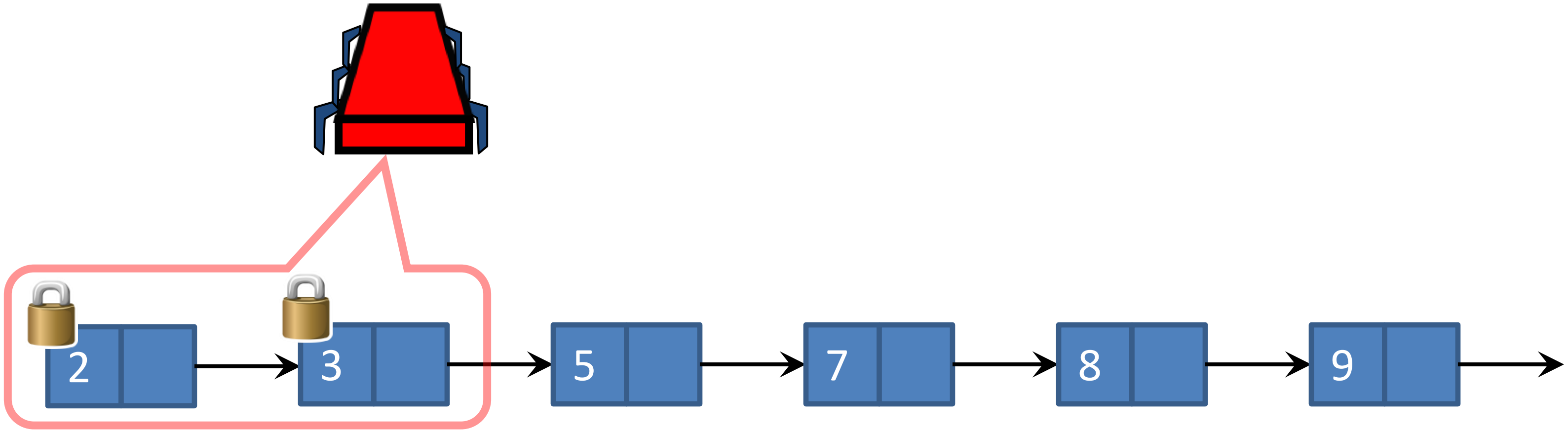

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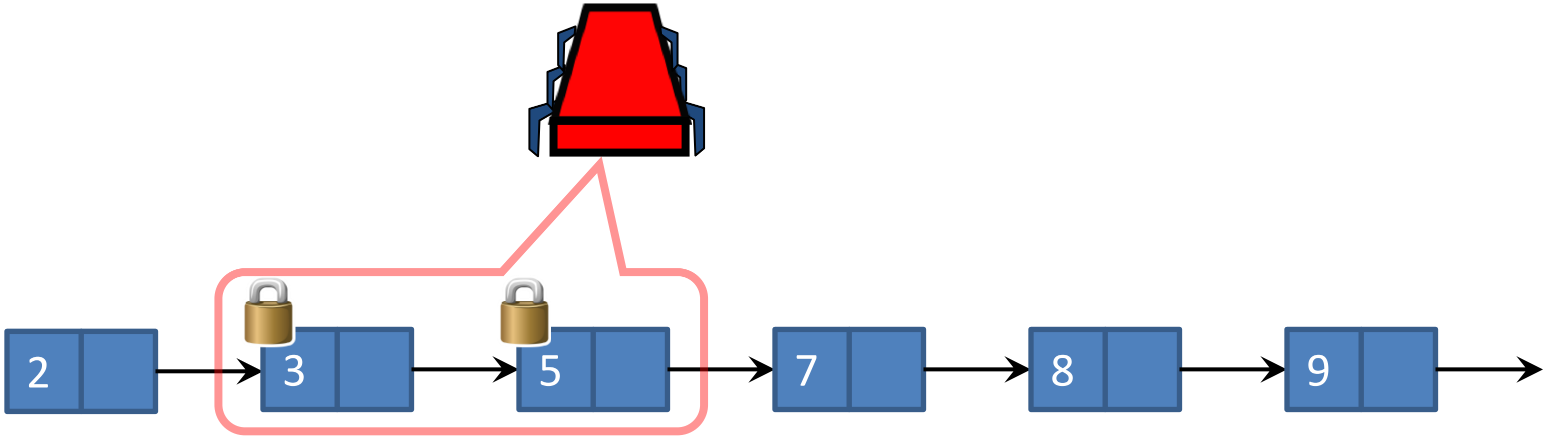

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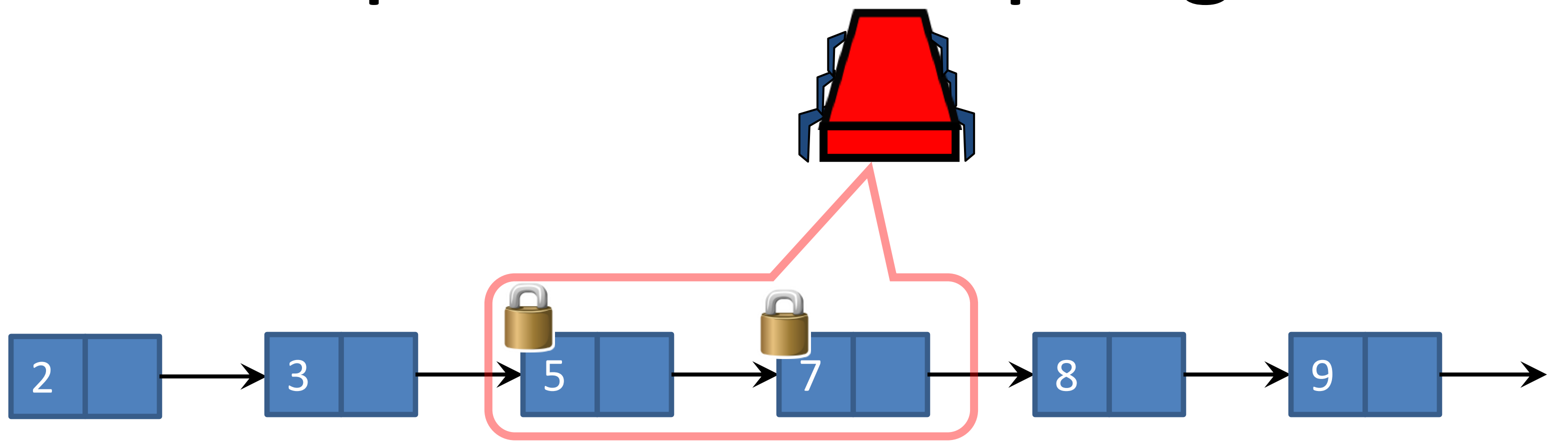

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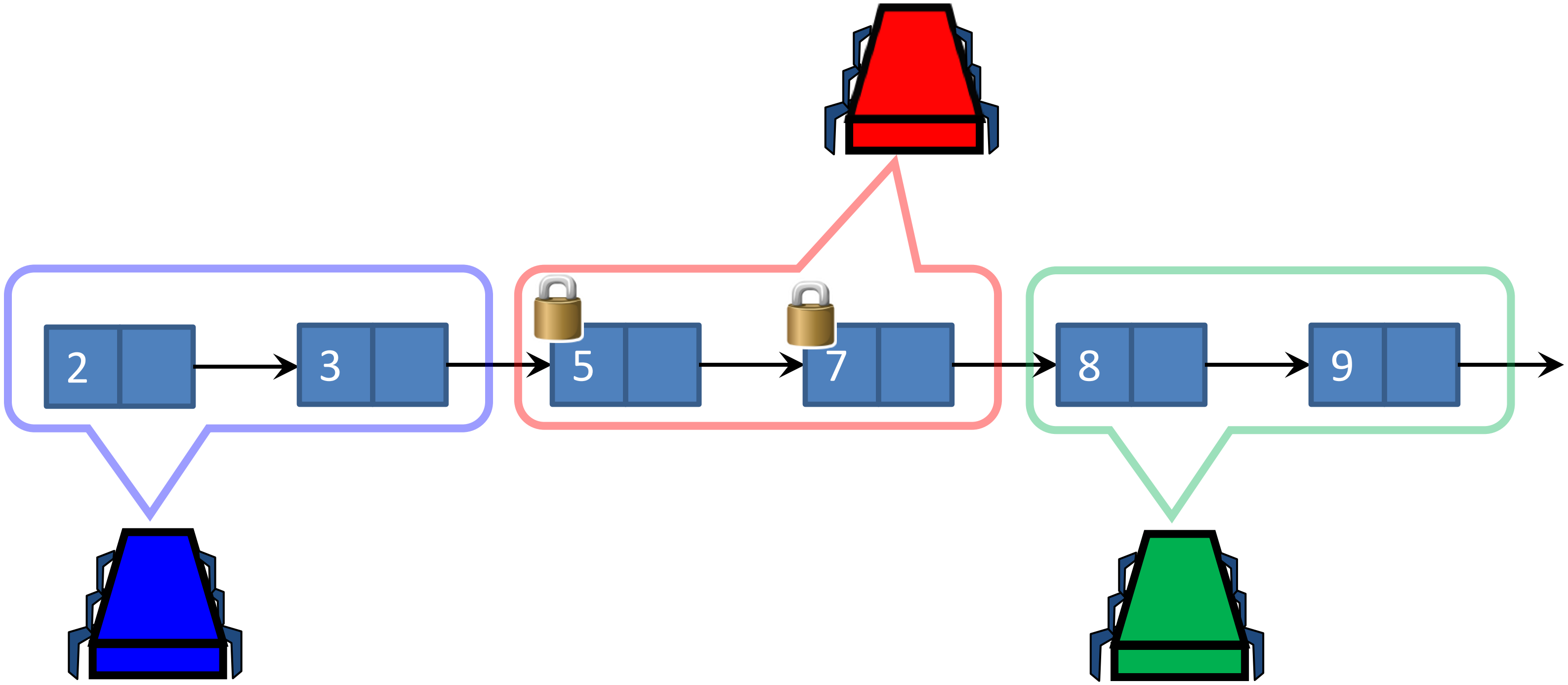

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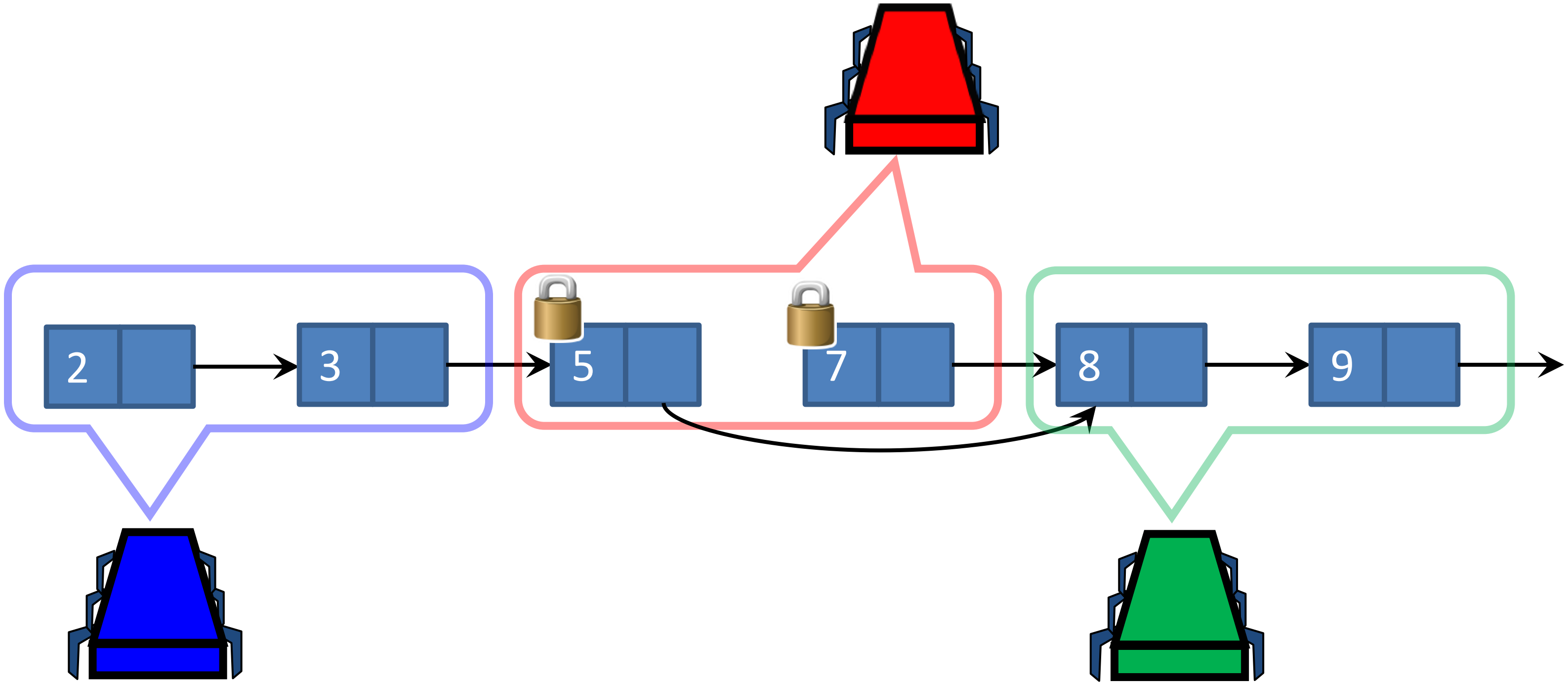

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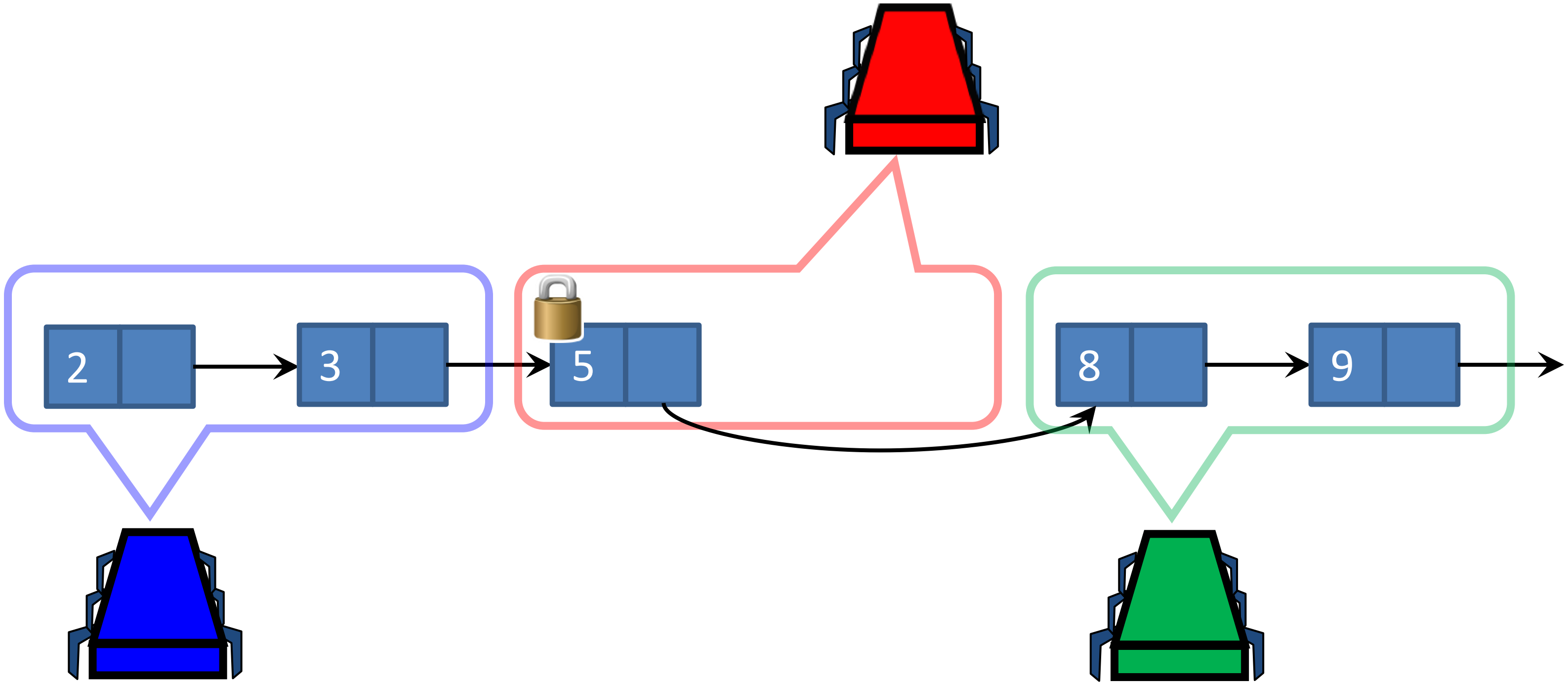

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Extensions of Separation Logic for Concurrent Programs

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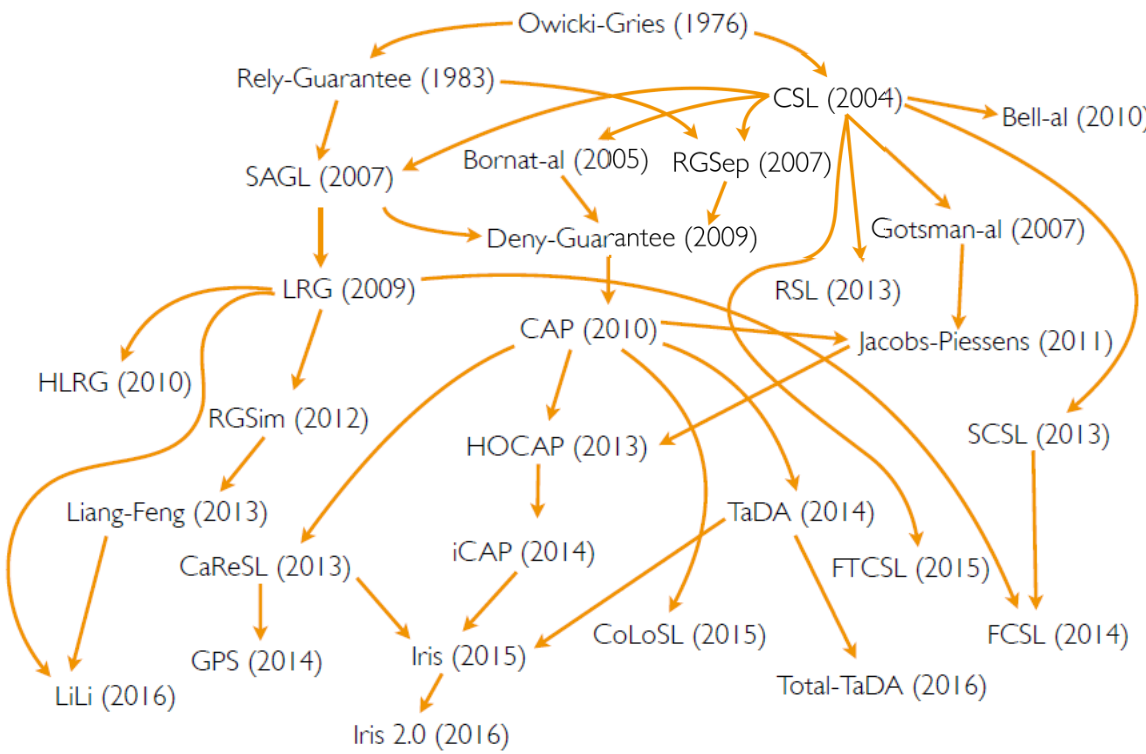
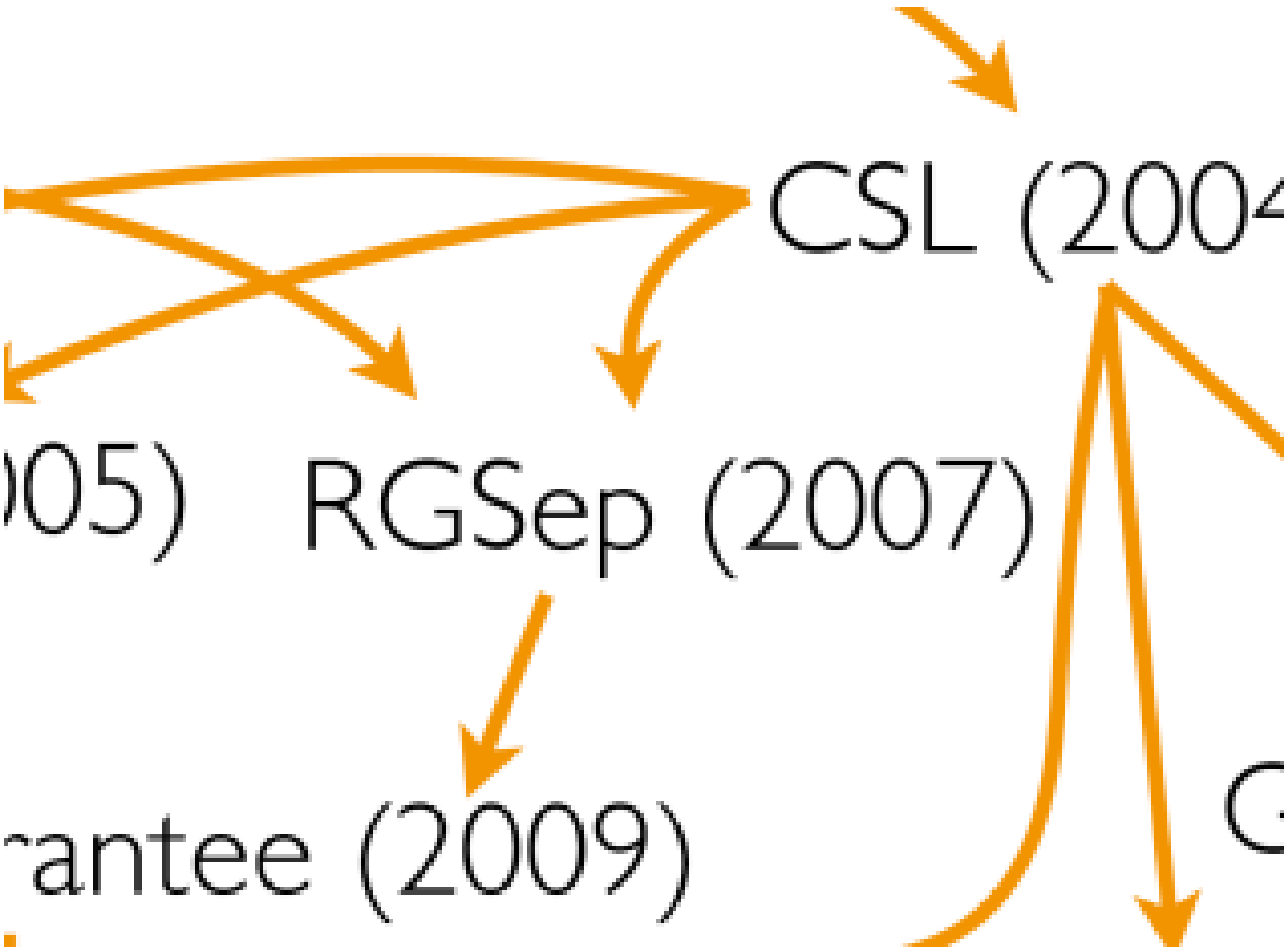


Fig. 1. CSL Family Tree (courtesy of Ilya Sergey)

Extensions of Separation Logic for Concurrent Programs



RGSep Primer

[courtesy of Viktor Vafeiadis]

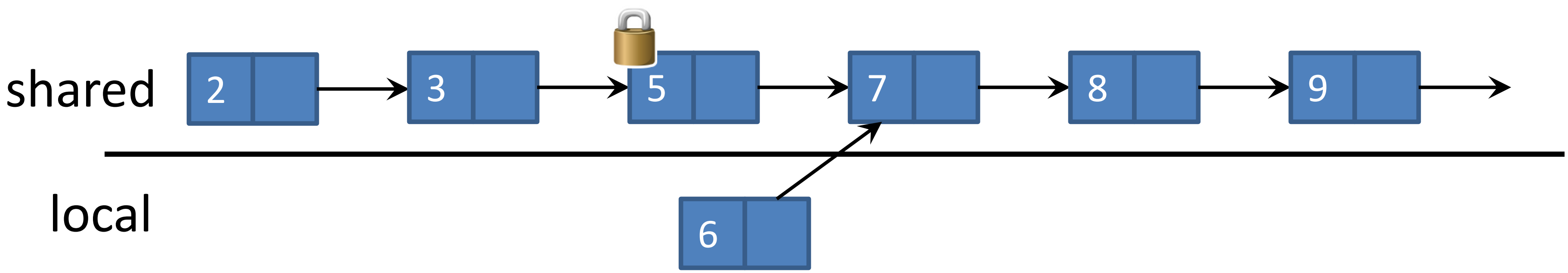
Program and Environment

- **Program:** the current thread being verified.
- **Environment:** all other threads of the system that execute in parallel with the thread being verified.
- **Interference:** The program interferes with the environment by modifying the shared state.

  Conversely, the environment interferes with the program by modifying the shared state.

Local & Shared State

- The total state is logically divided into two components:
  - ***Shared****:* accessible by all threads via synchronization
  - ***Local****:* accessible only by one thread, its owner



State of the lock-coupling list just before inserting a new node.

The node to be added is local because other threads cannot yet access it.

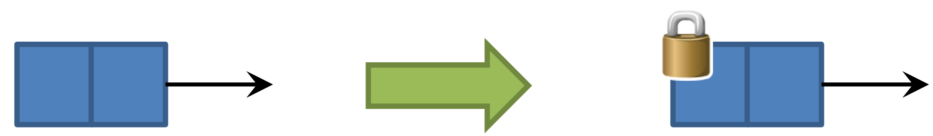
Program Specifications

- The specification of a program consists of two assertions (precondition & postcondition), and two sets of actions:
- **Rely:** Describes the interference that the program can tolerate from the environment; i.e. specifies how the environment can change the shared state.
- **Guarantee:** Describes the interference that the program imposes on its environment; i.e. specifies how the program can change the shared state.

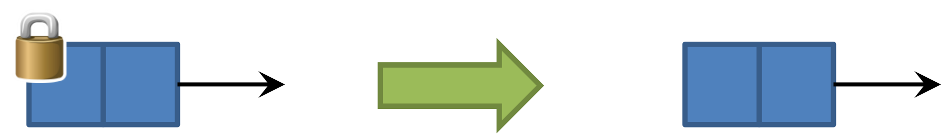
Rely/Guarantee Actions

Actions describe minimal atomic changes to the shared state.

Lock node



Unlock node

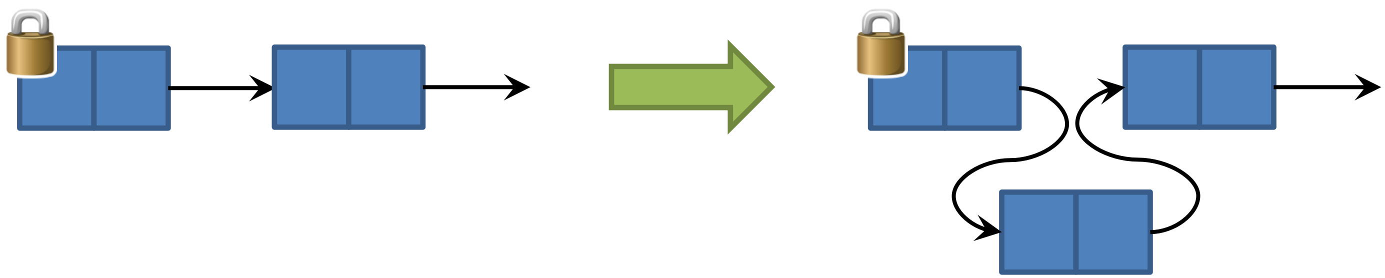


An action allows any part of the *shared state* that satisfies the LHS to be changed to a part satisfying the RHS, but the rest of the shared state must not be changed.

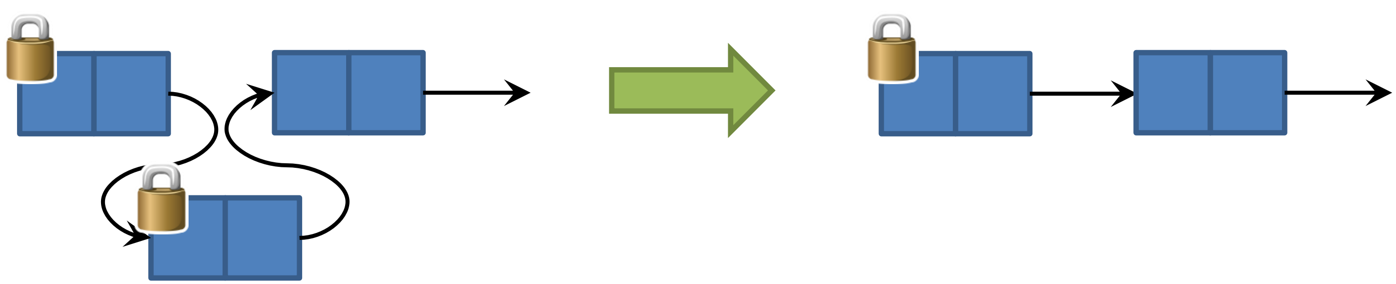
Rely/Guarantee Actions

Actions can adjust the boundary between local state and stared state.
This is also known as *transfer of ownership*.

Add node



Delete node



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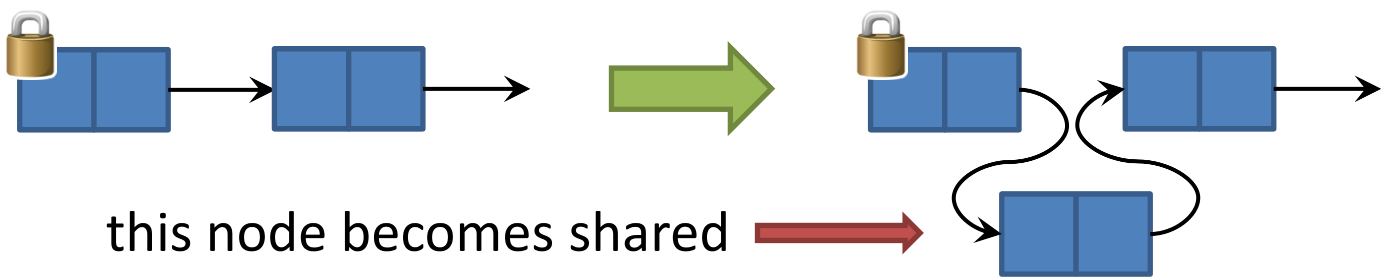
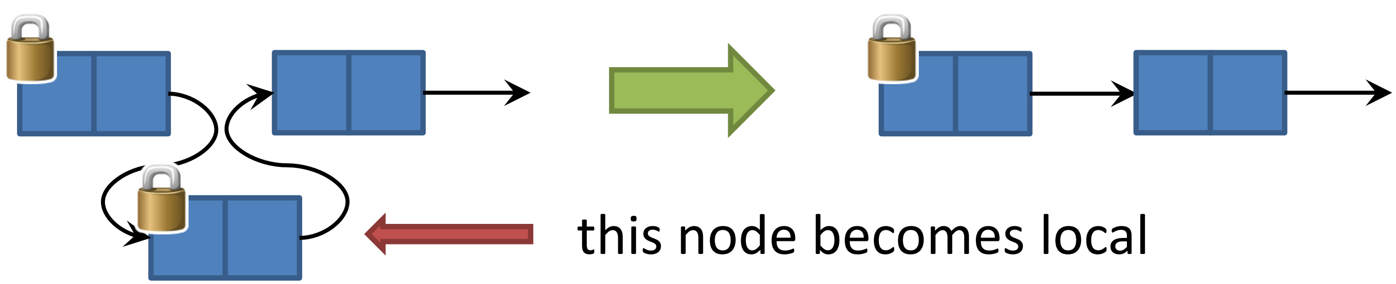
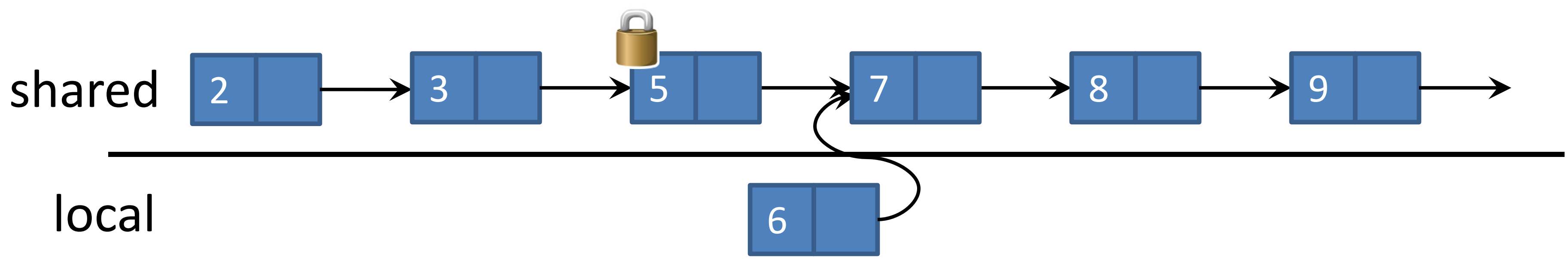


Delete node



Rely/Guarantee Actions: Lock Coupling List



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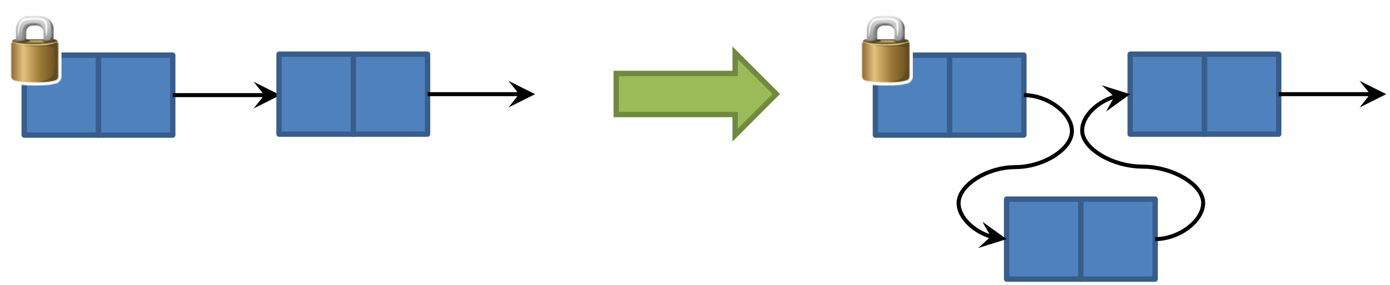

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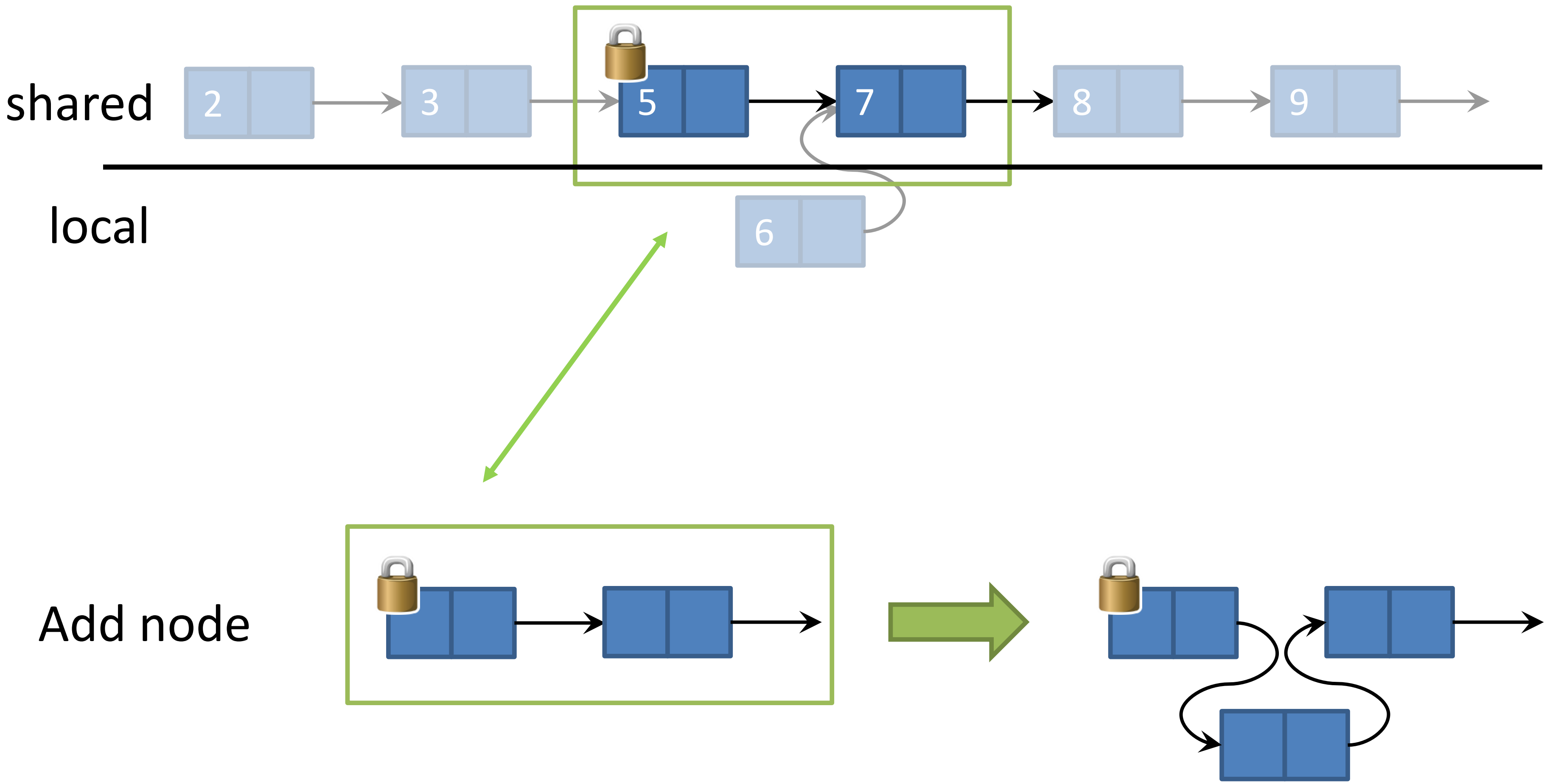

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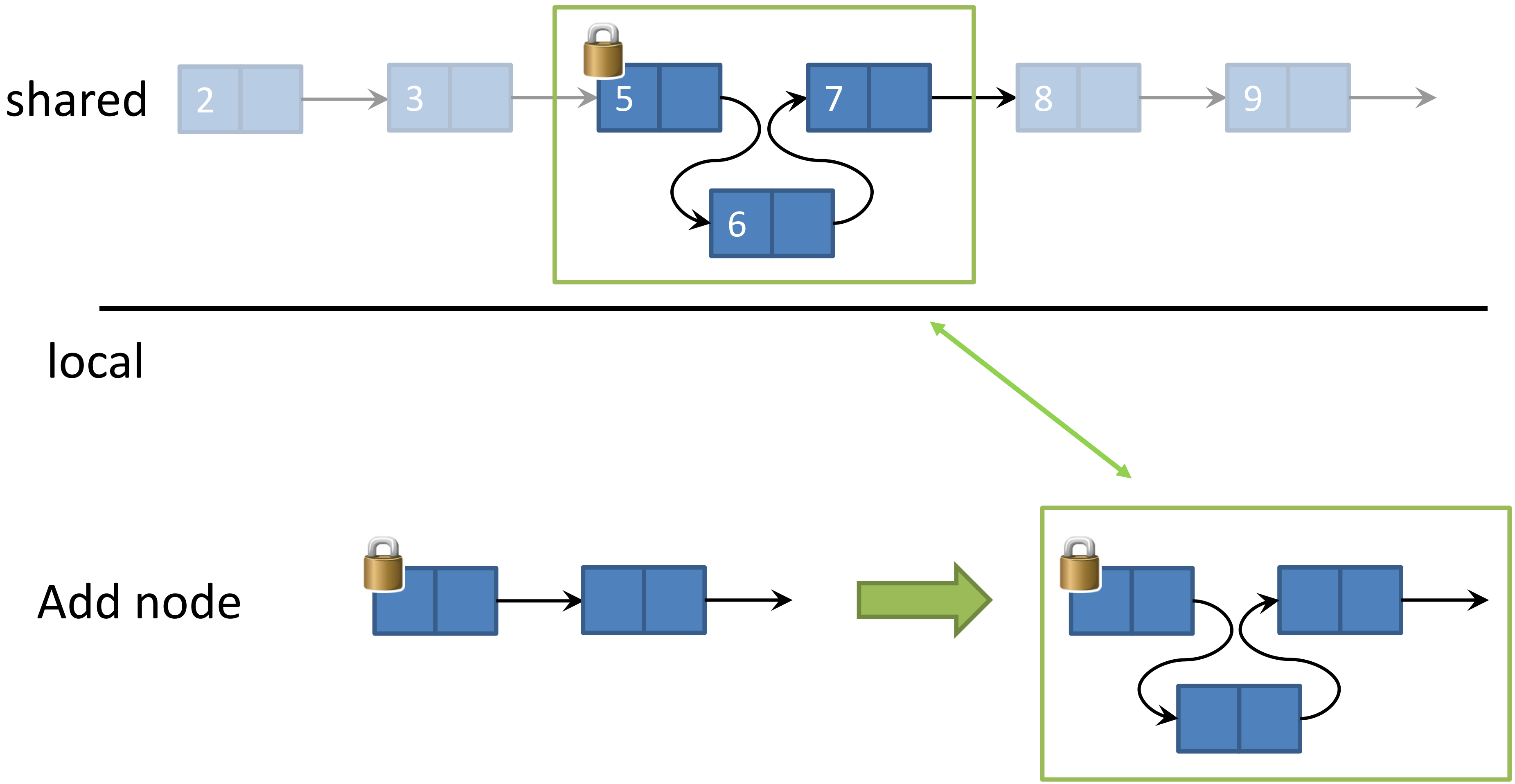
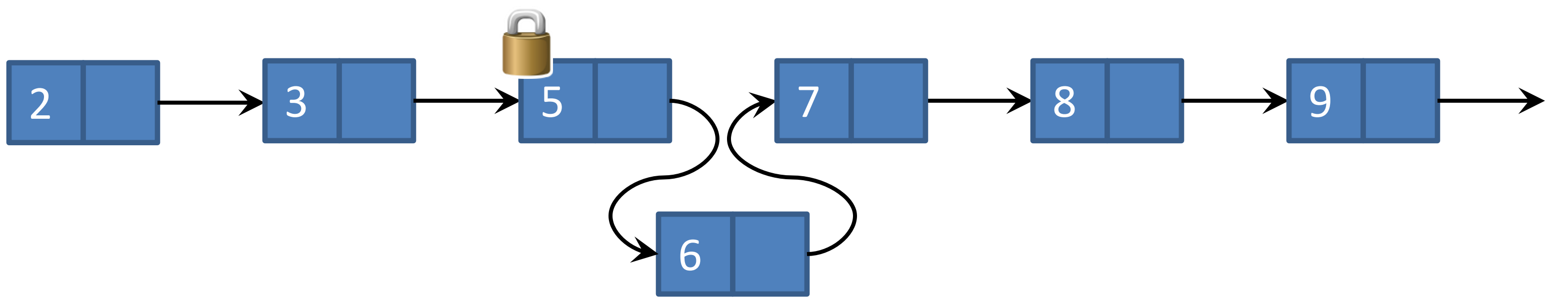


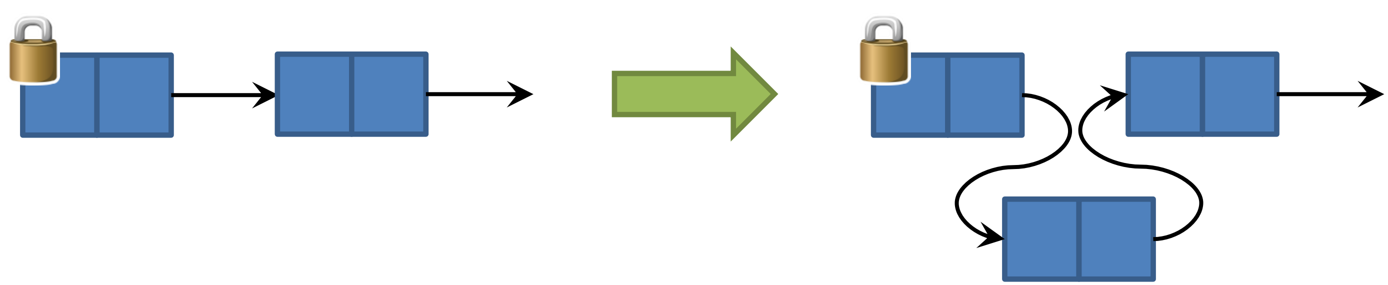
Rely/Guarantee Actions: Lock Coupling List

shared



local

Add node



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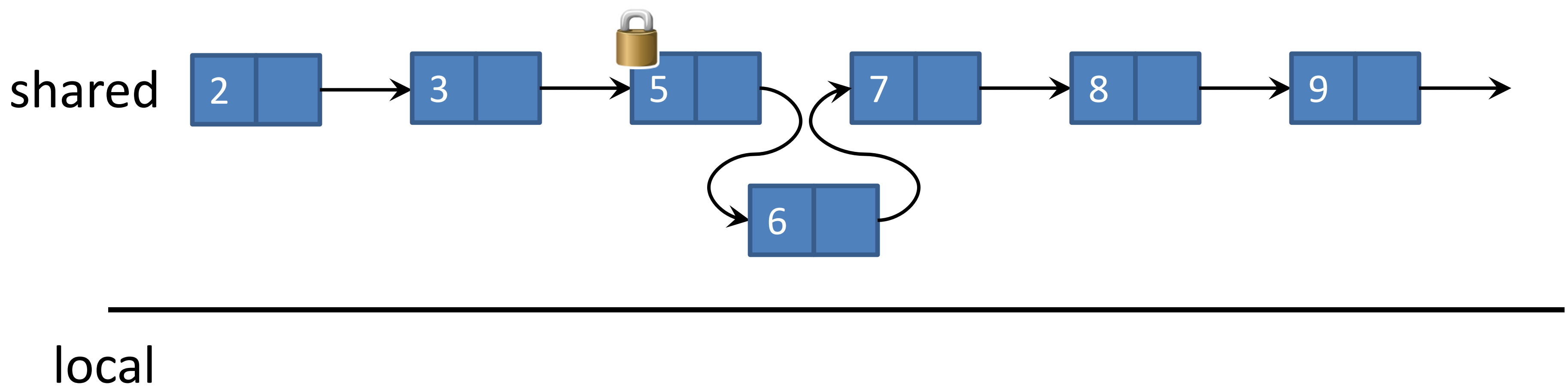

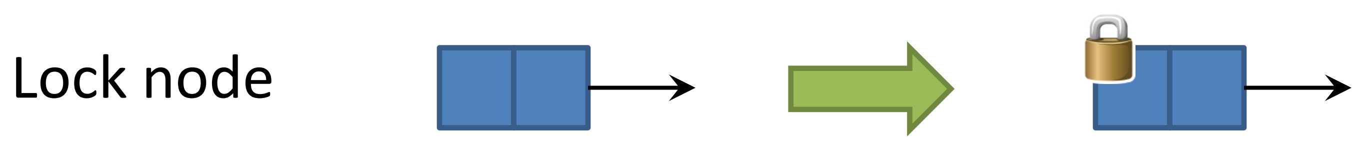

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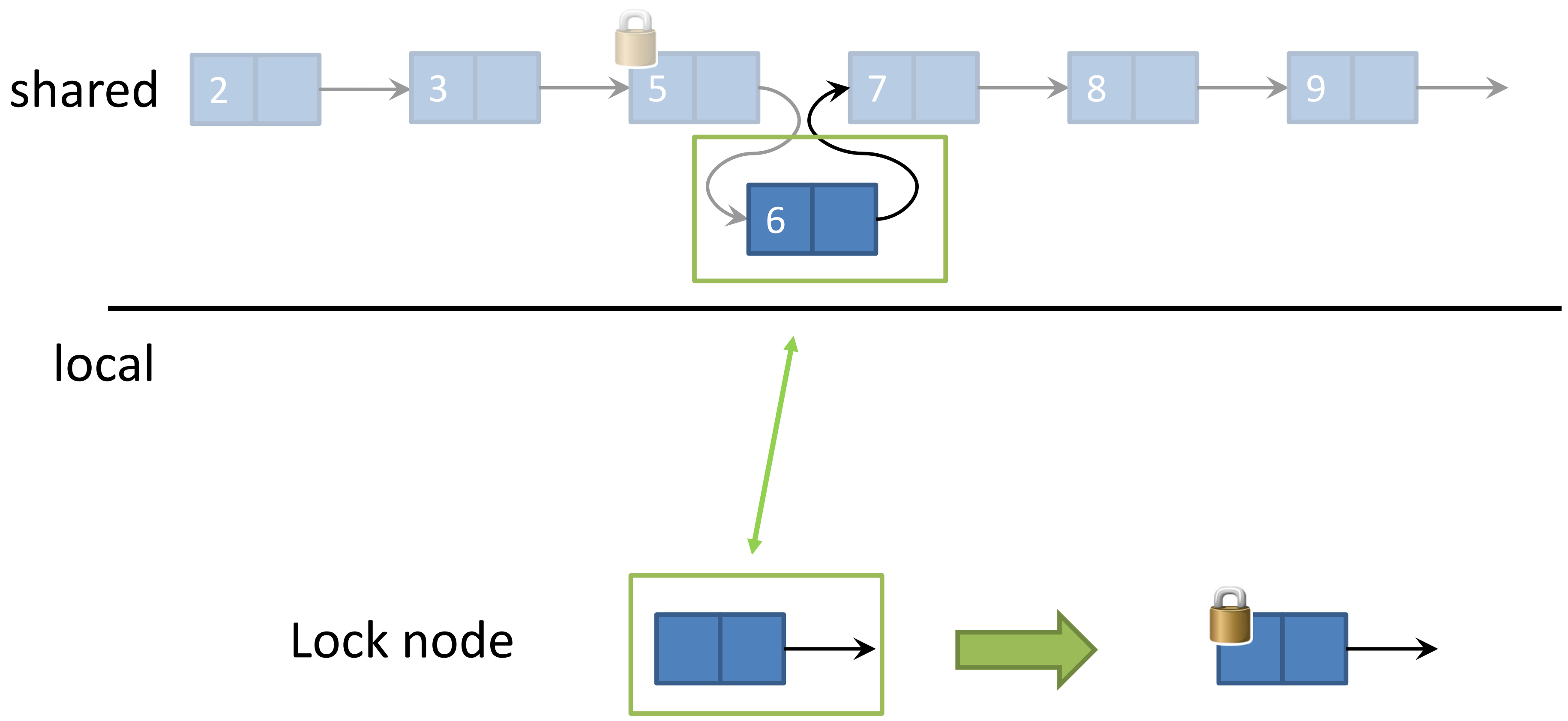

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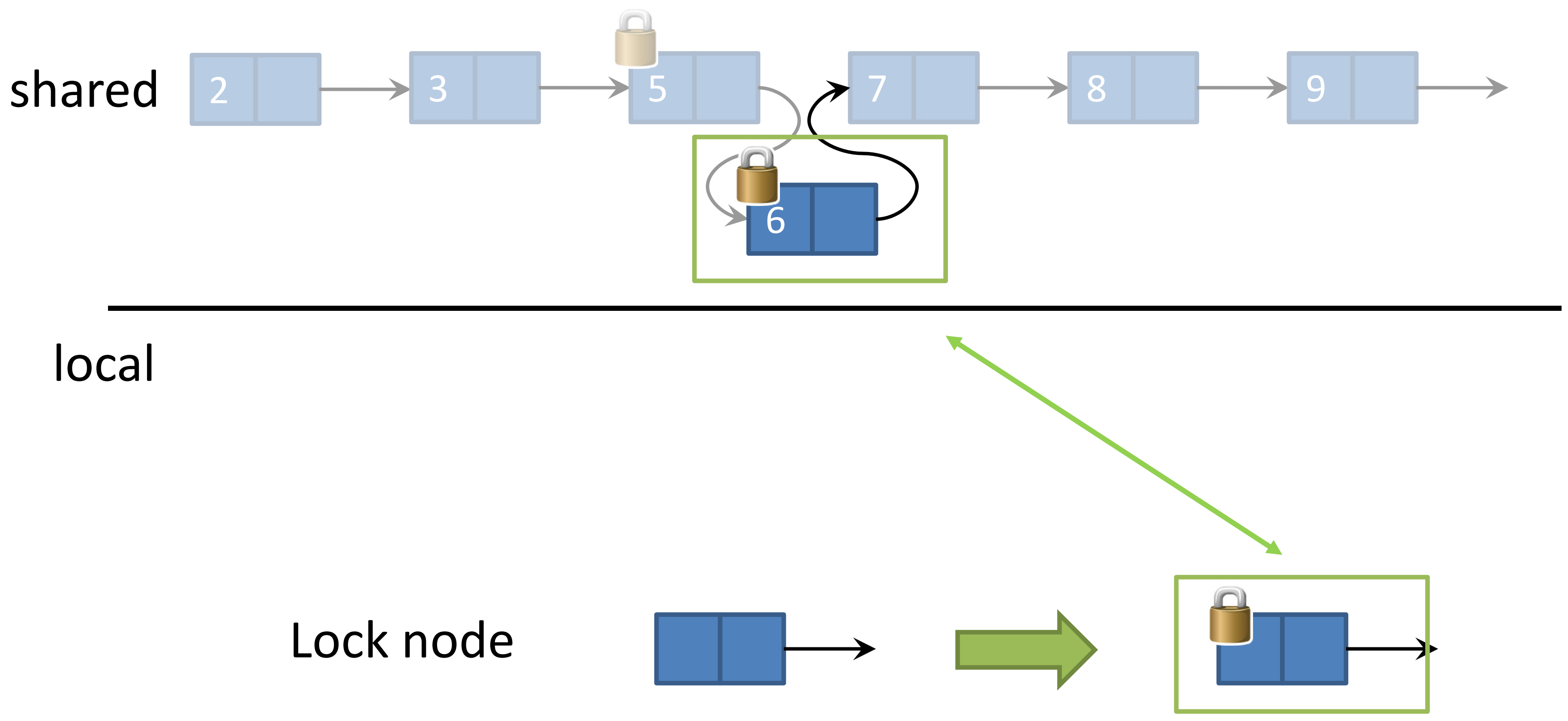
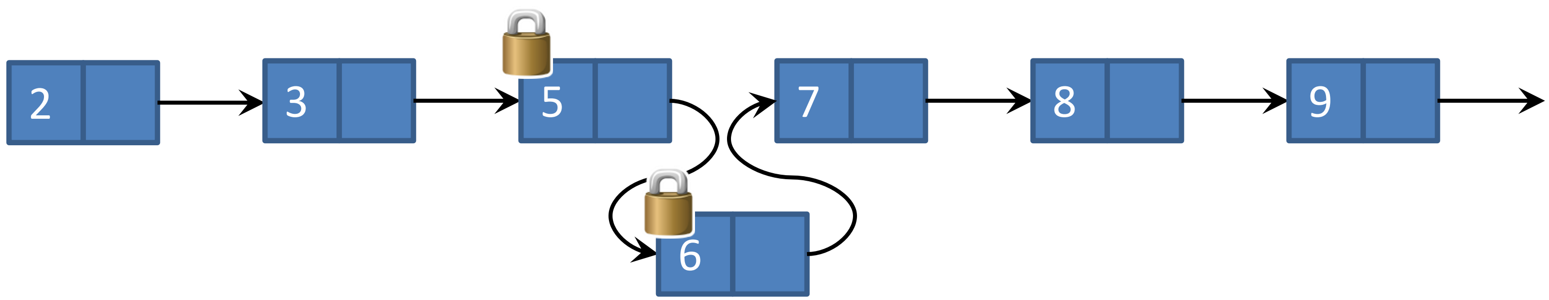


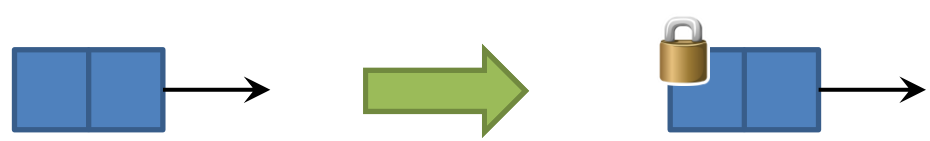
Rely/Guarantee Actions: Lock Coupling List

shared



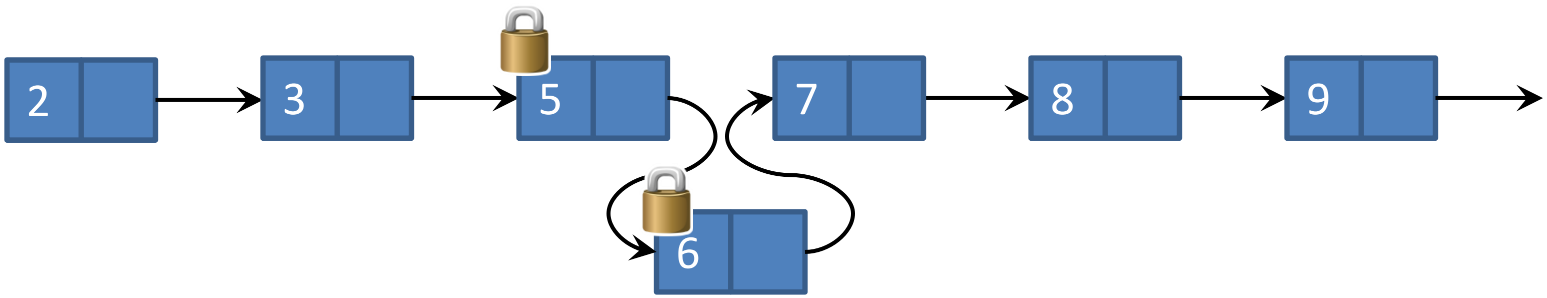
local

Lock node



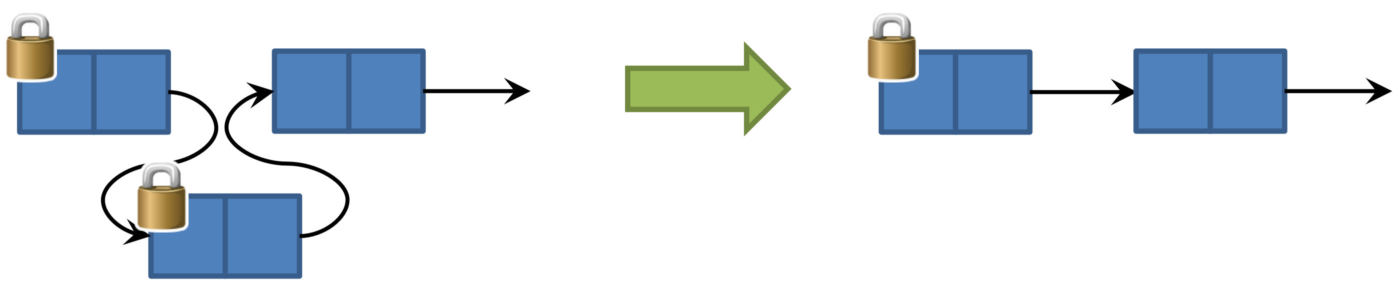
Rely/Guarantee Actions: Lock Coupling List

shared



local

Delete node



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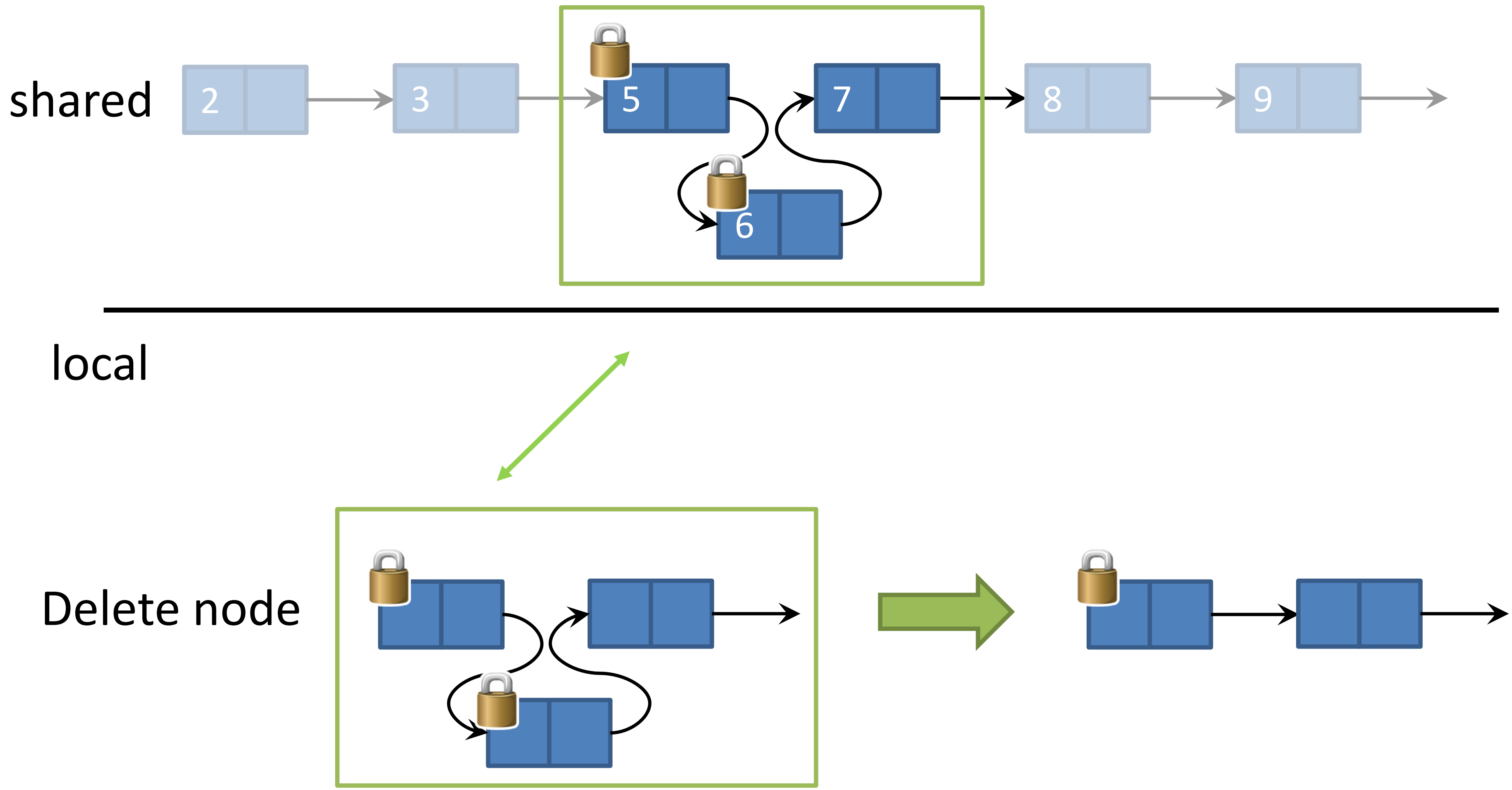

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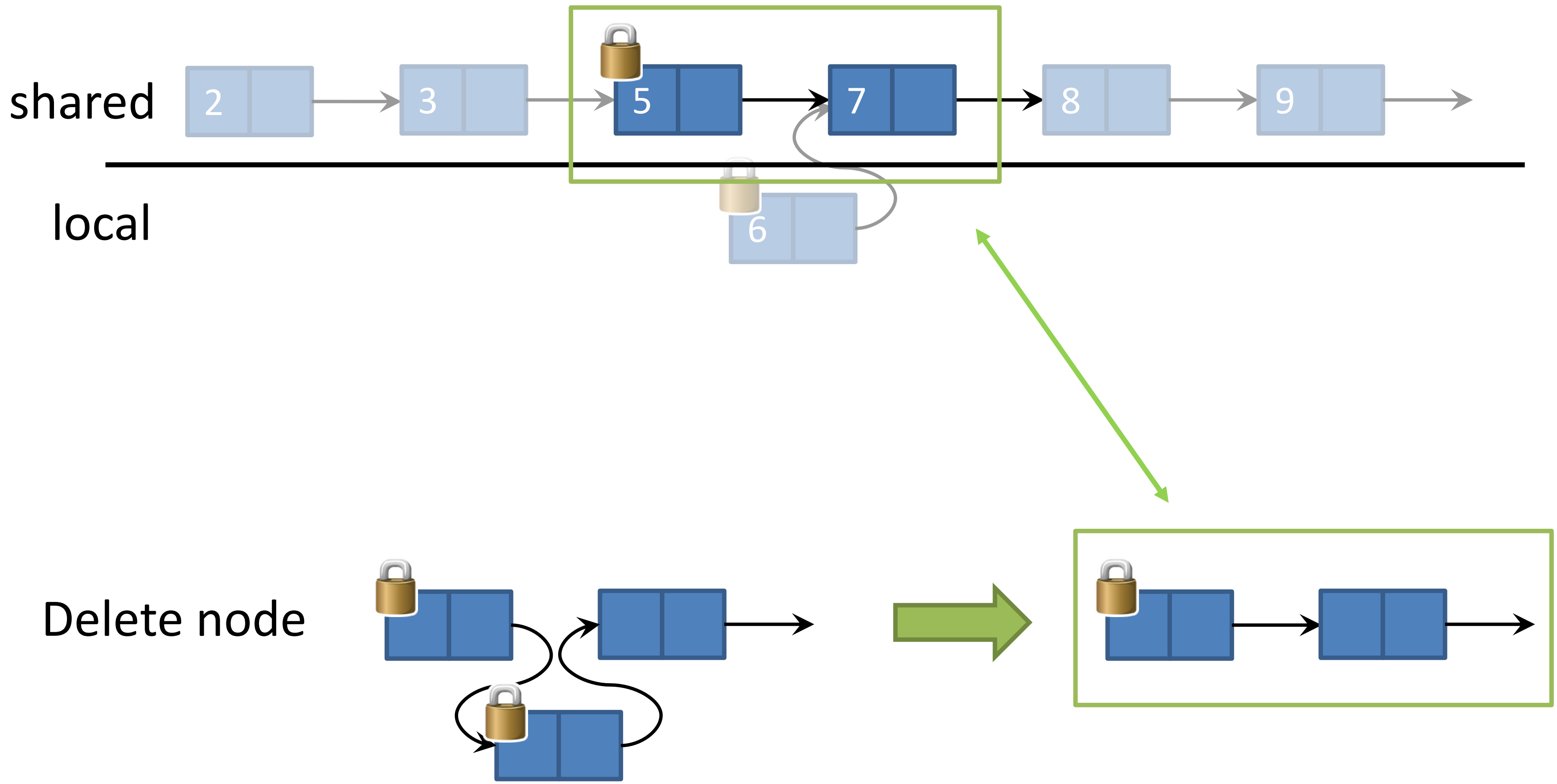

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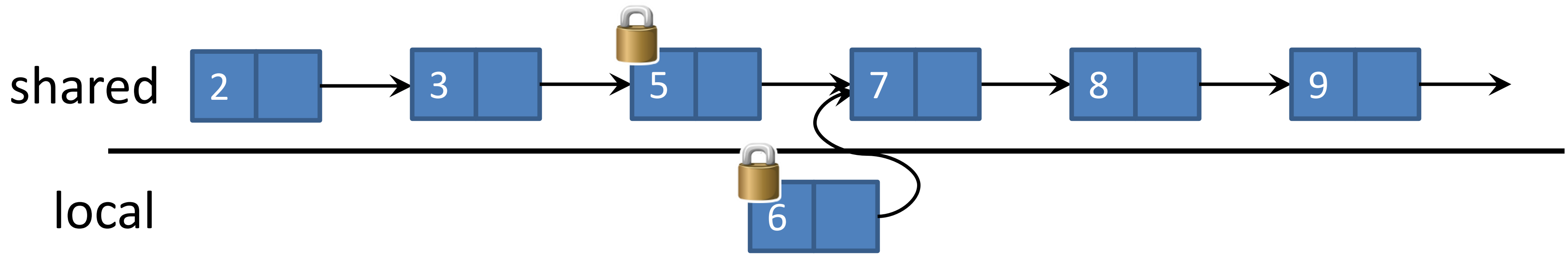

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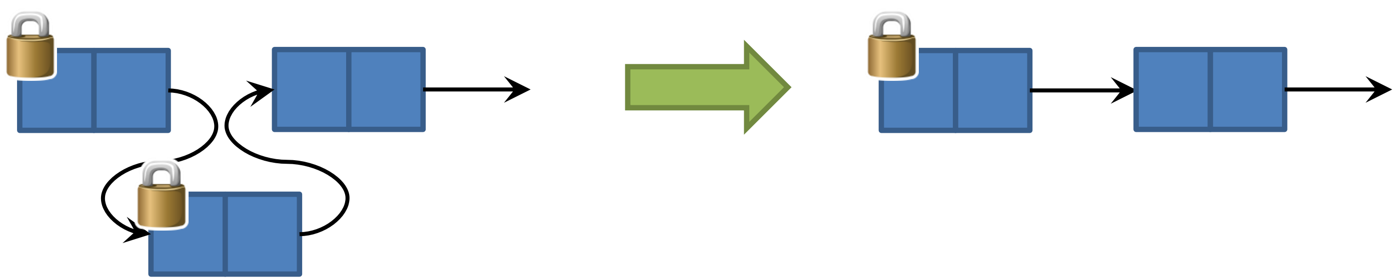

Assertion Syntax

- Separation Logic

$$P, Q ::= e = e \mid e \neq e \mid e \mapsto (\mathbf{f}: \mathbf{e}) \mid P * Q \mid \ldots$$

- Extended Logic

$$p, q ::= P \mid \boxed{P} \mid p * q \mid \ldots$$

local

shared

Assertion Semantics

- $l, s \vDash P \quad \Leftrightarrow \quad l \vDash_{SL} P$
- $l, s \vDash \boxed{P} \quad \Leftrightarrow \quad s \vDash_{SL} P$ and $l = \{\}$
- $l, s \vDash p * q \quad \Leftrightarrow$ exists l_1, l_2 :
  $l = l_1 \bullet l_2$ and $l_1, s \vDash p$ and $l_2, s \vDash q$

Assertion Semantics

- $l, s \models P \quad \Leftrightarrow \quad l \models_{SL} P$
- $l, s \models \boxed{P} \quad \Leftrightarrow \quad s \models_{SL} P$ and $l = \{\}$
- $l, s \models p * q \quad \Leftrightarrow \quad$ exists l_1, l_2 :
  $l = l_1 \bullet l_2$ and $l_1, s \models p$ and $l_2, s \models q$

split local state

Assertion Semantics

- $l, s \vDash P \quad\Leftrightarrow\quad l \vDash_{SL} P$
- $l, s \vDash \boxed{P} \quad\Leftrightarrow\quad s \vDash_{SL} P \text{ and } l = \{\}$
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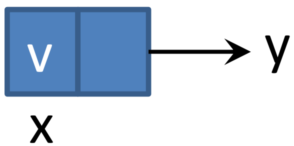
share global state

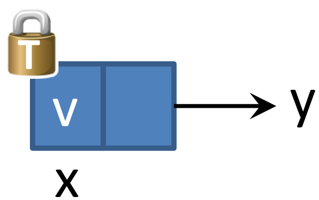
Assertions: Lock Coupling List

Unlocked node x holding value v and pointing to y



$x \mapsto (0, v, y)$

Node x holding value v and pointing to y, locked by thread T

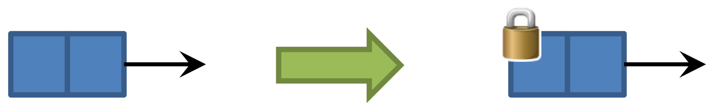


$x \mapsto (T, v, y)$

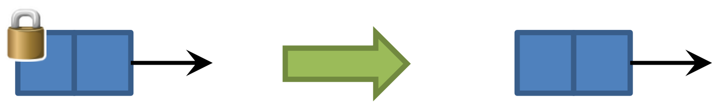
List segment from x to y of possibly locked nodes

`lseg`(x, y)

Rely/Guarantee Actions: Lock Coupling List



$x \mapsto (0, v, y) \rightarrow x \mapsto (T, v, y)$



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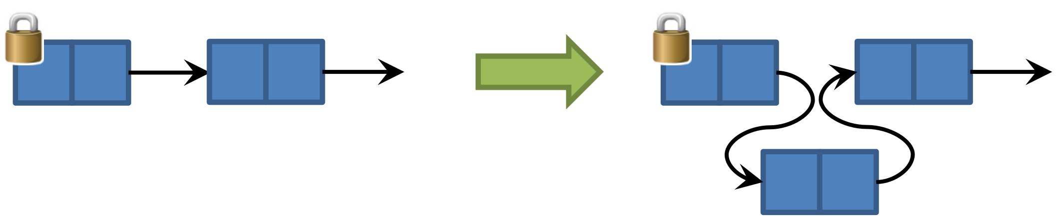
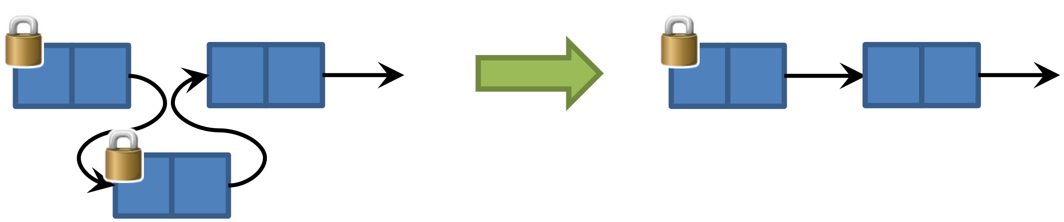


$x \mapsto (T, v, y) \rightarrow x \mapsto (T, v, z) * z \mapsto (0, w, y)$



$x \mapsto (T, v, z) * z \mapsto (T, w, y) \rightarrow x \mapsto (T, v, y)$

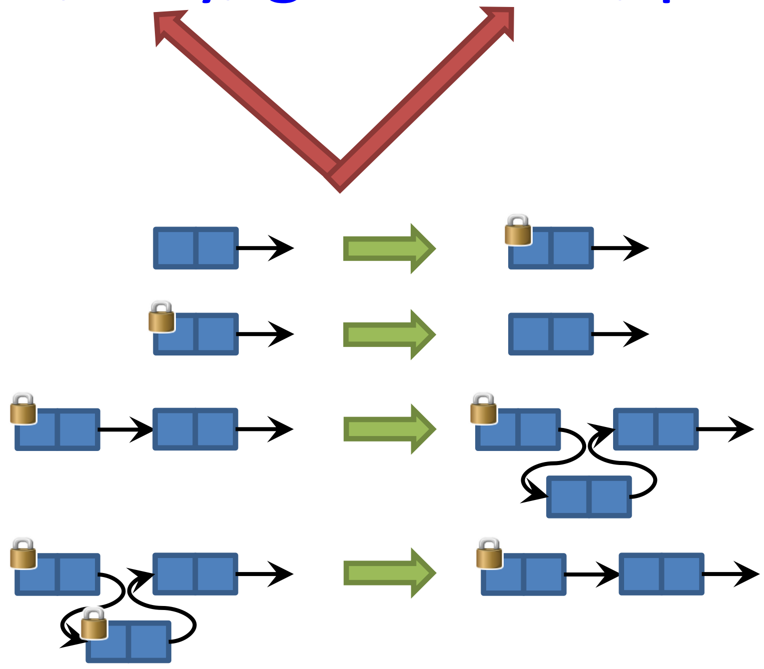
Programs: Syntax

- Basic commands c:
  - noop: **skip**
  - guard: **assume**(b)
  - heap write: [x] := y
  - heap read: x := [y]
  - allocation: x := **new**()
  - deallocation: **free**(x)
  - …
- Commands $C \in$ Com:
  - basic commands: c
  - seq. composition: C_1; C_2
  - nondet. choice: $C_1 + C_2$
  - looping: C*
  - atomic com.: **atomic** C
  - par. composition: $C_1 \mid C_2$

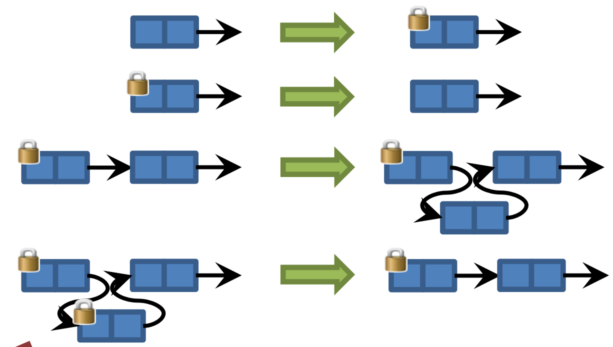
Rely/Guarantee Judgements

$\vdash$ C **sat** (p, R, G, q)

(precondition, rely, guarantee, postcondition)



Parallel Composition Rule



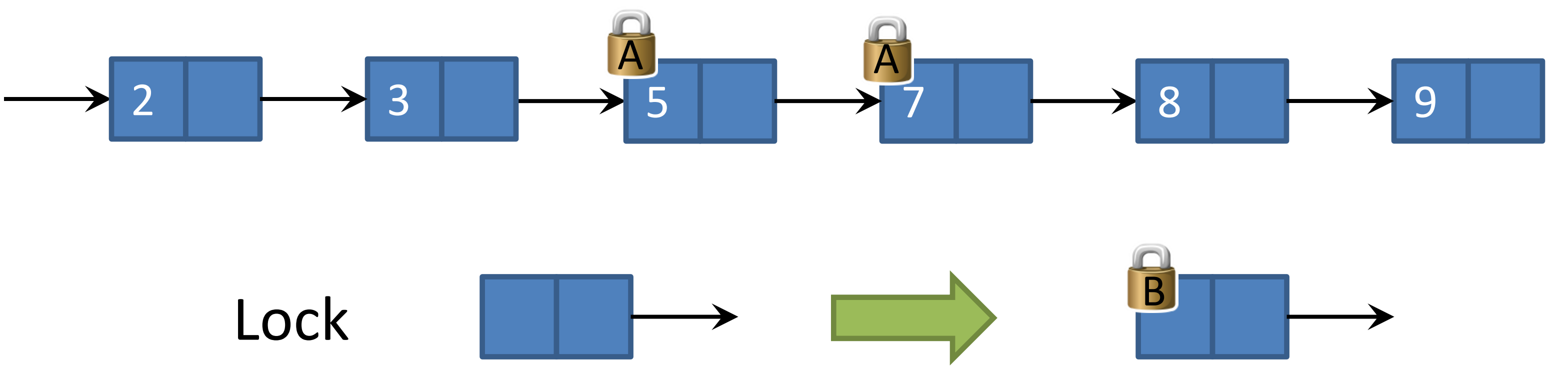
$$\vdash C_1 \ \mathbf{sat}\ (p_1, R \cup G_2, G_1, q_1)$$

$$\vdash C_2 \ \mathbf{sat}\ (p_2, R \cup G_1, G_2, q_2)$$

$$\overline{\vdash (C_1 \mid C_2)\ \mathbf{sat}\ (p_1 * p_2, R, G_1 \cup G_2, q_1 * q_2)}$$

Stability

- An assertion is *stable* iff it is preserved under interference by other threads.
- **Example**:



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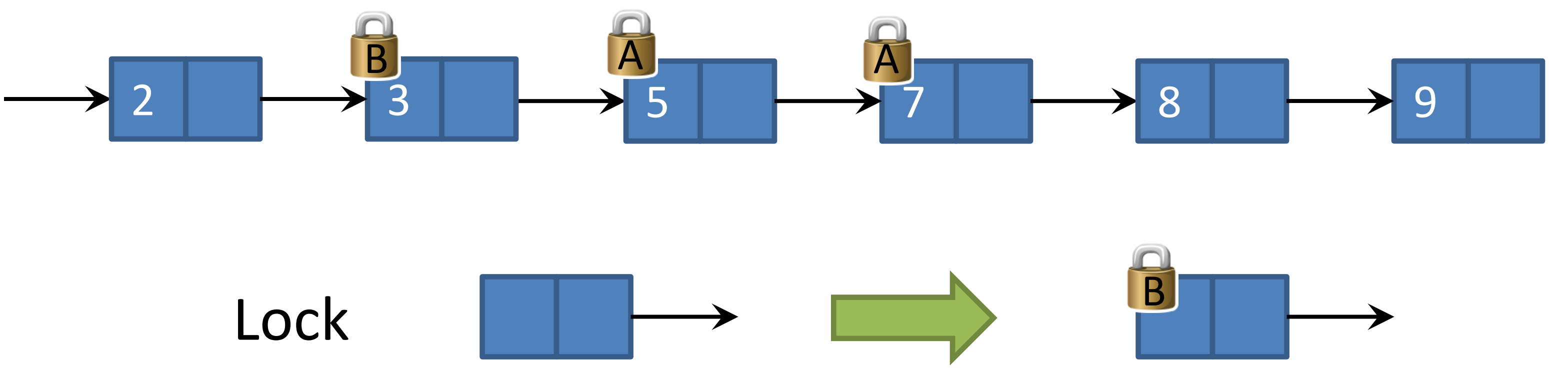

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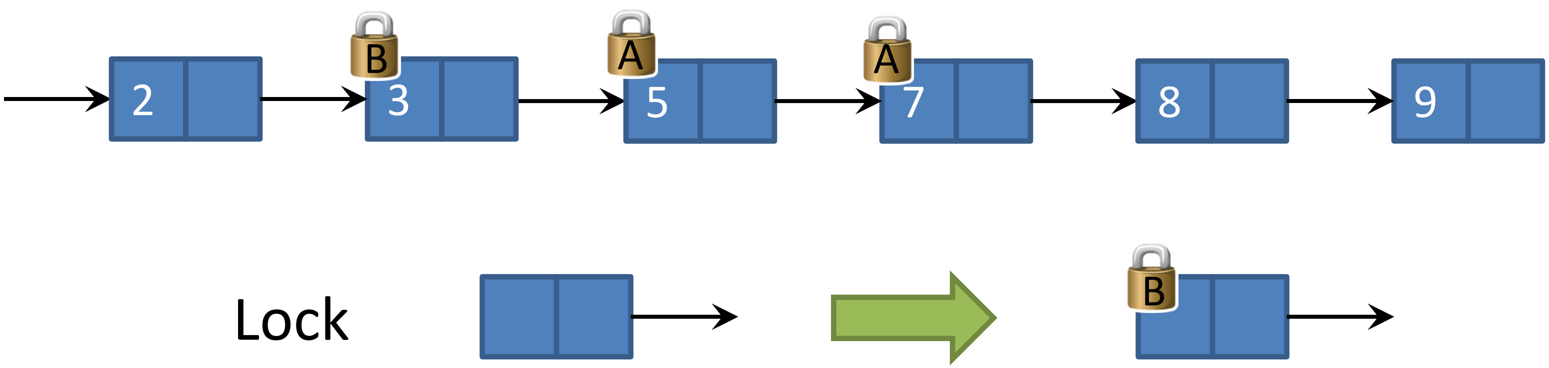

not stable!

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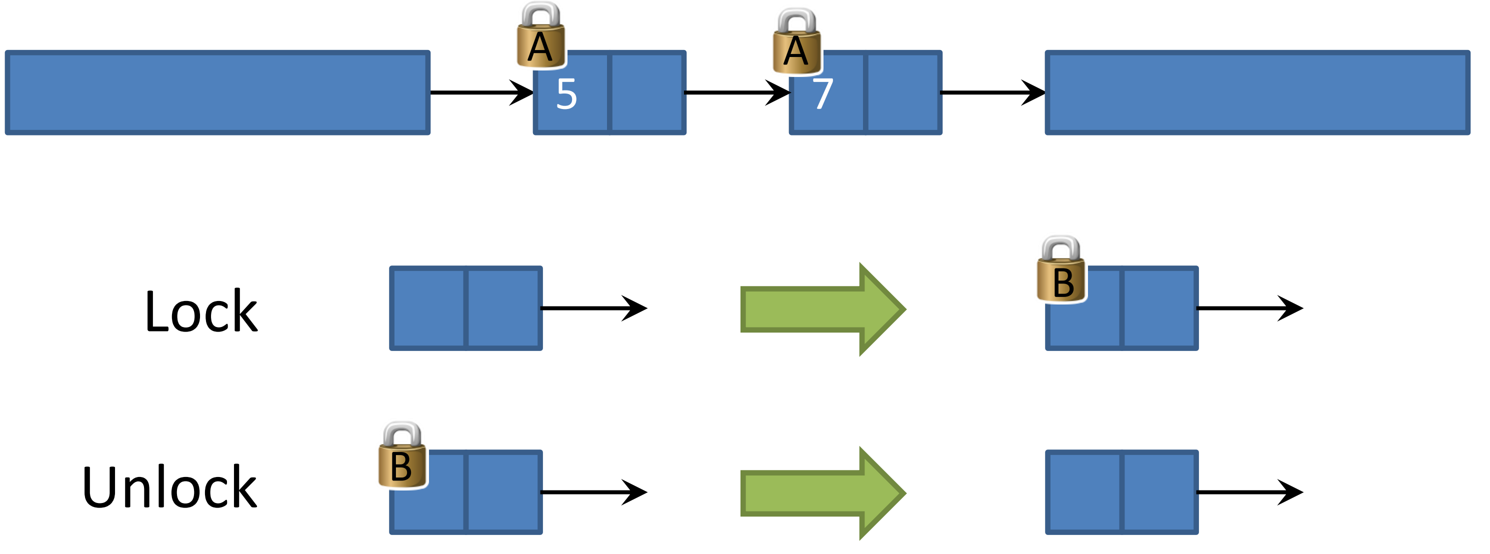

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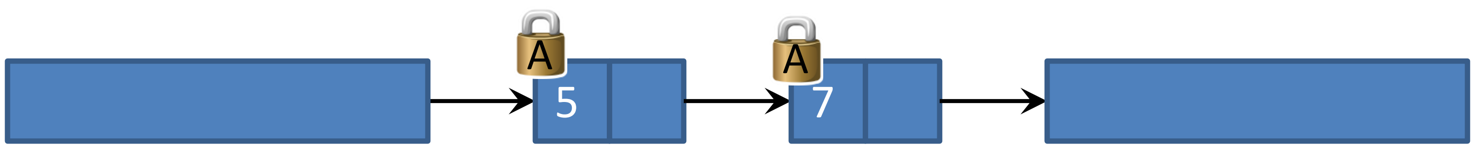
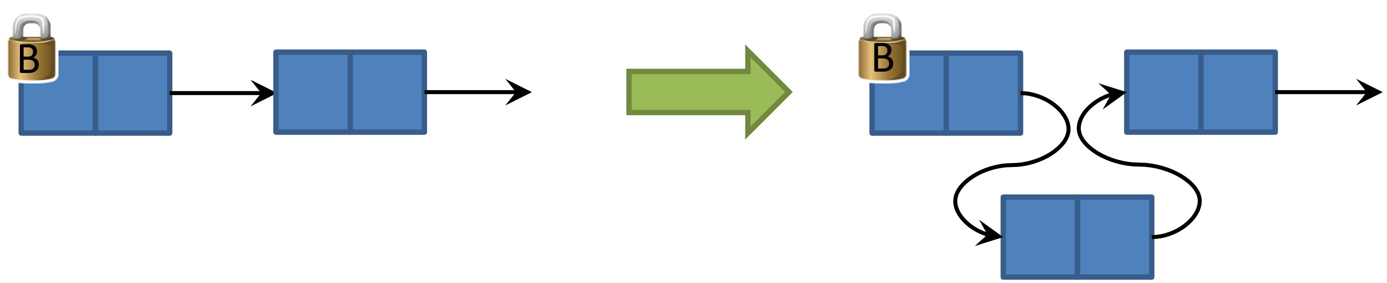


Add



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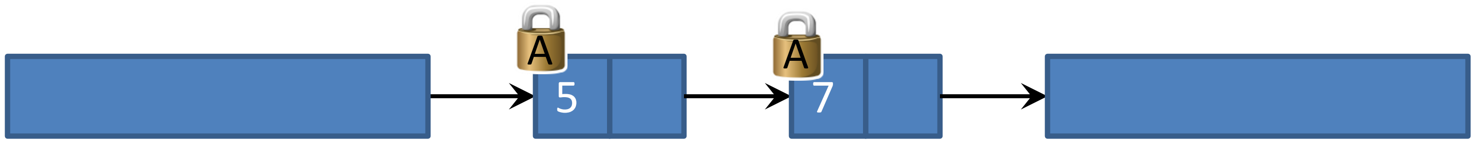

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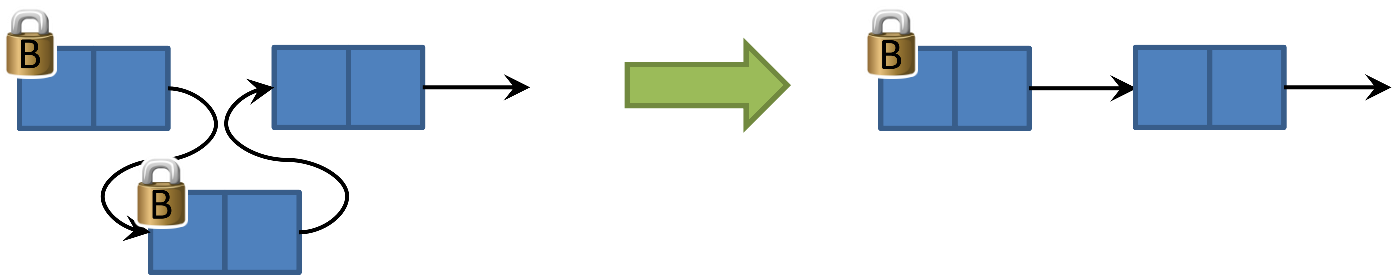

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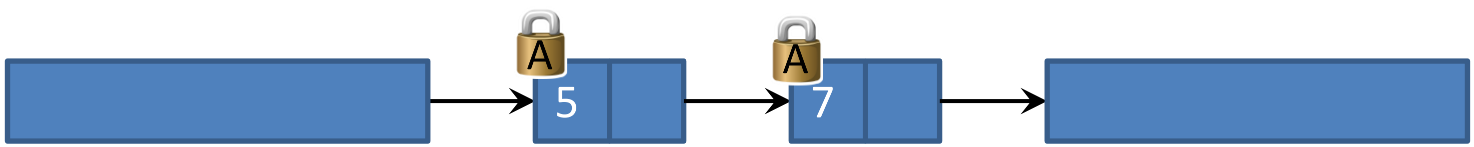

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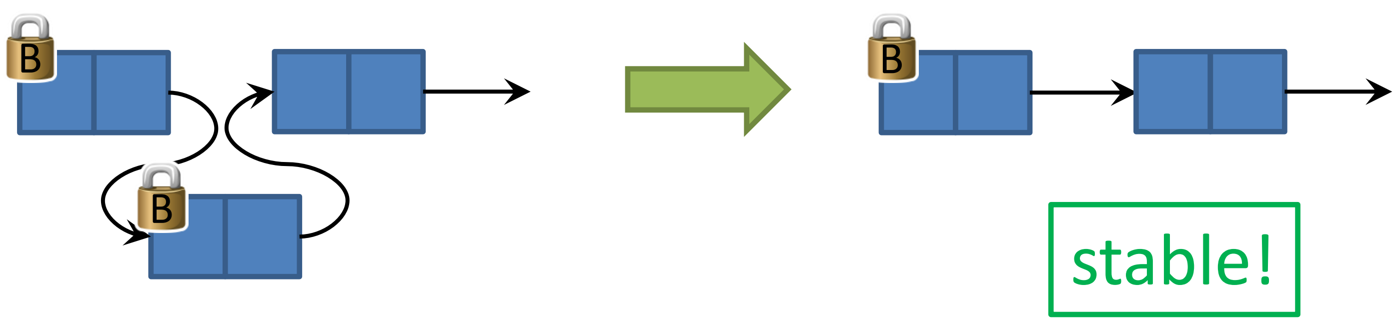
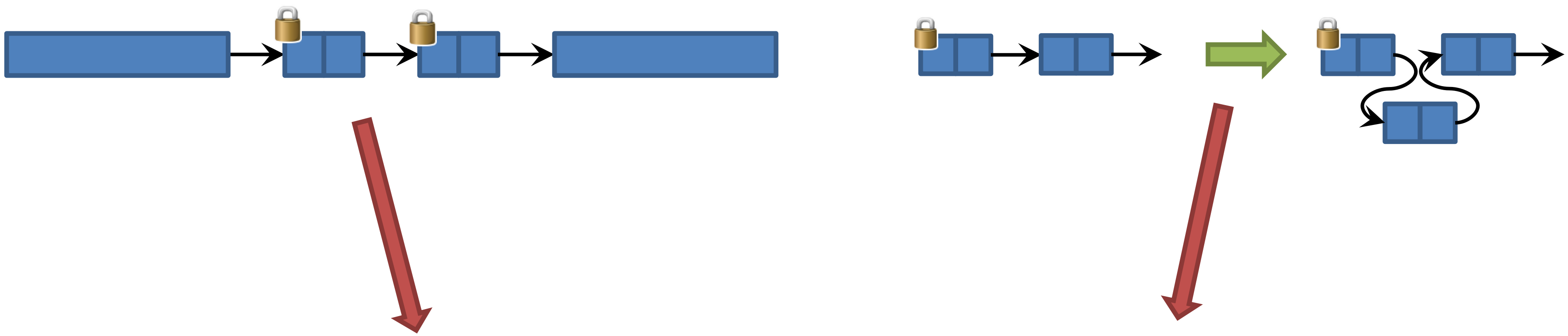


Stability (Formally)



$\boxed{S}$ stable under $P \rightarrow Q$

iff

$(P \mathbin{-\!\circledast} S) * Q \models S$

where $P \mathbin{-\!\circledast} S \;:=\; \neg\,(\neg\, P \mathbin{-\!*} \neg\, S)$

Atomic Commands

$$\frac{\vdash \{\ P\ \}\ \mathsf{C}\ \{\ Q\ \}}{\vdash (\mathtt{atomic}\ \mathsf{C})\ \mathbf{sat}\ (P, \emptyset, \emptyset, Q)}$$

Atomic Commands

reduction to sequential SL

$$\frac{\vdash \{\, P \,\}\; C \;\{\, Q \,\}}{\vdash (\mathbf{atomic}\; C)\; \mathbf{sat}\; (P, \emptyset, \emptyset, Q)}$$

only local state

Atomic Commands

$$\frac{\vdash \{\, P \,\}\ C\ \{\, Q \,\}}{\vdash (\texttt{atomic}\ C)\ \textbf{sat}\ (P, \emptyset, \emptyset, Q)}$$

$$\frac{p, q \text{ stable under } R \qquad \vdash (\texttt{atomic}\ C)\ \textbf{sat}\ (p, \emptyset, G, q)}{\vdash (\texttt{atomic}\ C)\ \textbf{sat}\ (p, R, G, q)}$$

Atomic Commands

$$\frac{P_2, Q_2 \text{ precise} \qquad P_2 \to Q_2 \in G \qquad \vdash (\texttt{atomic}\ C)\ \textbf{sat}\ (P_1 * P_2, \emptyset, \emptyset, Q_1 * Q_2)}{\vdash (\texttt{atomic}\ C)\ \textbf{sat}\ (P_1 * \boxed{P_2 * F}, \emptyset, G, Q_1 * \boxed{Q_2 * F})}$$

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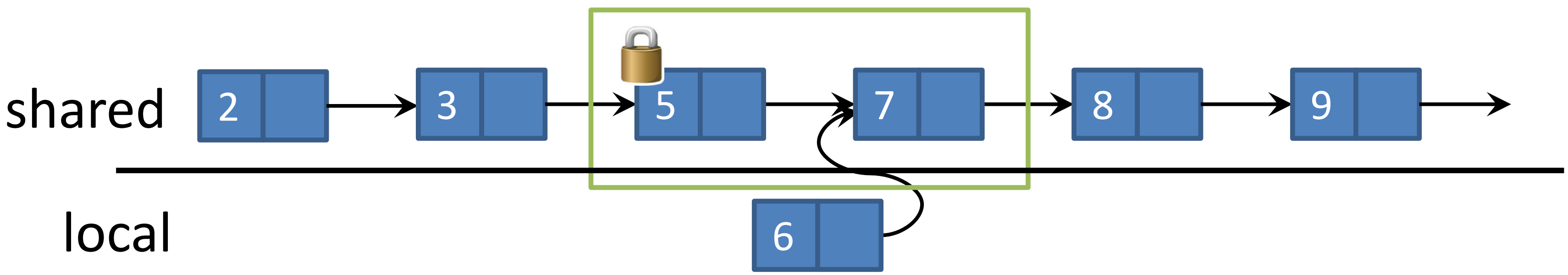
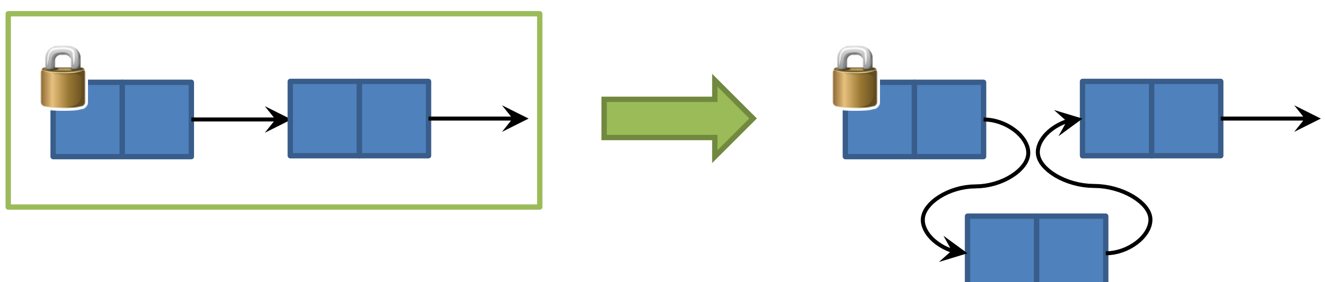


Add node



Atomic Commands

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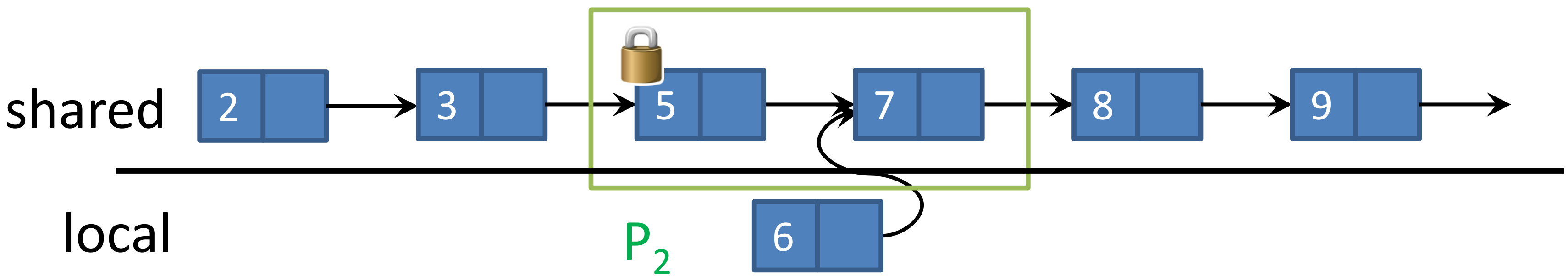
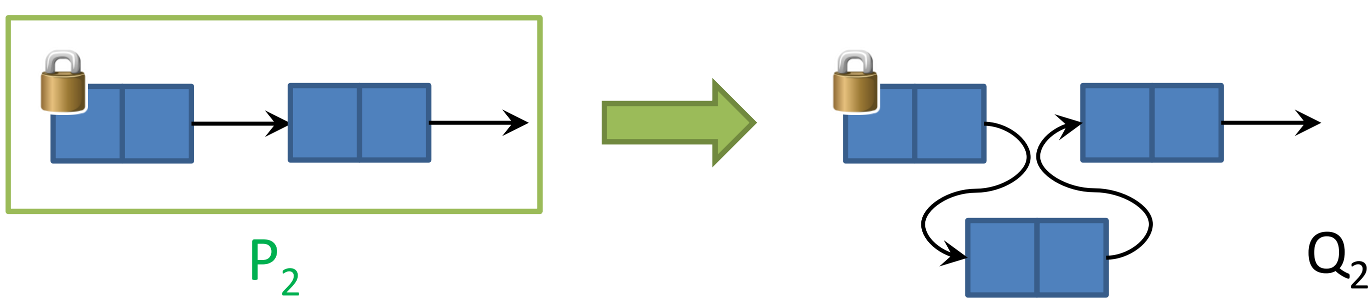
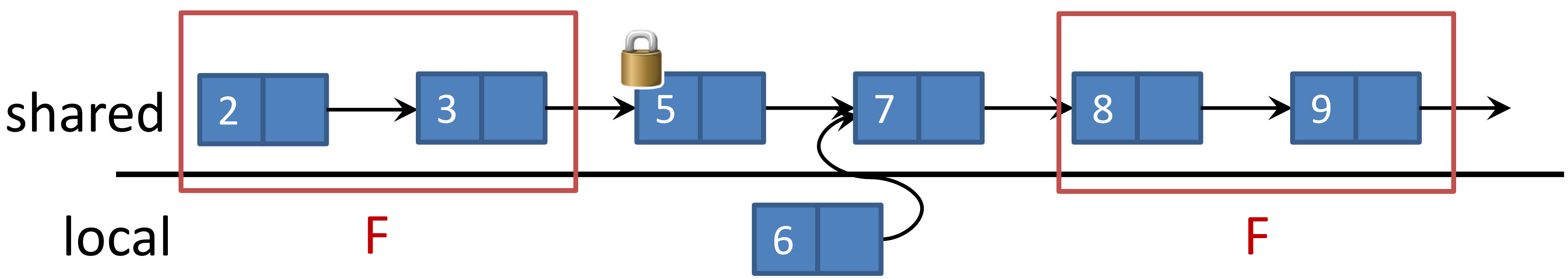
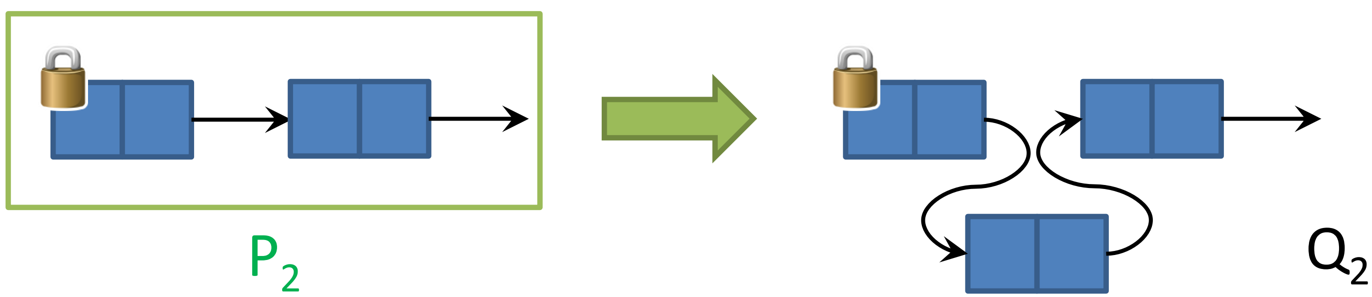




Atomic Commands

$$\frac{\begin{array}{c} P_2, Q_2 \text{ precise} \qquad P_2 \to Q_2 \in G \\ \vdash (\mathbf{atomic}\ C)\ \mathbf{sat}\ (P_1 * P_2, \emptyset, \emptyset, Q_1 * Q_2) \end{array}}{\vdash (\mathbf{atomic}\ C)\ \mathbf{sat}\ (P_1 * \boxed{P_2 * F}, \emptyset, G, Q_1 * \boxed{Q_2 * F})}$$





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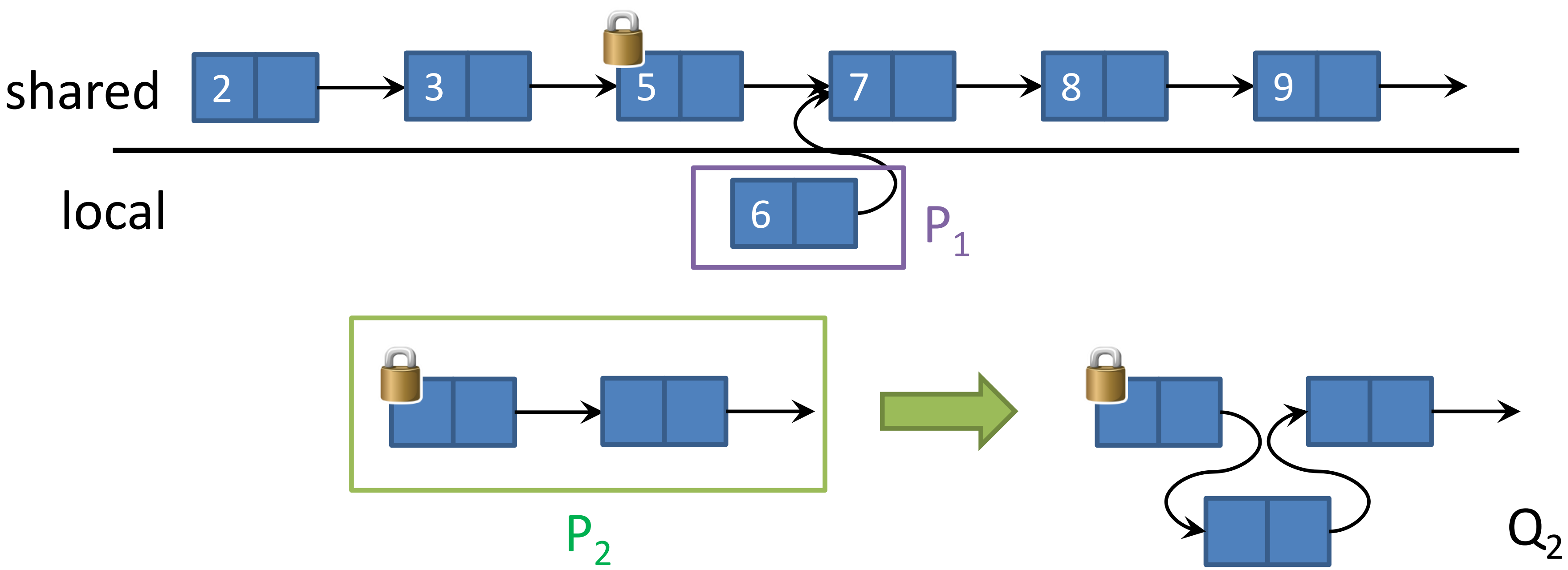

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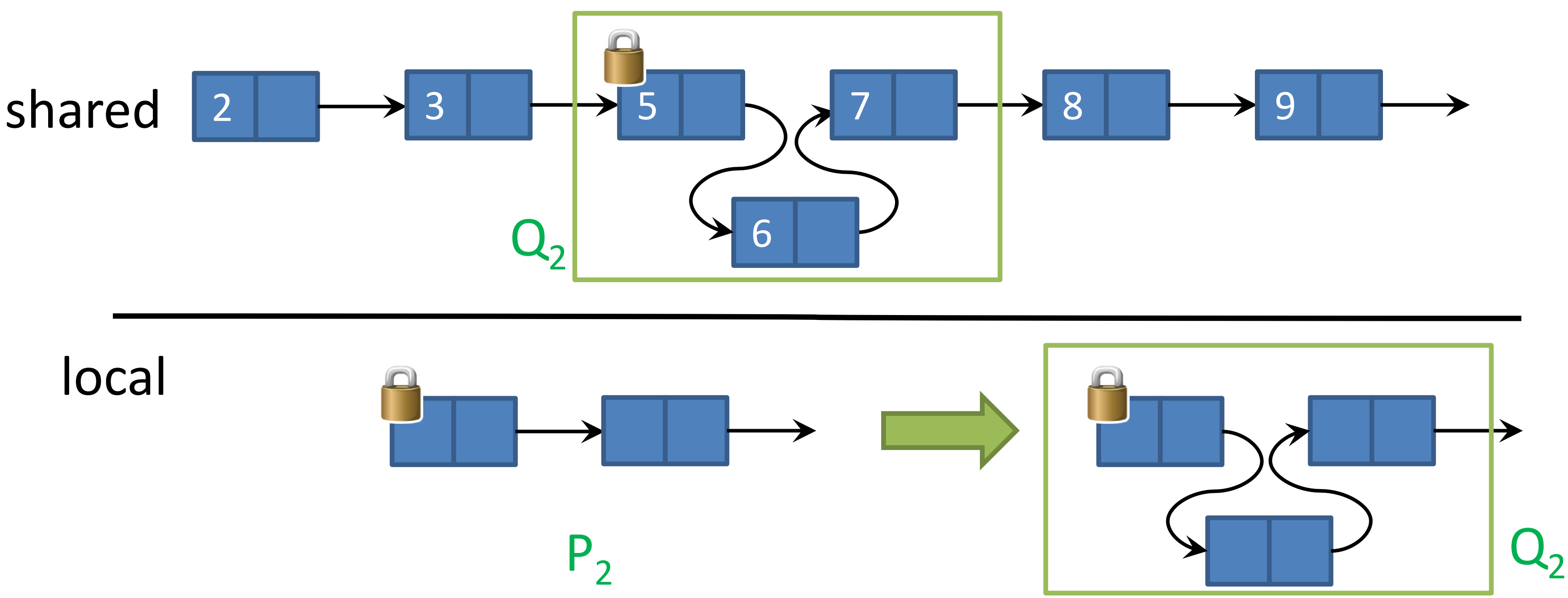

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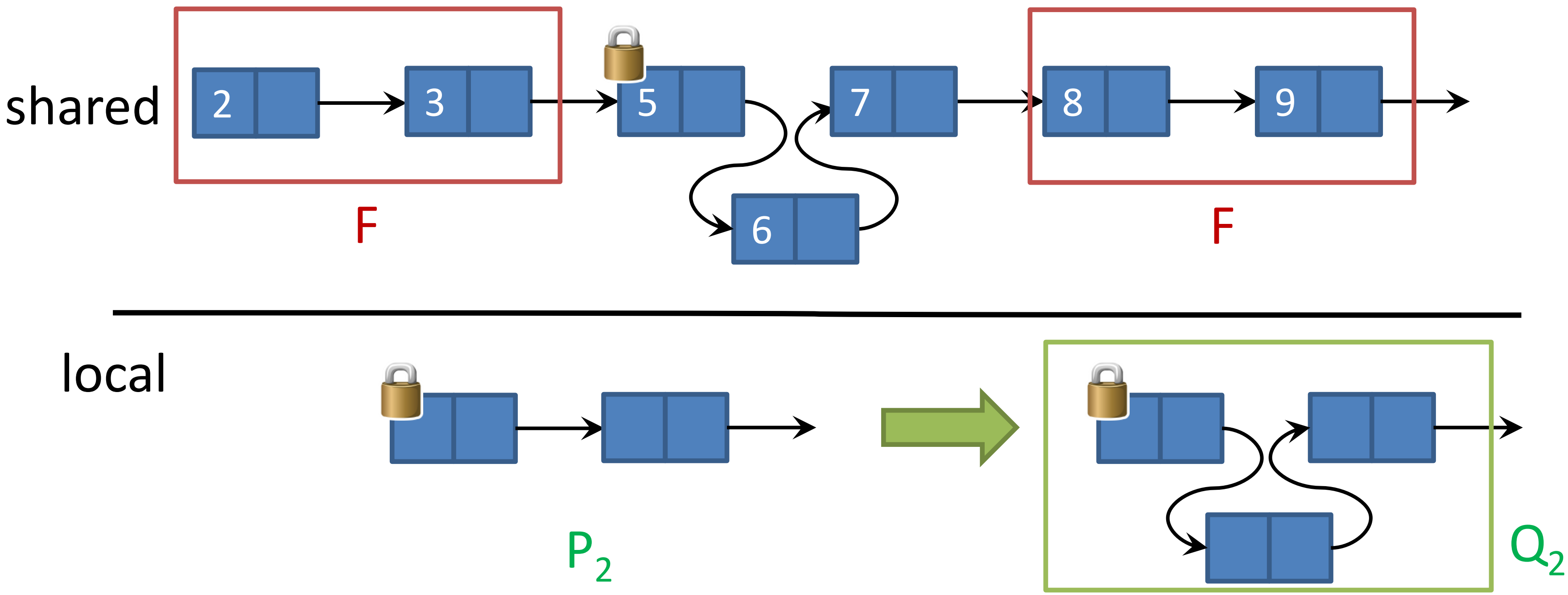

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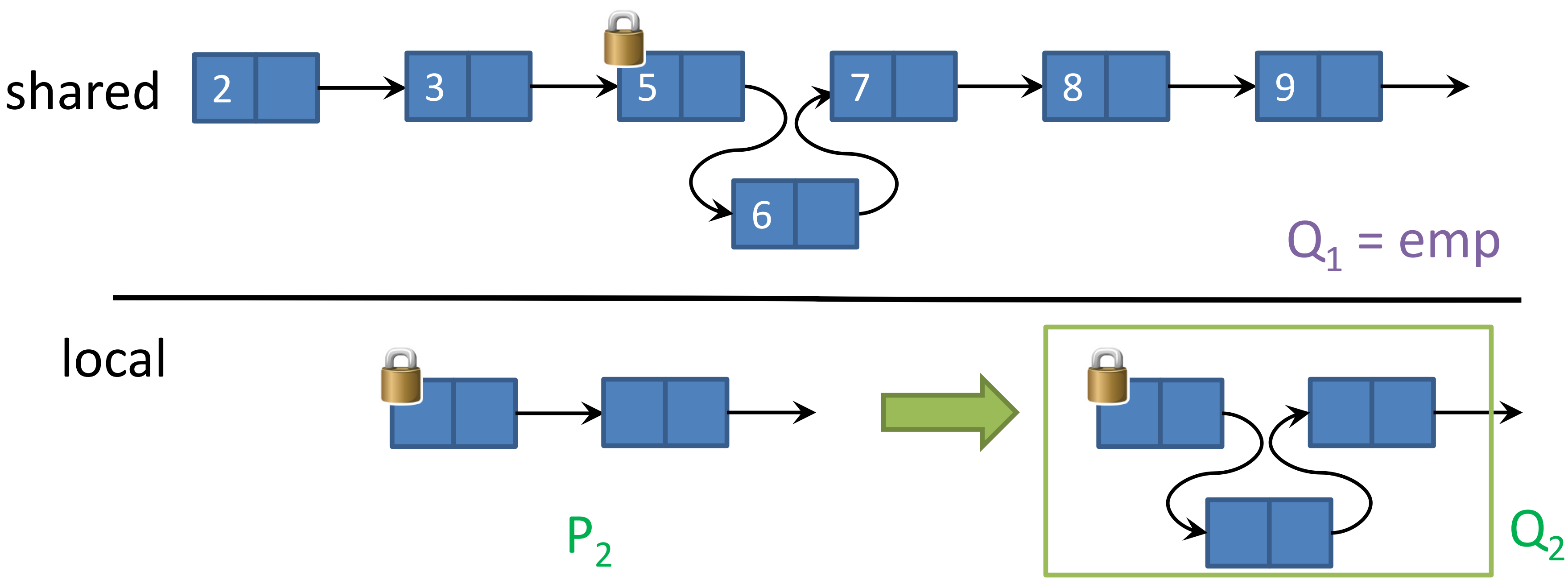
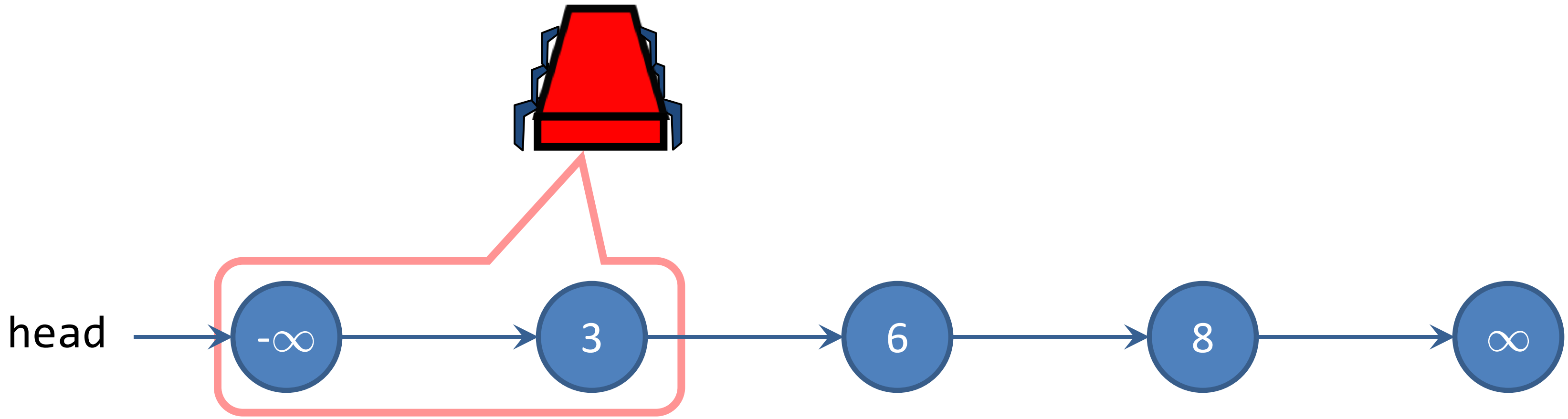


Challenge: Harris' Non-blocking List



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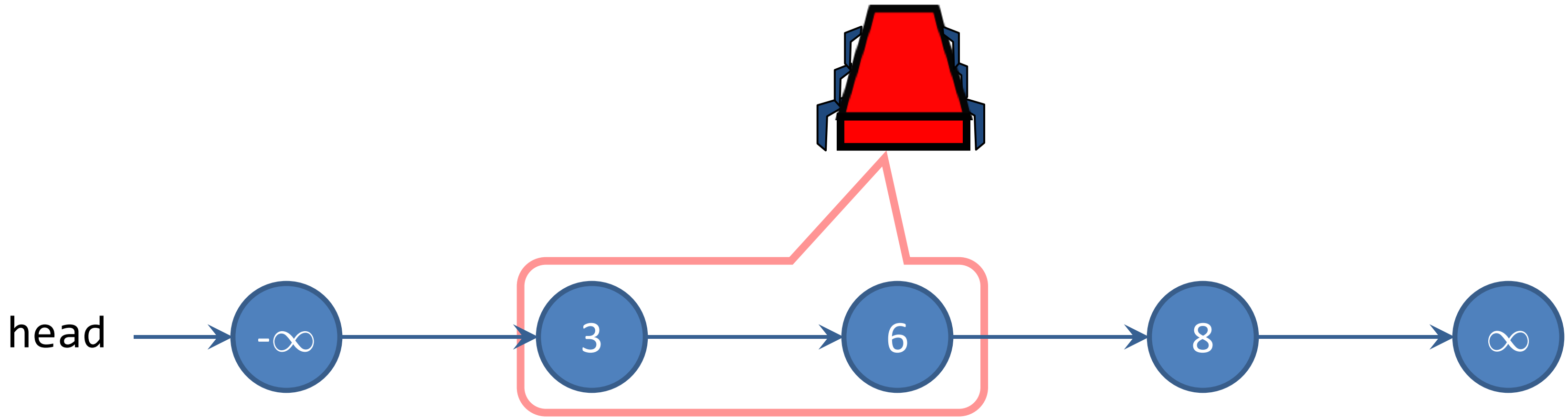

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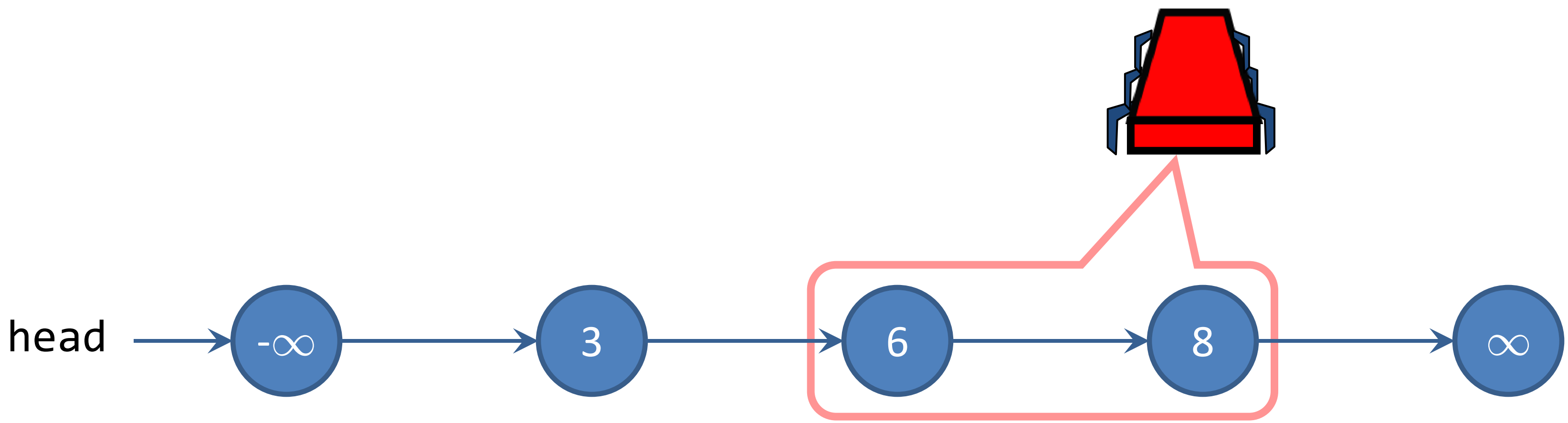

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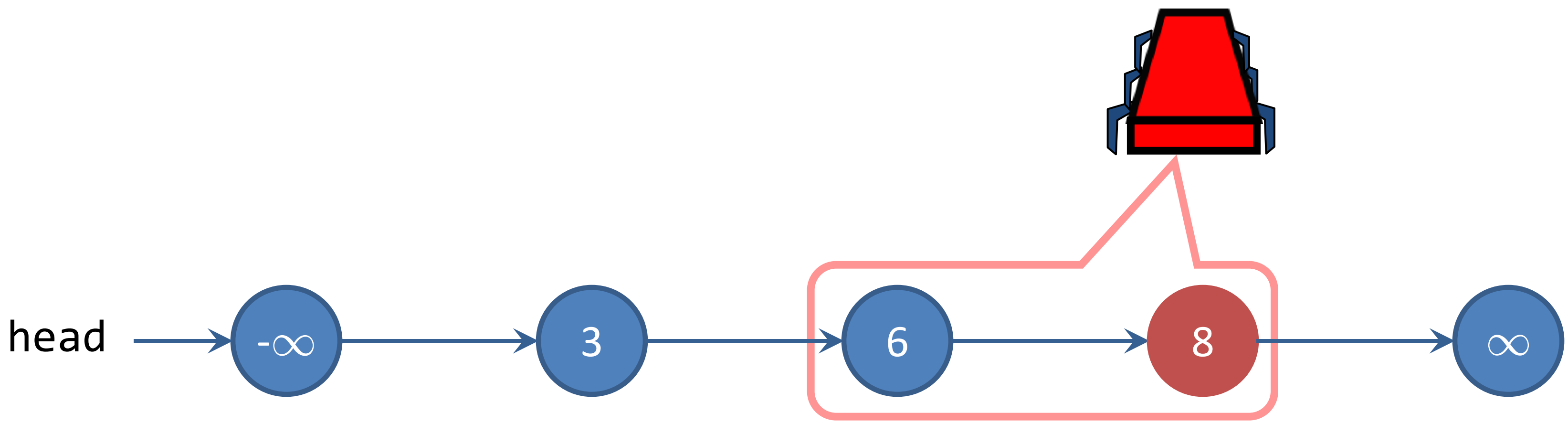

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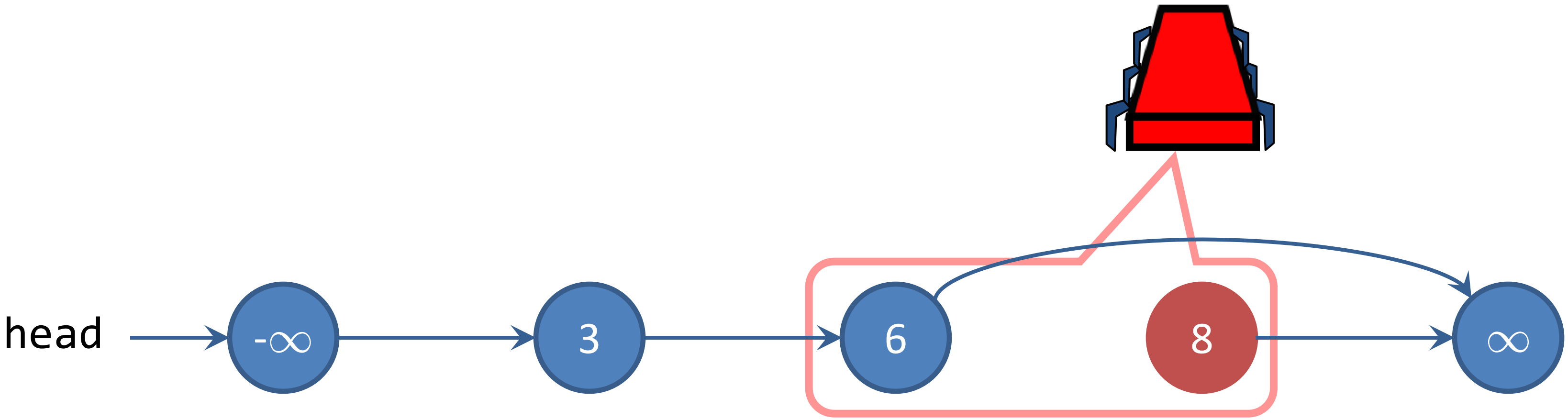

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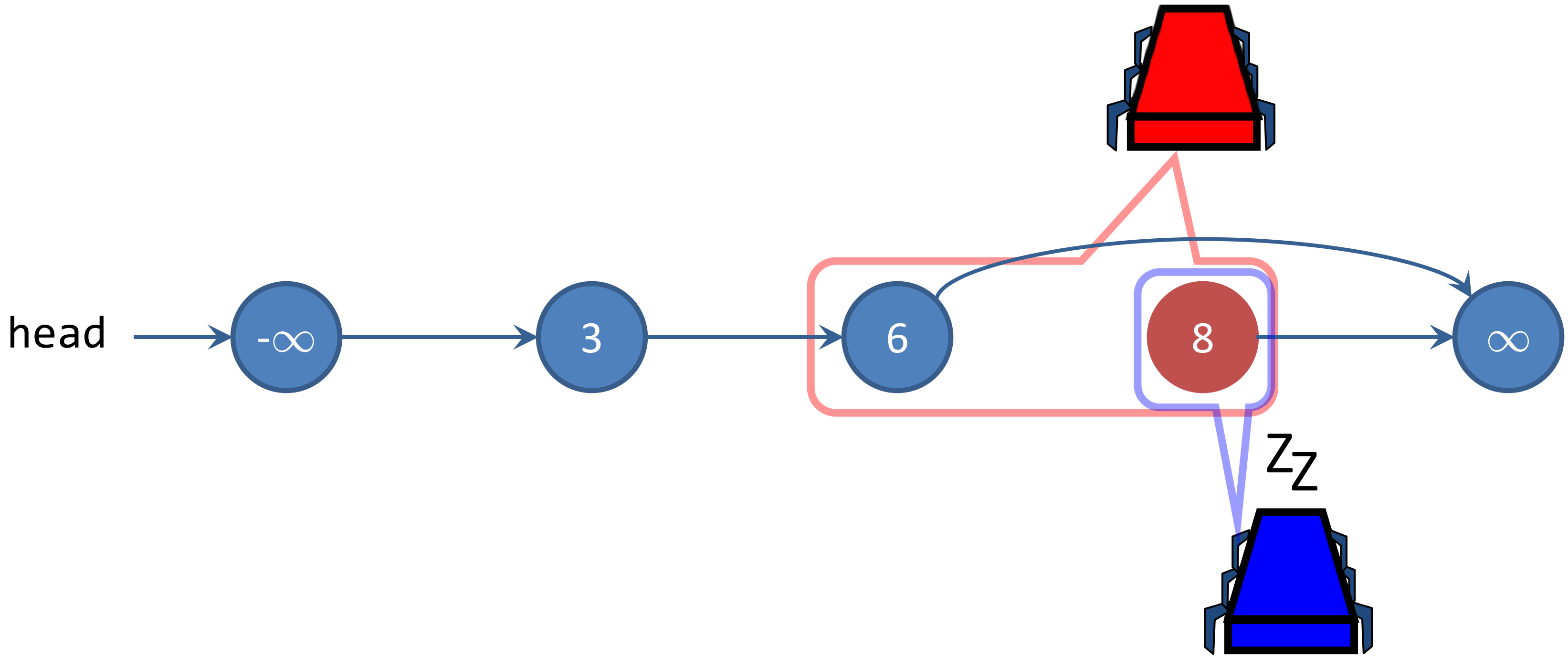

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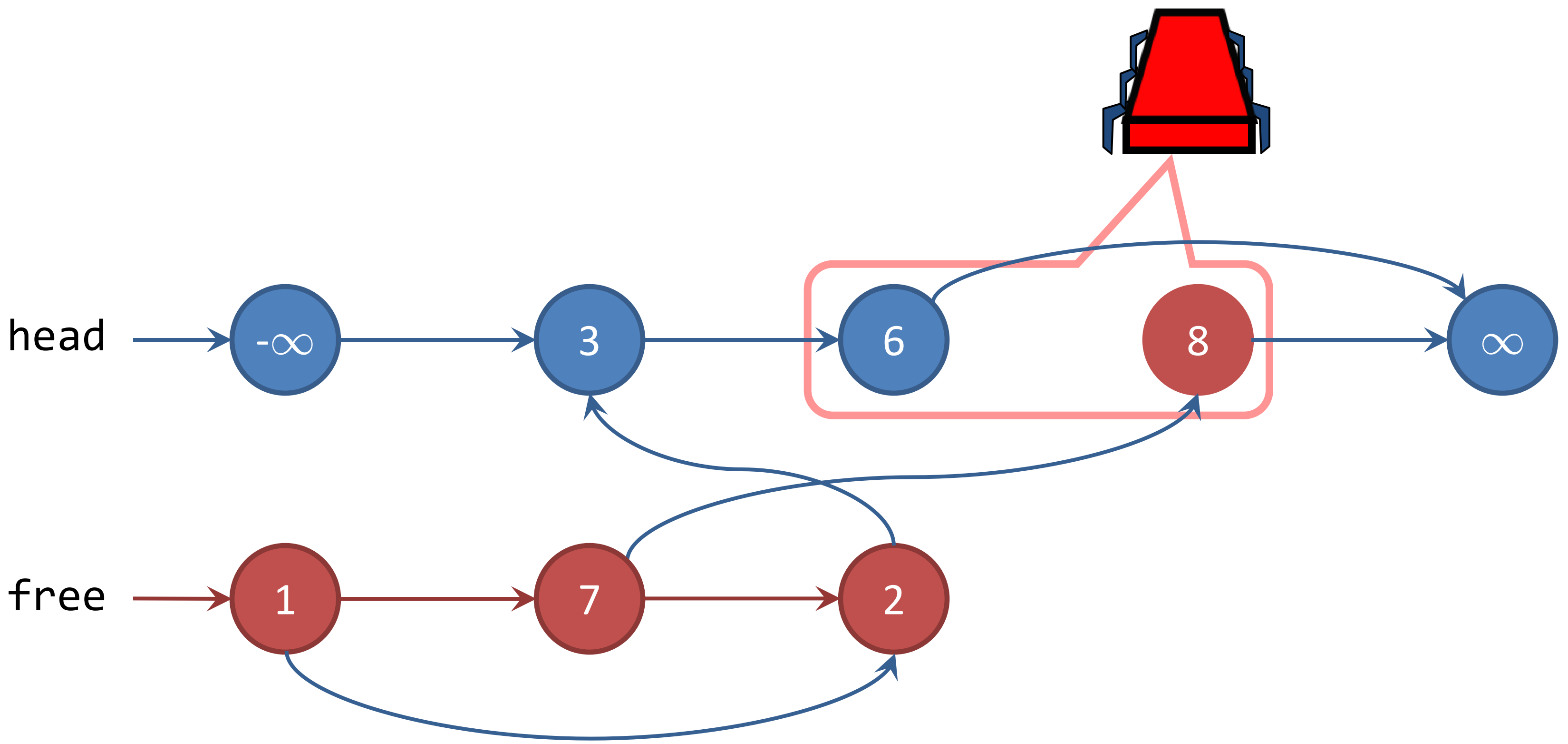

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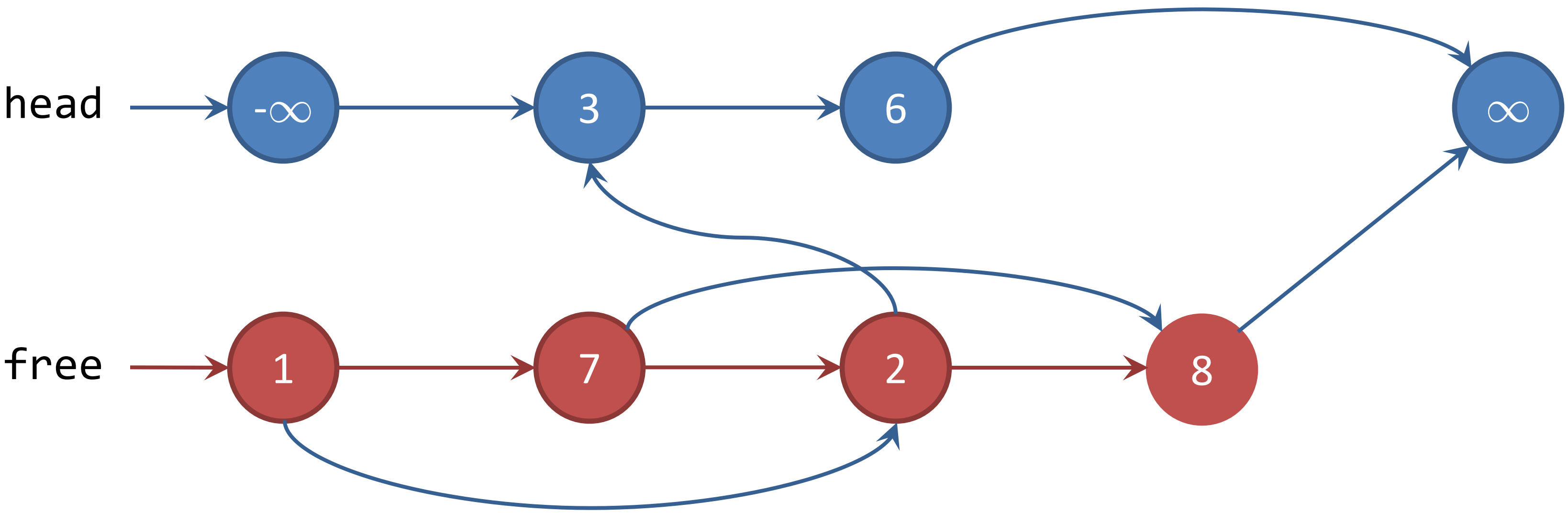

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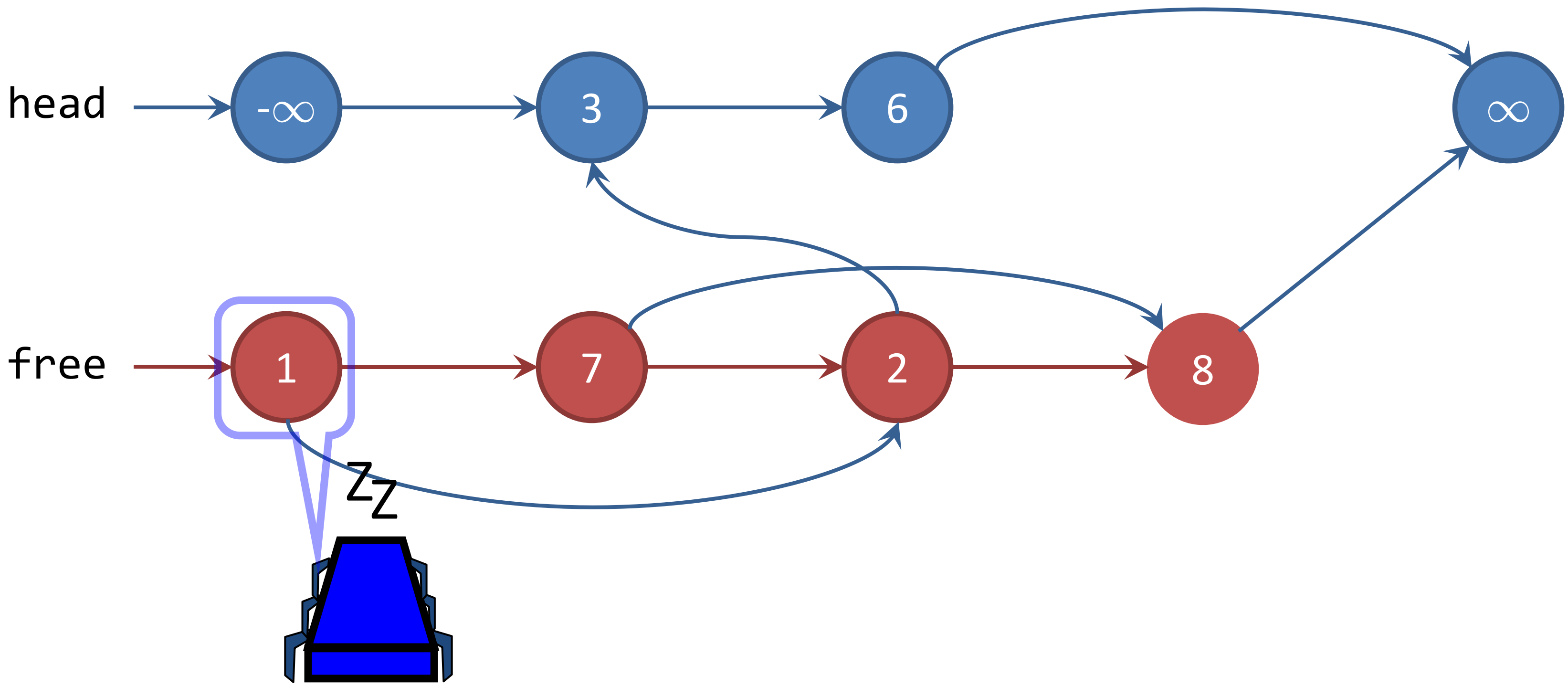

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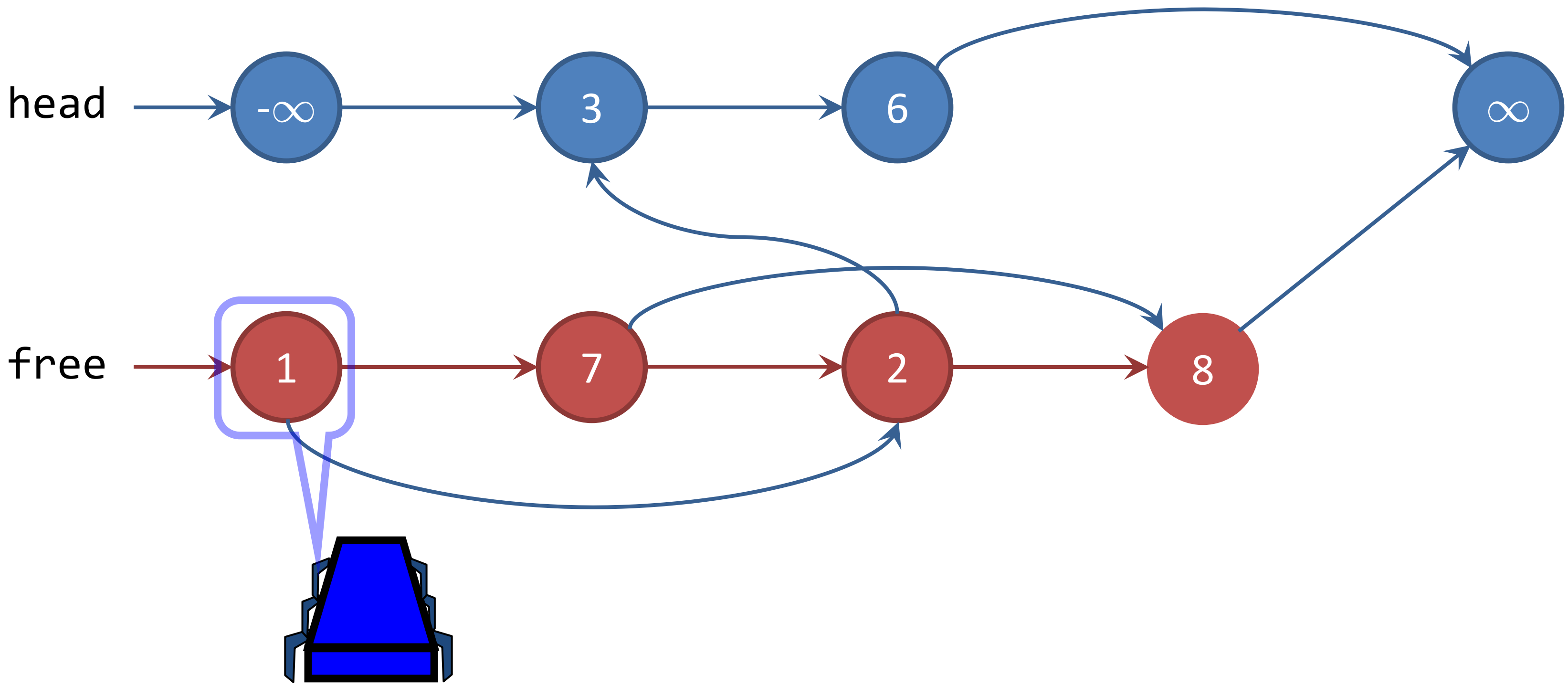

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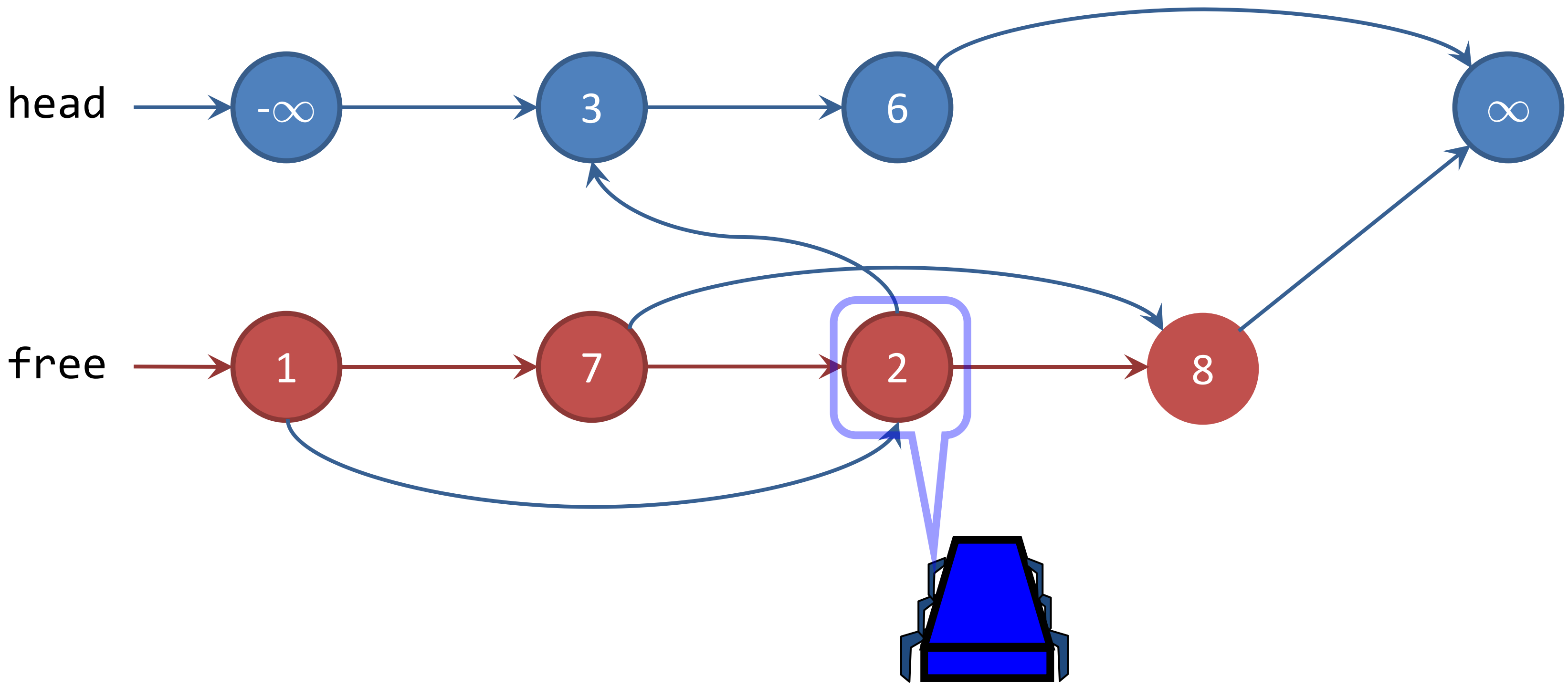

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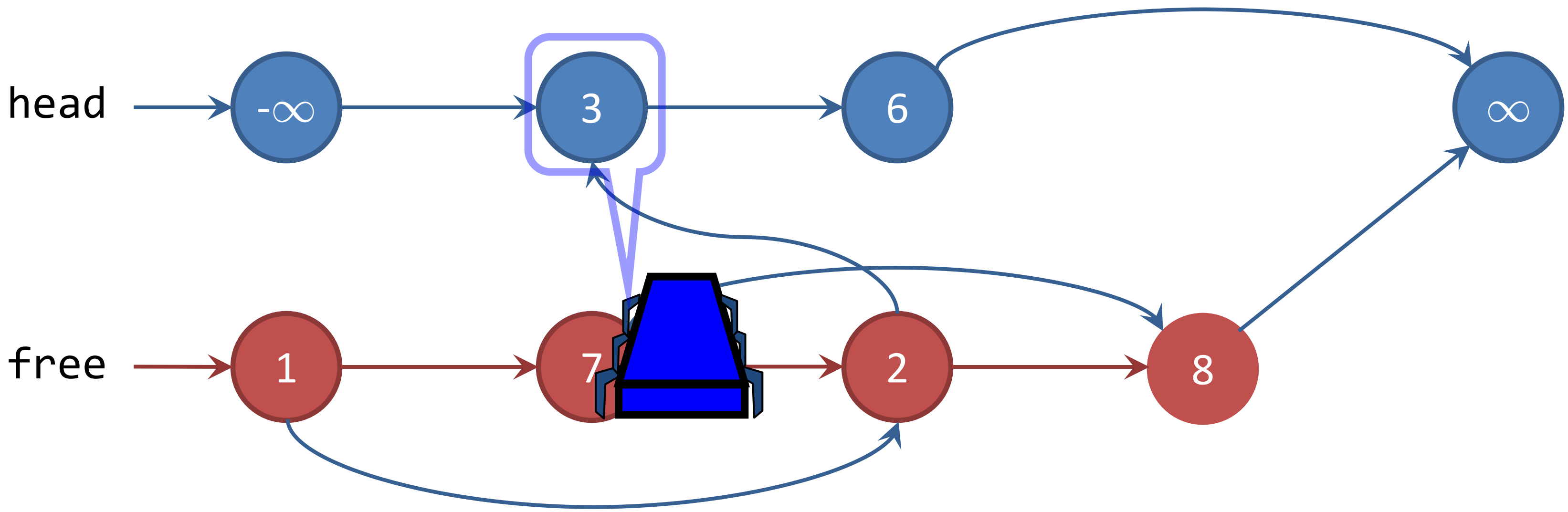

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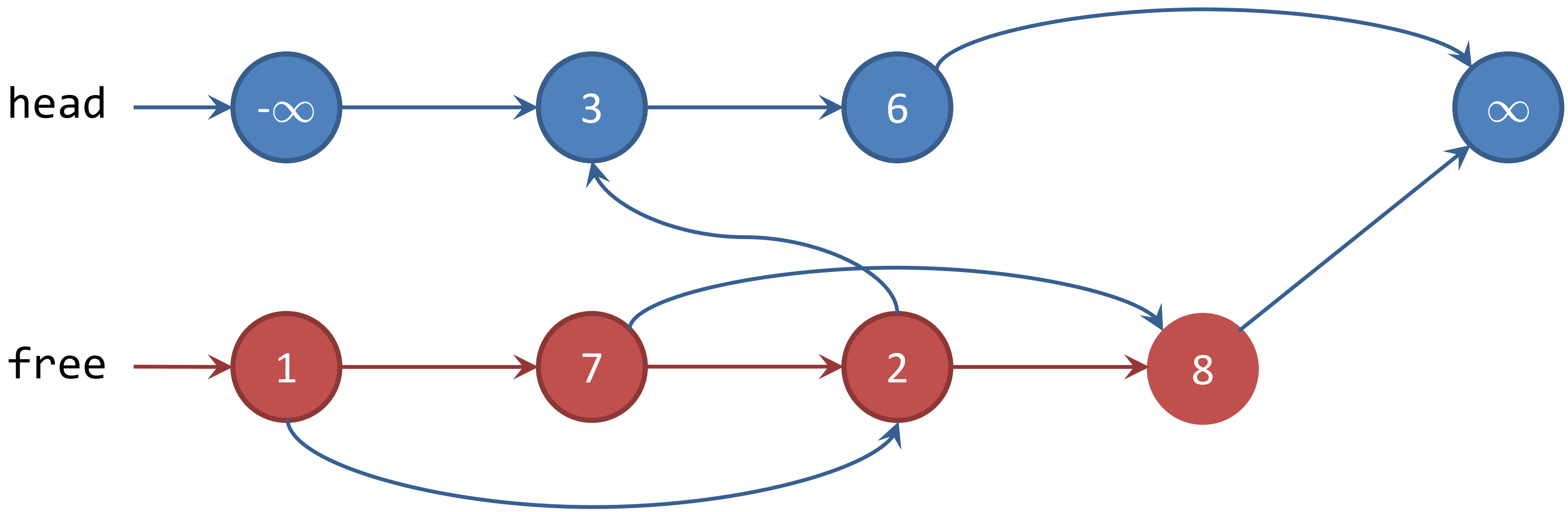


Flow Interfaces

joint work with Siddharth Krishna and Dennis Shasha

Goal

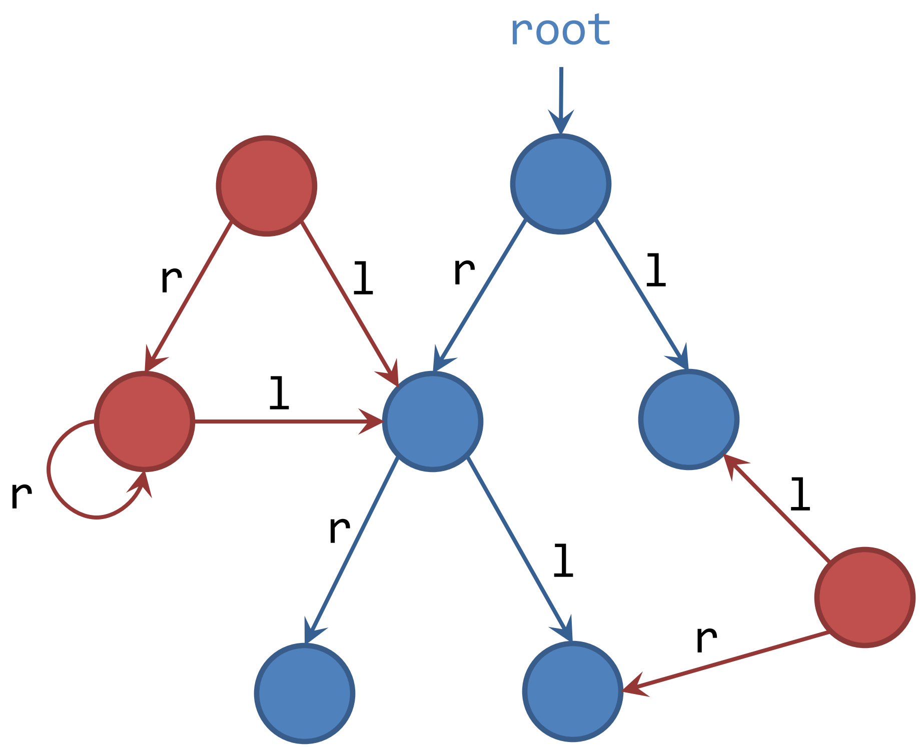
- Data structure abstractions that
  - can handle unbounded sharing and overlays
  - treat structural and data constraints uniformly
  - do not encode specific traversal strategies
  - provide data-structure-agnostic composition and decomposition rules
  - remain within general theory of separation logic

$\Rightarrow$ Flow Interfaces

High-Level Idea

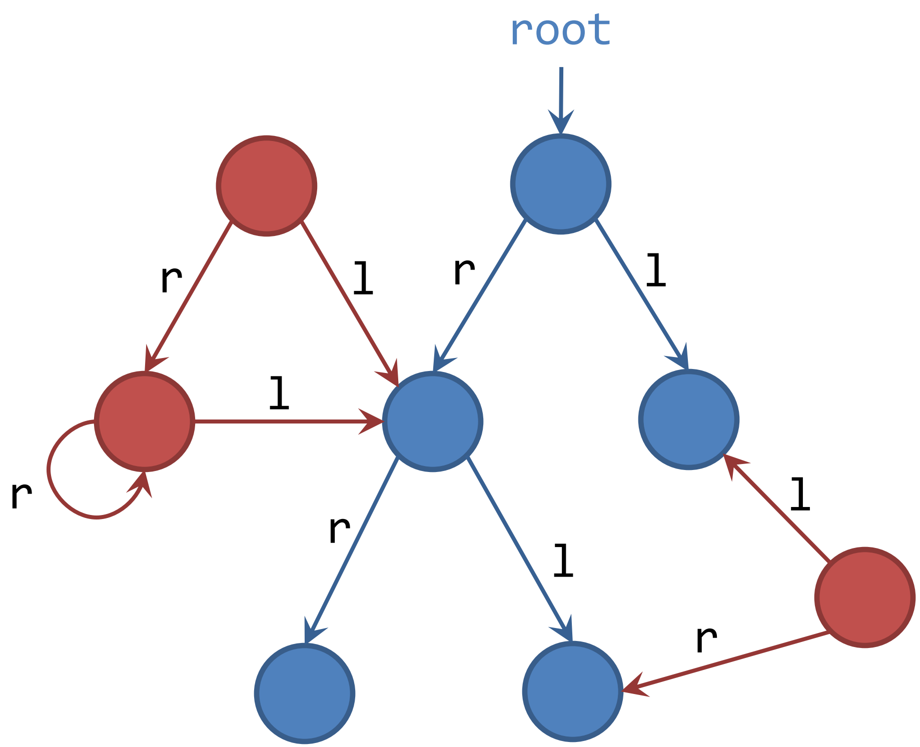
- Express all data structure invariants in terms of a local condition, satisfied by each node.
  - Local condition may depend on a quantity of the graph that is calculated inductively over the entire graph (the flow).
- Introduce a notion of graph composition that preserves local invariants of global flows.
- Introduce a generic *good graph* predicate that abstracts a heap region satisfying the local flow condition (the flow interface).

Local Data Structure Invariants with Flows



Can we express the property that `root` points to a tree as a local condition of each node in the graph?

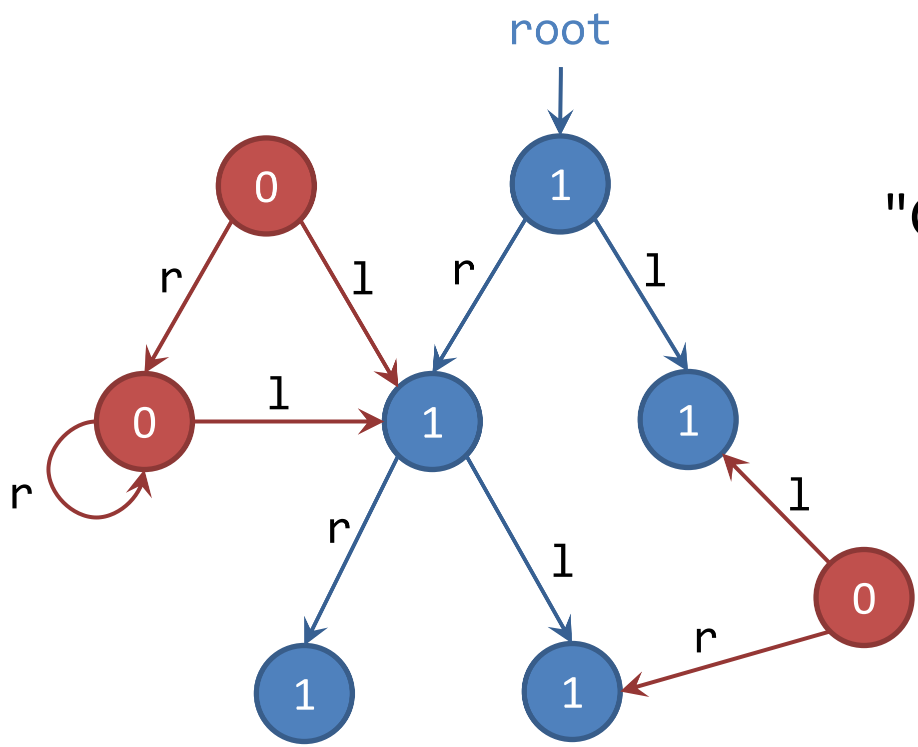
Local Data Structure Invariants with Flows



Path counting!

Can we express the property that `root` points to a tree as a local condition of each node in the graph?

Local Data Structure Invariants with Flows



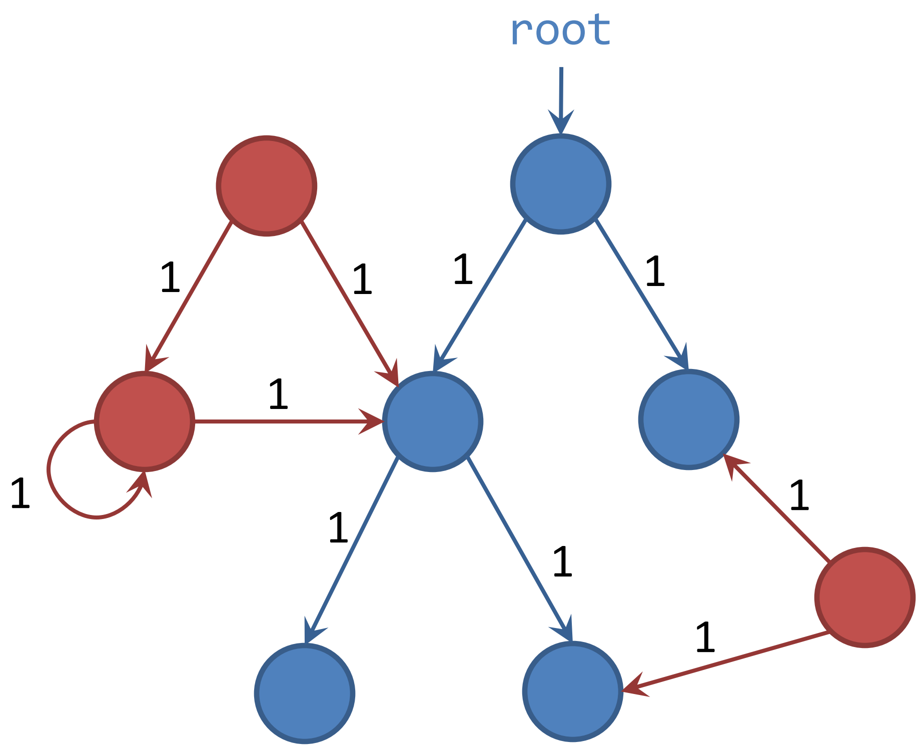
$\forall\, n \in N.\ pc(\texttt{root}, n) \leq 1$

"G contains a tree rooted at root"

Can we express the property that root points to a tree as a local condition of each node in the graph?

Flows

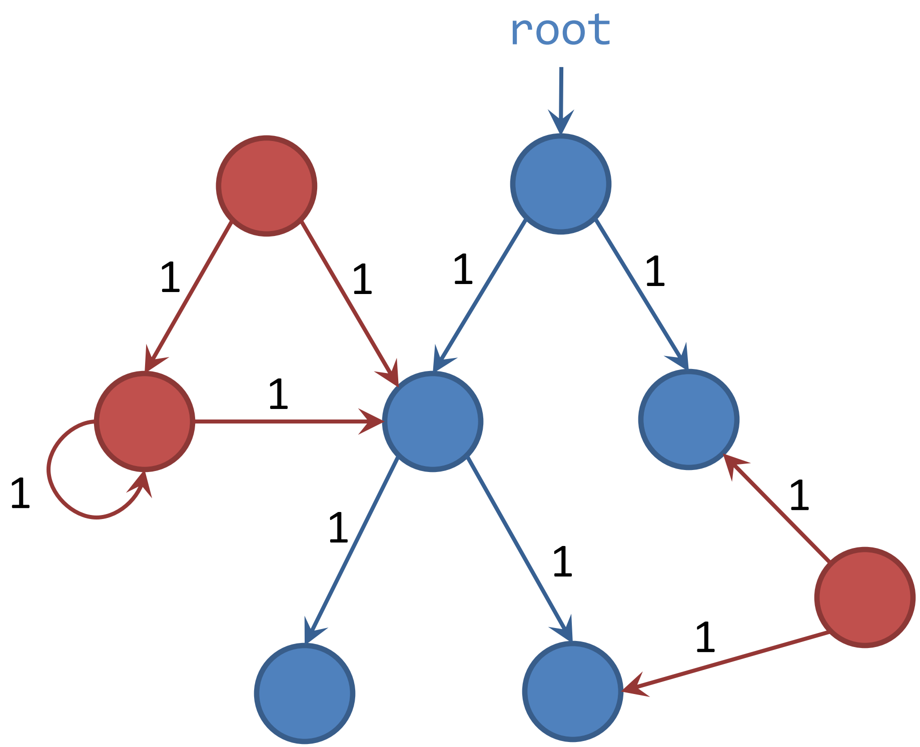
Step 1: Defining the Flow Graph



Label each edge in the graph with an element from some *flow domain* $(D, \sqsubseteq, +, \cdot, 0, 1)$

Flows

Step 1: Defining the Flow Graph



Requirements of flow domain:

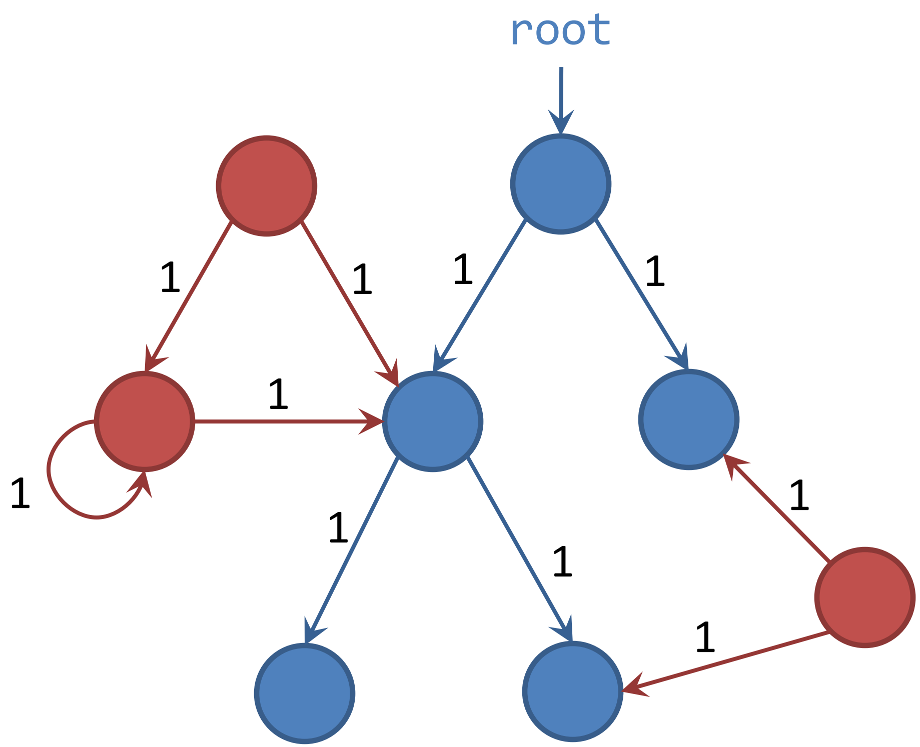
- (D, +, ·, 0, 1) is a semiring
- (D, $\sqsubseteq$) is ω-cpo with smallest element 0
- + and · are continuous

Path counting flow domain:
$(\mathbb{N} \cup \{\infty\}, \leq, +, \cdot, 0, 1)$

Label each edge in the graph with an element from some *flow domain* (D, $\sqsubseteq$, +, ·, 0, 1)

Flows

Step 1: Defining the Flow Graph



Flow graph $G = (N, e)$

- N finite set of nodes
- $e: N \times N \to D$

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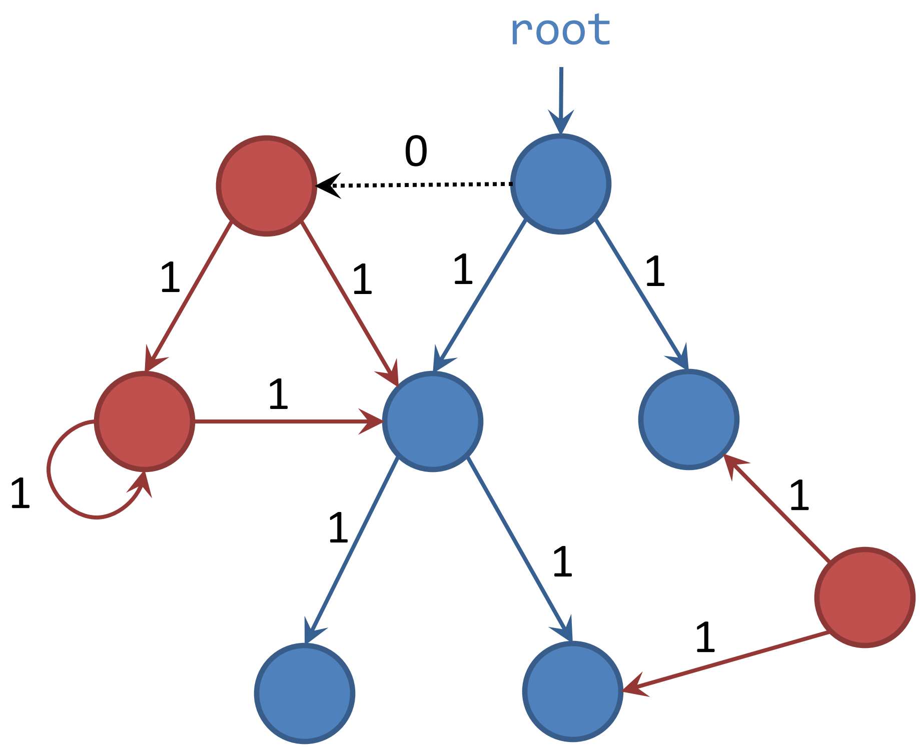
Flows

Step 1: Defining the Flow Graph



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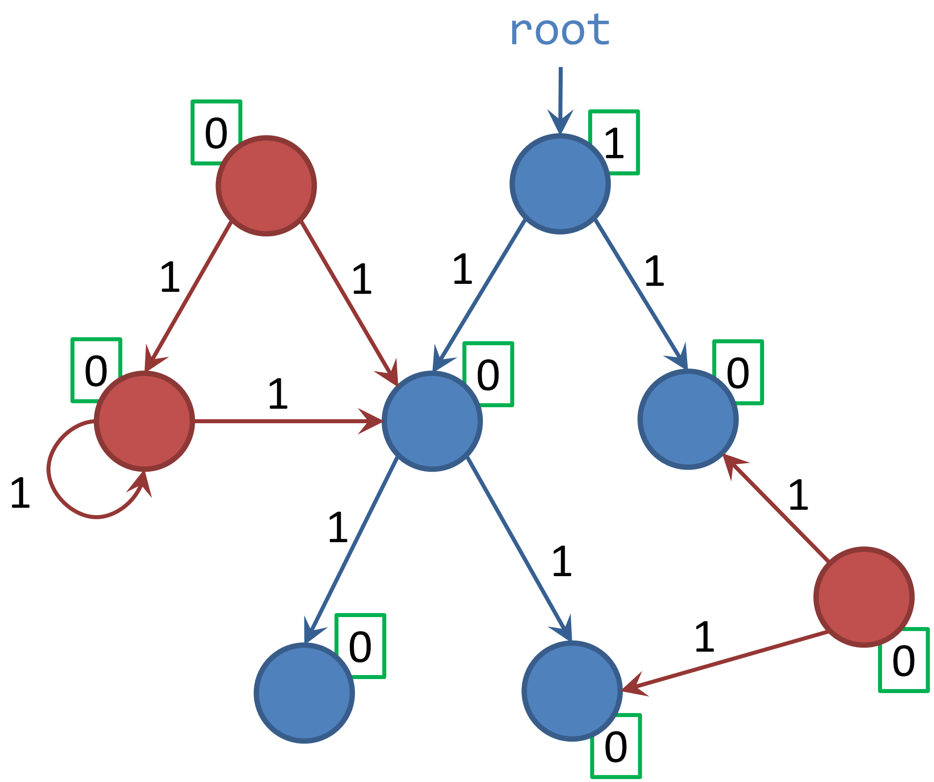
Flows

Step 2: Define the Inflow

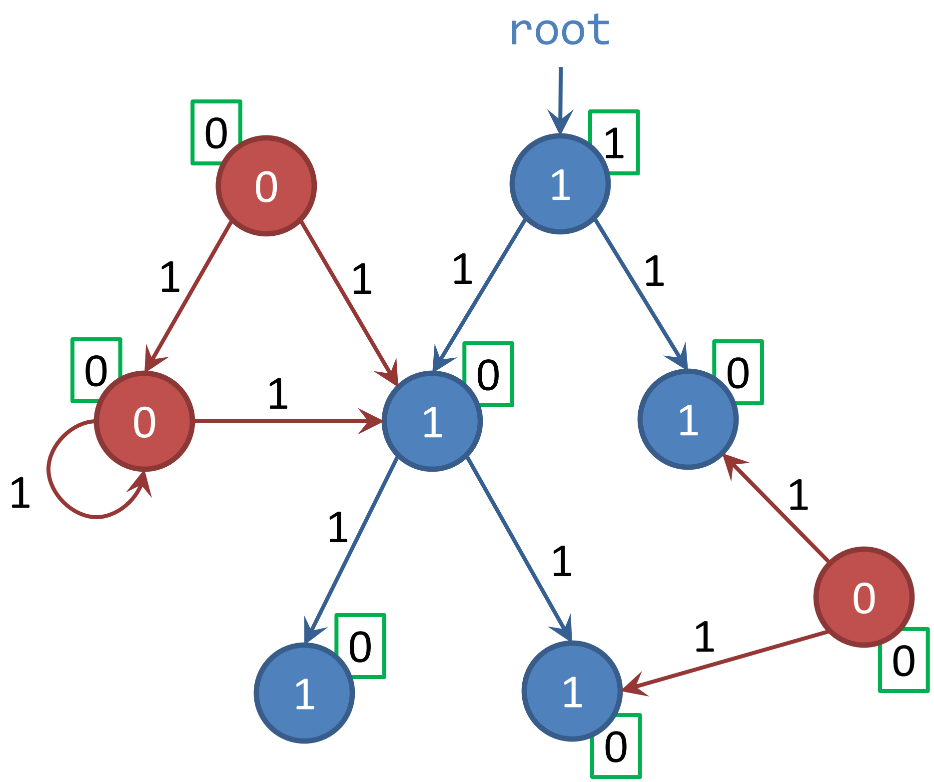


$$\color{green}{in}_{\color{blue}{\texttt{root}}}(n) = \begin{cases} 1, n = \color{blue}{\texttt{root}} \\ 0, n \neq \color{blue}{\texttt{root}} \end{cases}$$

Label each node using an *inflow in*: N → D

Flows

Step 3: Calculate the flow



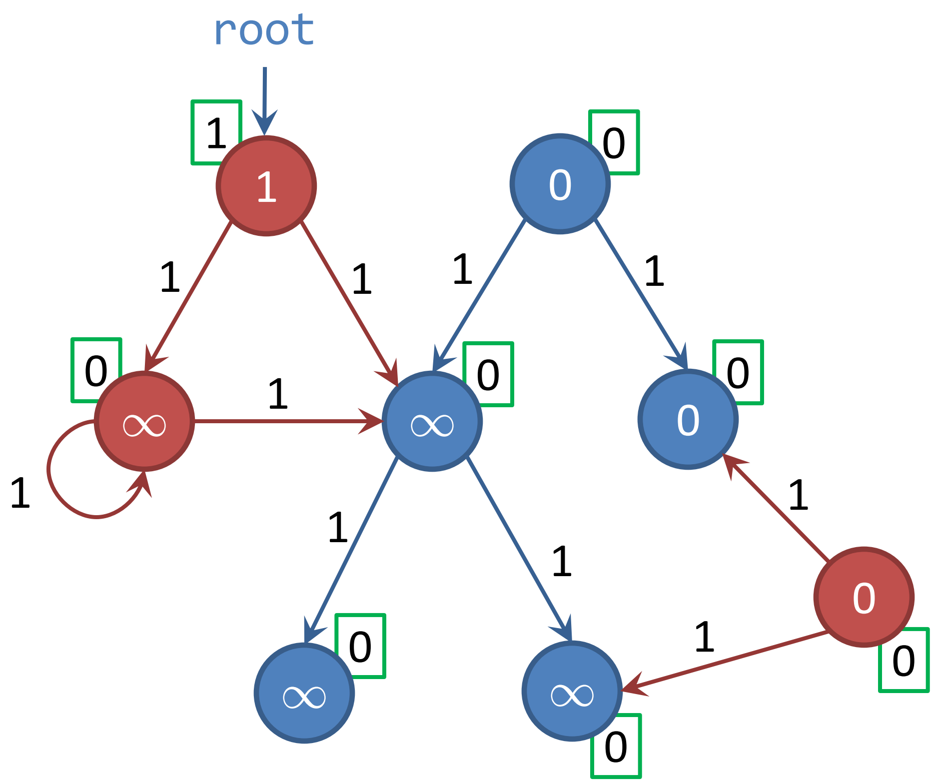
Flow graph G = (N, e)

flow$(in, G) : N \to D$

flow$(in, G) = \text{lfp}\left(\lambda C.\,\lambda n \in N.\, in(n) + \sum_{n' \in N} C(n') \cdot e(n', n)\right)$

Flows

Step 3: Calculate the flow



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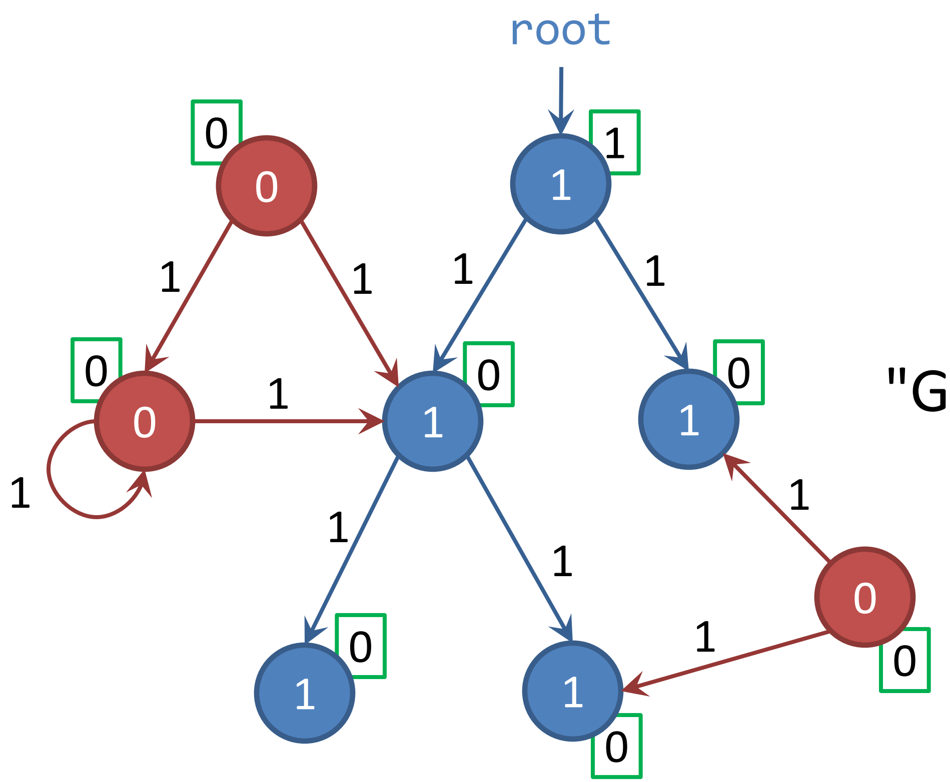

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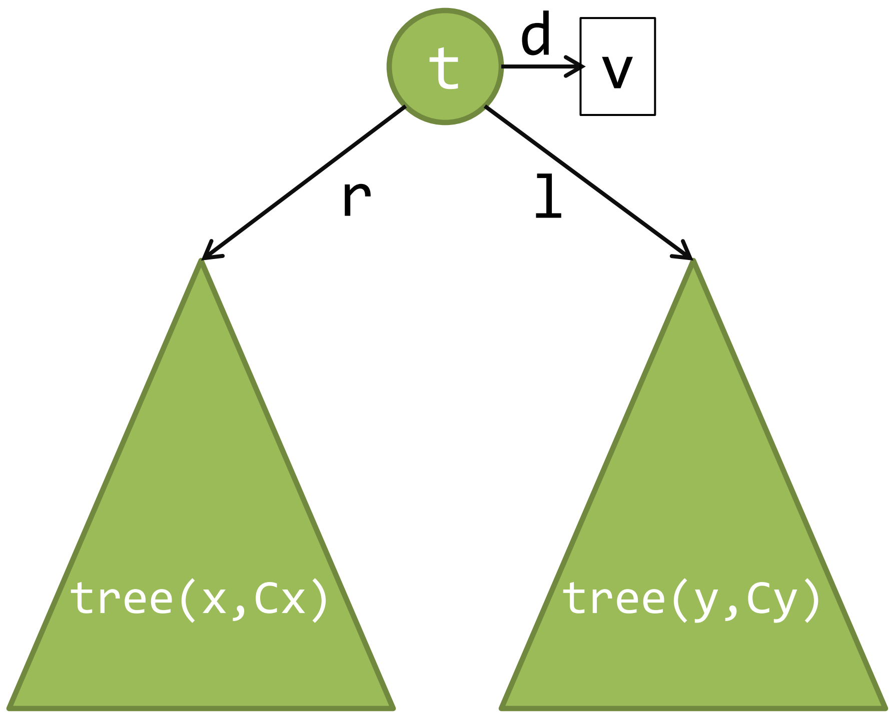
Data Constraints

```
predicate tree(t: Node, C: Set<Int>) {
  t == null ∧ emp ∧ C = ∅ ∨
  ∃ v, x, y, Cx, Cy ::
    t↦(d:v,r:x,l:y) * tree(x,Cx) * tree(y,Cy) ∧
    C = {v}∪Cx∪Cy ∧ v>Cx ∧ v<Cy
}
```



Data Invariants

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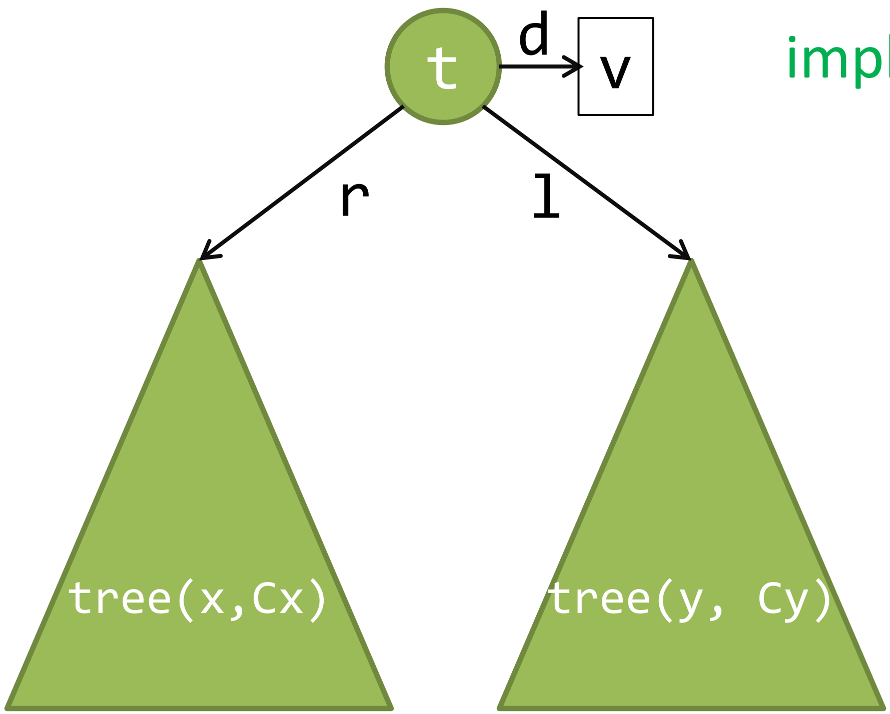

implies $Cx \cap Cy = \emptyset$

Data Invariants

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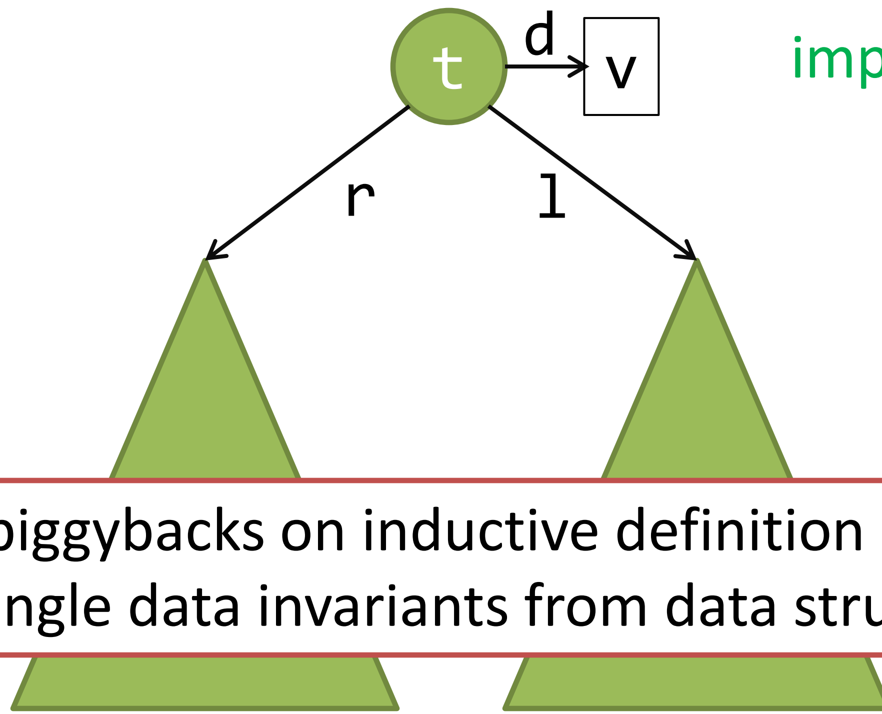
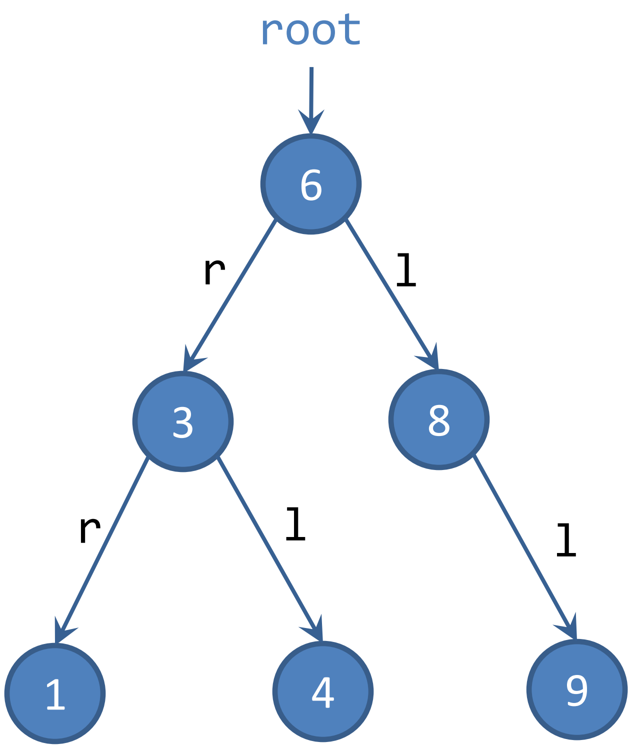


implies $Cx \cap Cy = \emptyset$

Data invariant piggybacks on inductive definition of the tree.
⇒ hard to entangle data invariants from data structure specifics.

Inset Flows

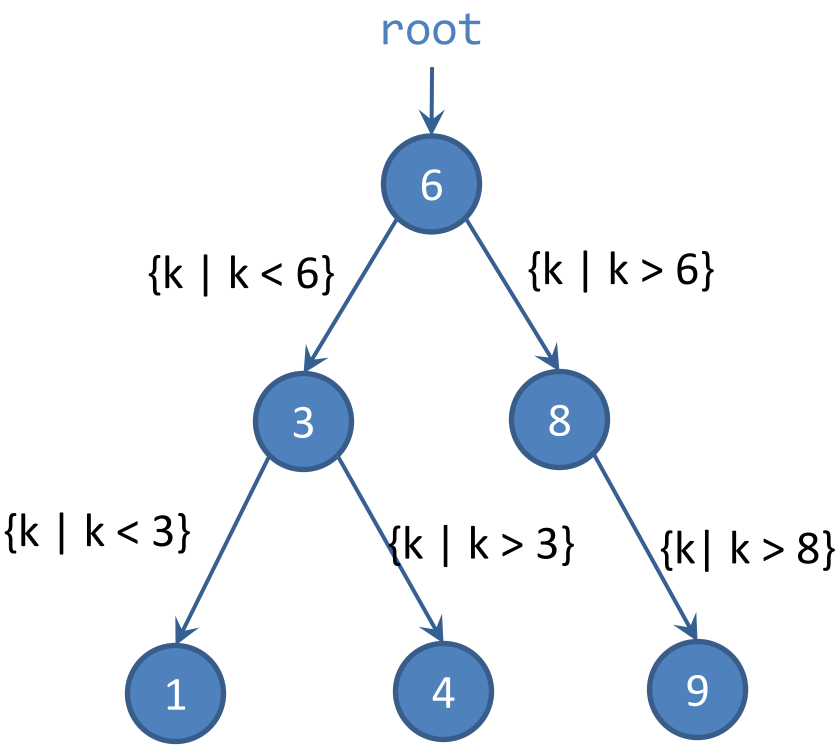


KS: the set of all search keys
e.g. KS = $\mathbb{Z}$

Inset flow domain:
$(2^{\mathrm{KS}}, \subseteq, \cup, \cap, \emptyset, \mathrm{KS})$

Label each edge with the set of keys that follow that edge in a search (edgeset).

Inset Flows



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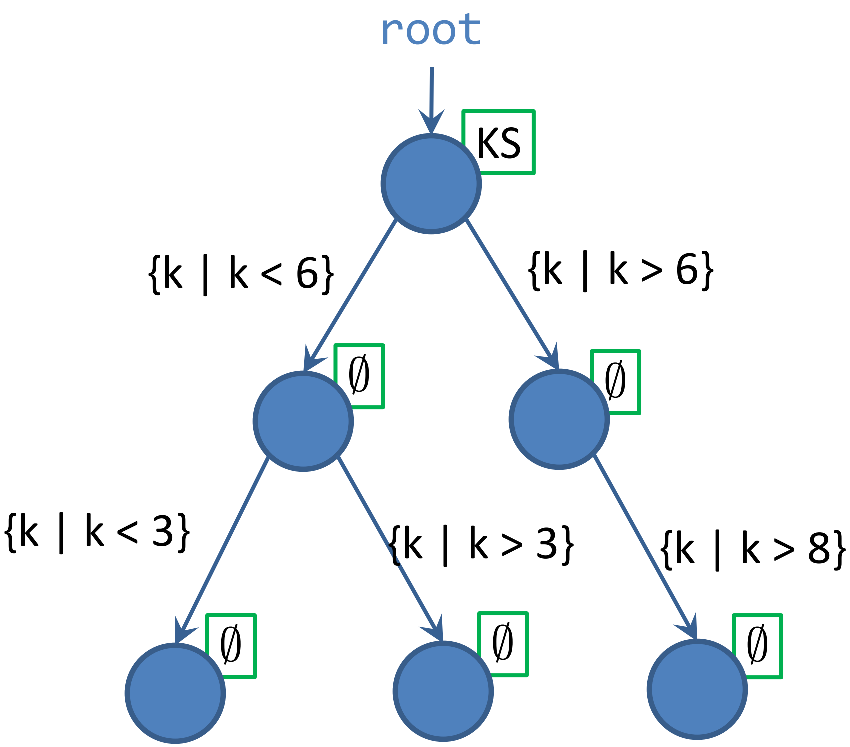

KS: the set of all search keys
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Inset flow domain:
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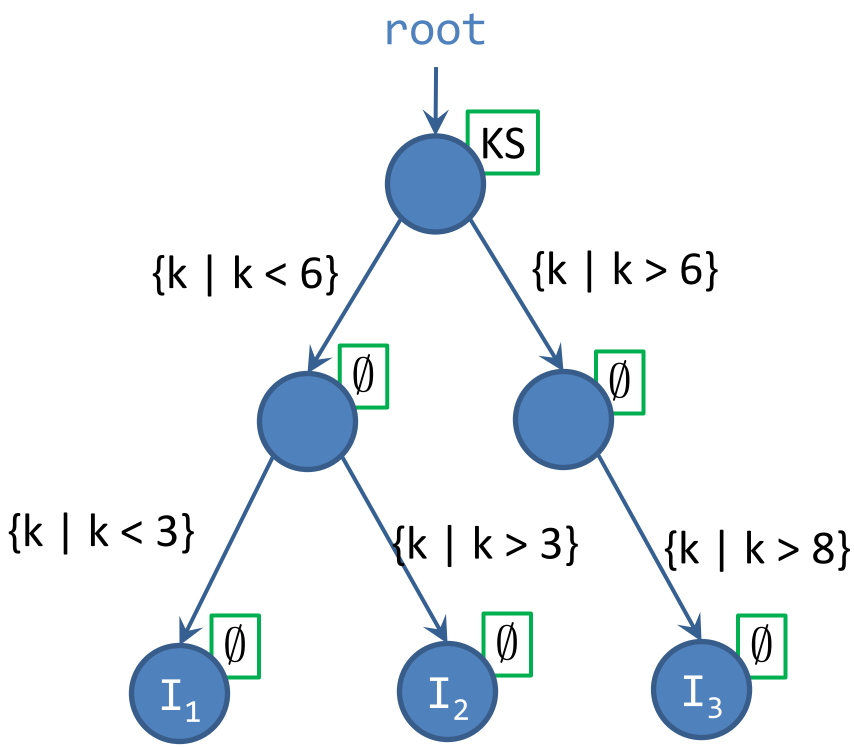
Set inflow *in* of `root` to KS and to $\emptyset$ for all other nodes.

Inset Flows



$I_1 = \{k \mid 3 < k\}$

$I_2 = \{k \mid 3 < k < 6\}$

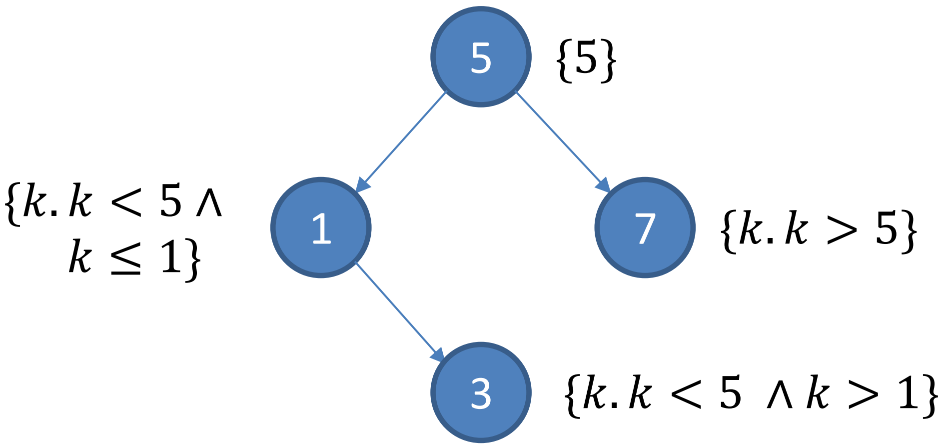
$I_3 = \{k \mid 8 < k\}$

flow(in, G)(n) is the *inset* of node n, i.e., the set of keys k such that a search for k will traverse node n.

From Insets to Keysets

$$outset(G)(n) = \bigcup_{n \in N} e(n, n')$$

$$keyset(in, G)(n) = inset(in, G)(n) \setminus outset(G)(n)$$



keyset(in, G)(n) is the set of keys that could be in n

Verifying Concurrent Search Data Structures

- Local data structure invariants
  - edgesets are disjoint for each n: $\{e(n,n')\}_{n' \in N}$ are disjoint
  - keyset of each n covers n's contents: $C(G)(n) \subseteq \text{keyset}(in, G)(n)$
- Observation: disjoint inflows imply disjoint keysets
  - If $\{in(n)\}_{n \in N}$ are disjoint (e.g. G has a single root)
  - then $\{\text{keyset}(in,G)(n)\}_{n \in N}$ are disjoint

$\Rightarrow$ Can be used to prove linearizability of concurrent search data structures in a data-structure-agnostic fashion

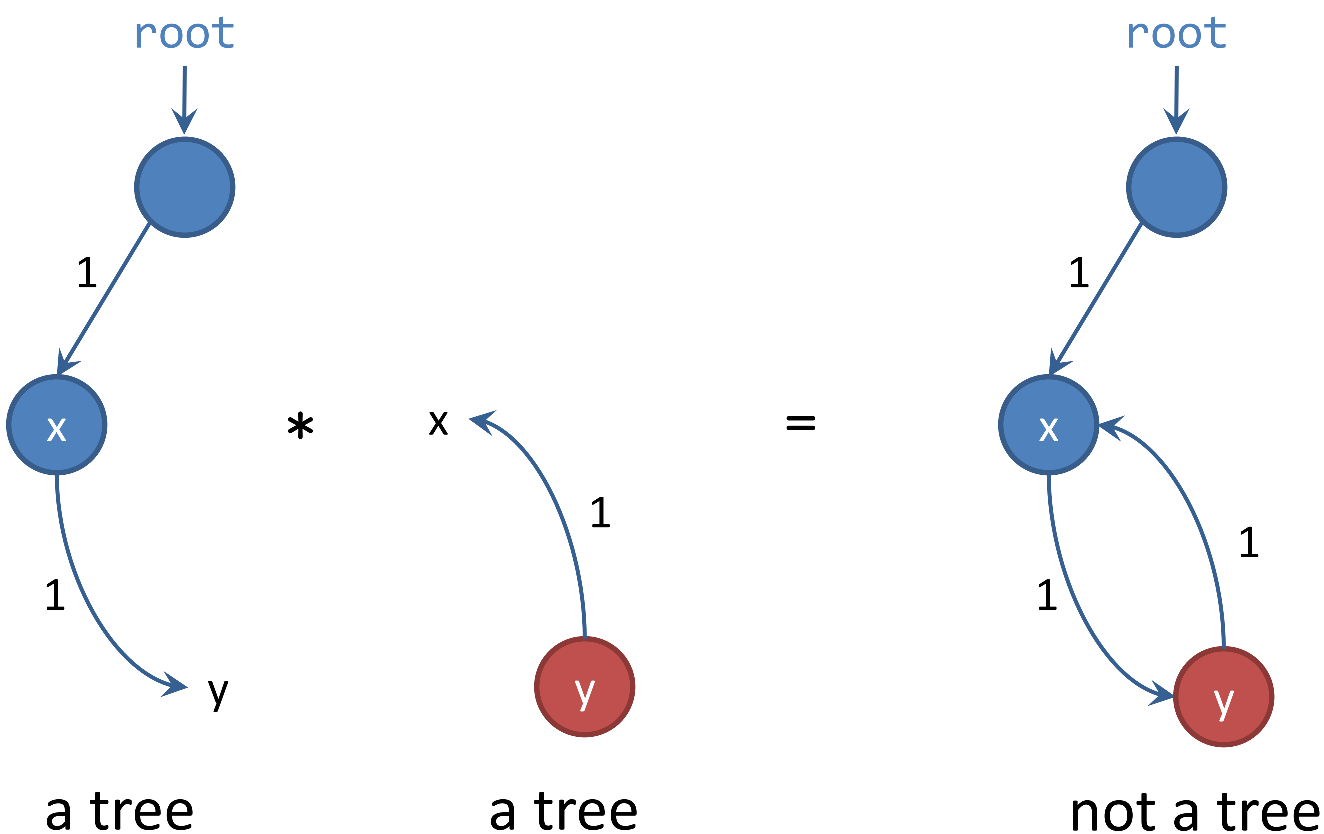
[Shasha and Goodman, 1988]

Compositional Reasoning

Can we reason compositionally about flows and flow graphs à la SL?

Flow Graph Composition

- Standard SL Composition (disjoint union) is too weak:



Flow Interface Graph

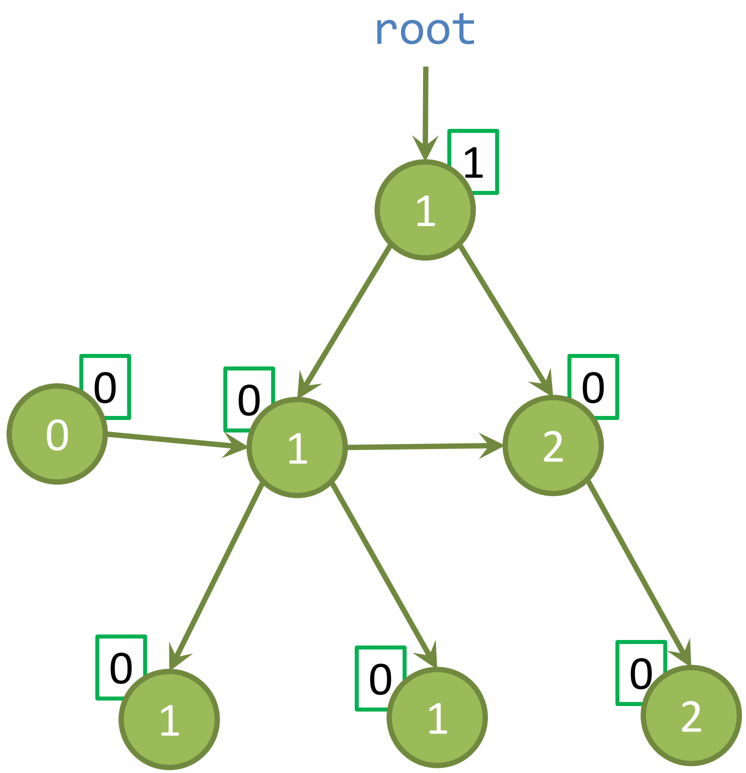
(in, G) is a *flow interface graph* iff

- $G = (N, N_o, \lambda, e)$ is a partial graph with
  - N the set of internal nodes of the graph
  - N_o the set of external nodes of the graph
  - λ: N $\rightarrow$ A a node labeling function
  - e: N x ($N \cup N_o$) $\rightarrow$ D is an edge function
- in: $N \rightarrow D$ is an inflow

Inflow *in* specifies rely of G on its context.

Flow Interface Graph Composition

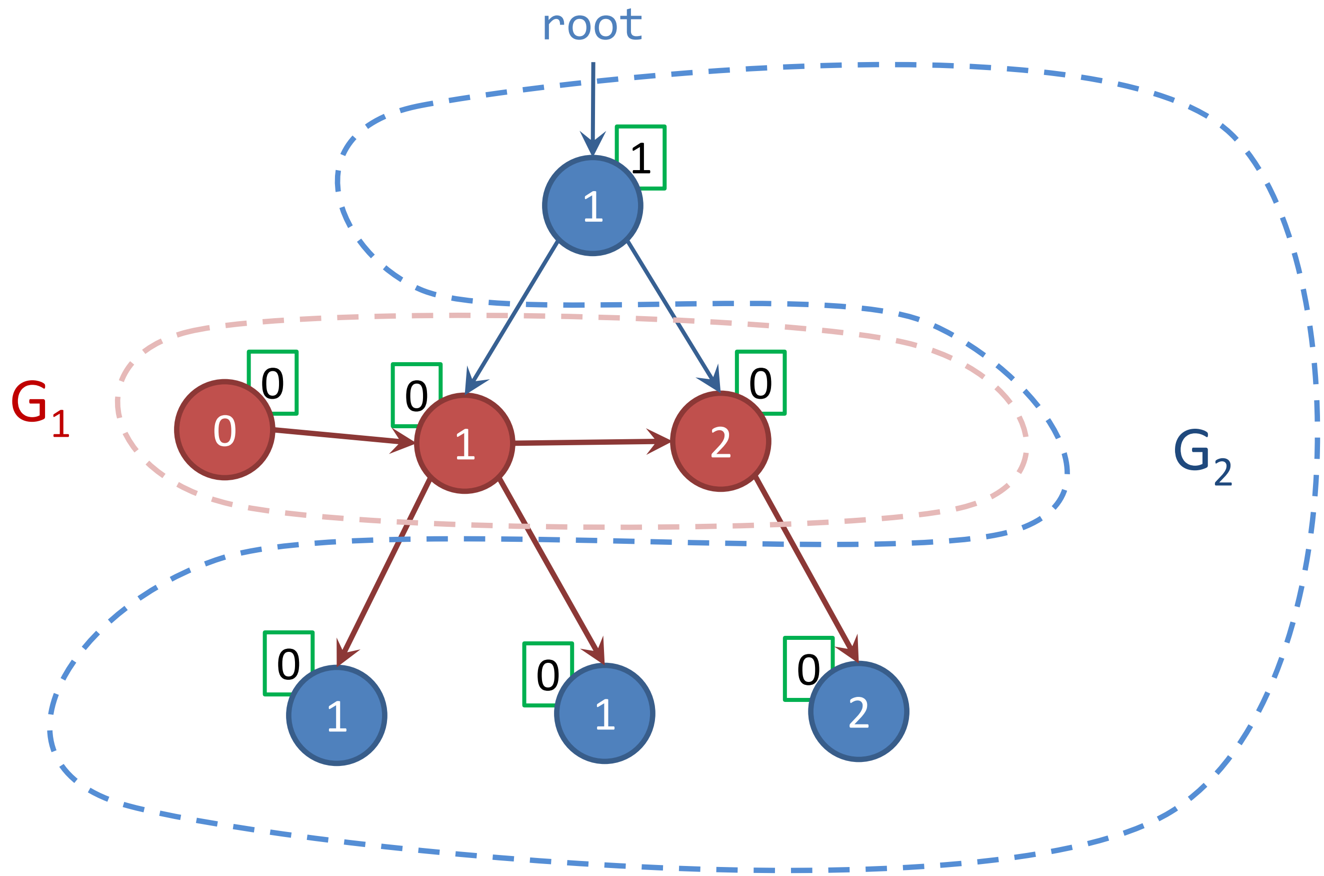
(in, G)



Flow Interface Graph Composition

$(in, G) = (in_1, G_1) \bullet (in_2, G_2)$

$in_1 = ?$, $in_2 = ?$



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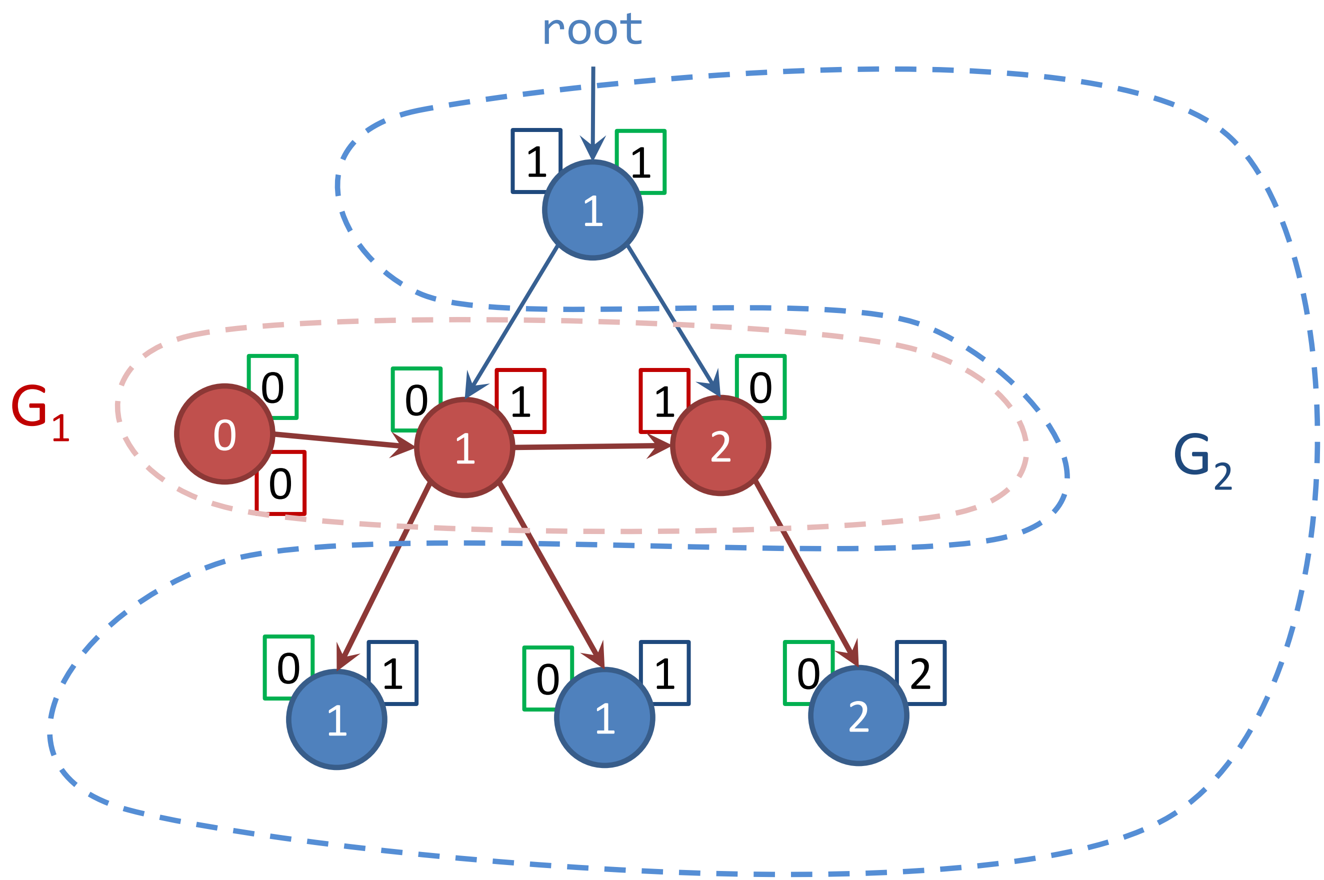

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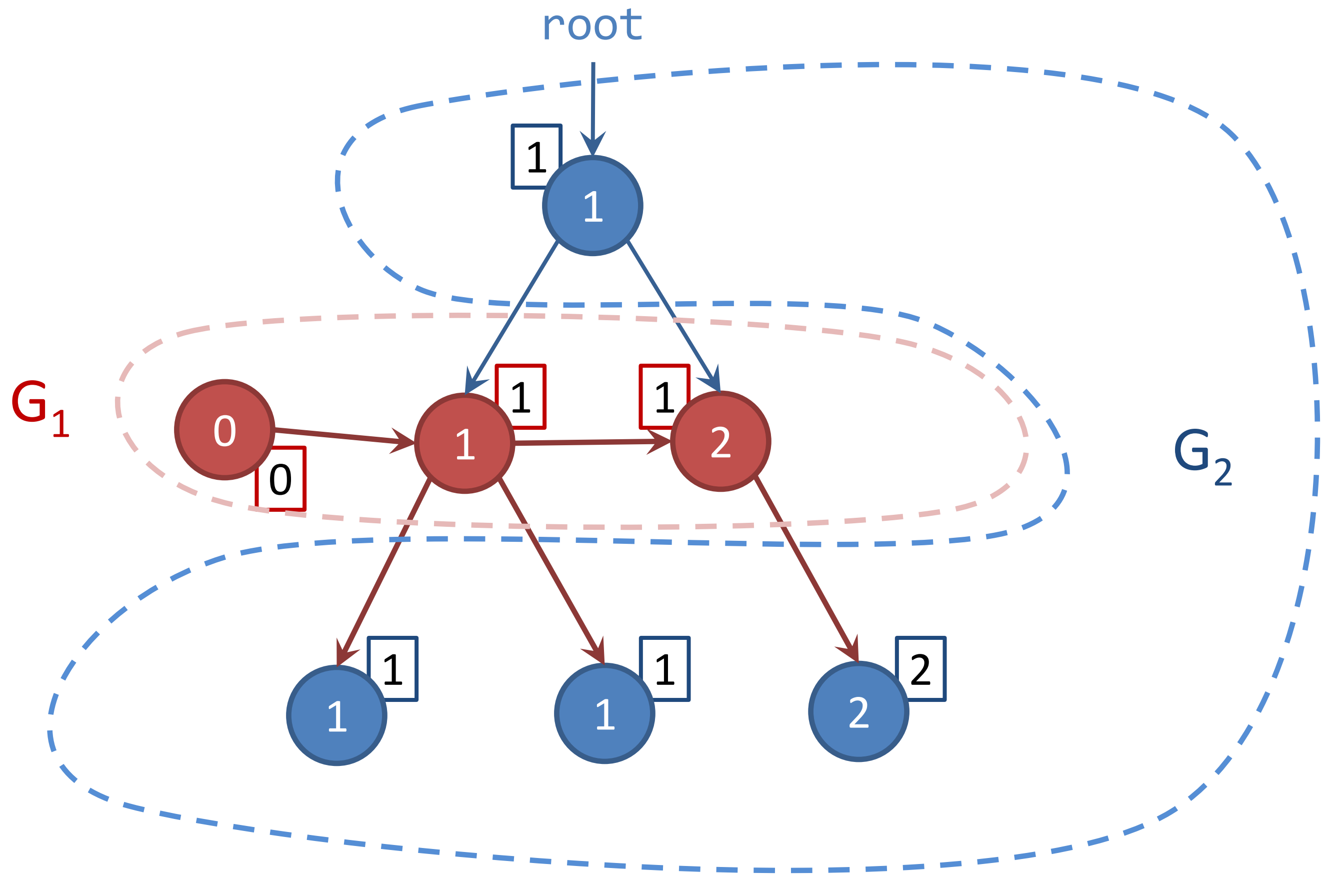

Flow Interface Graph Composition

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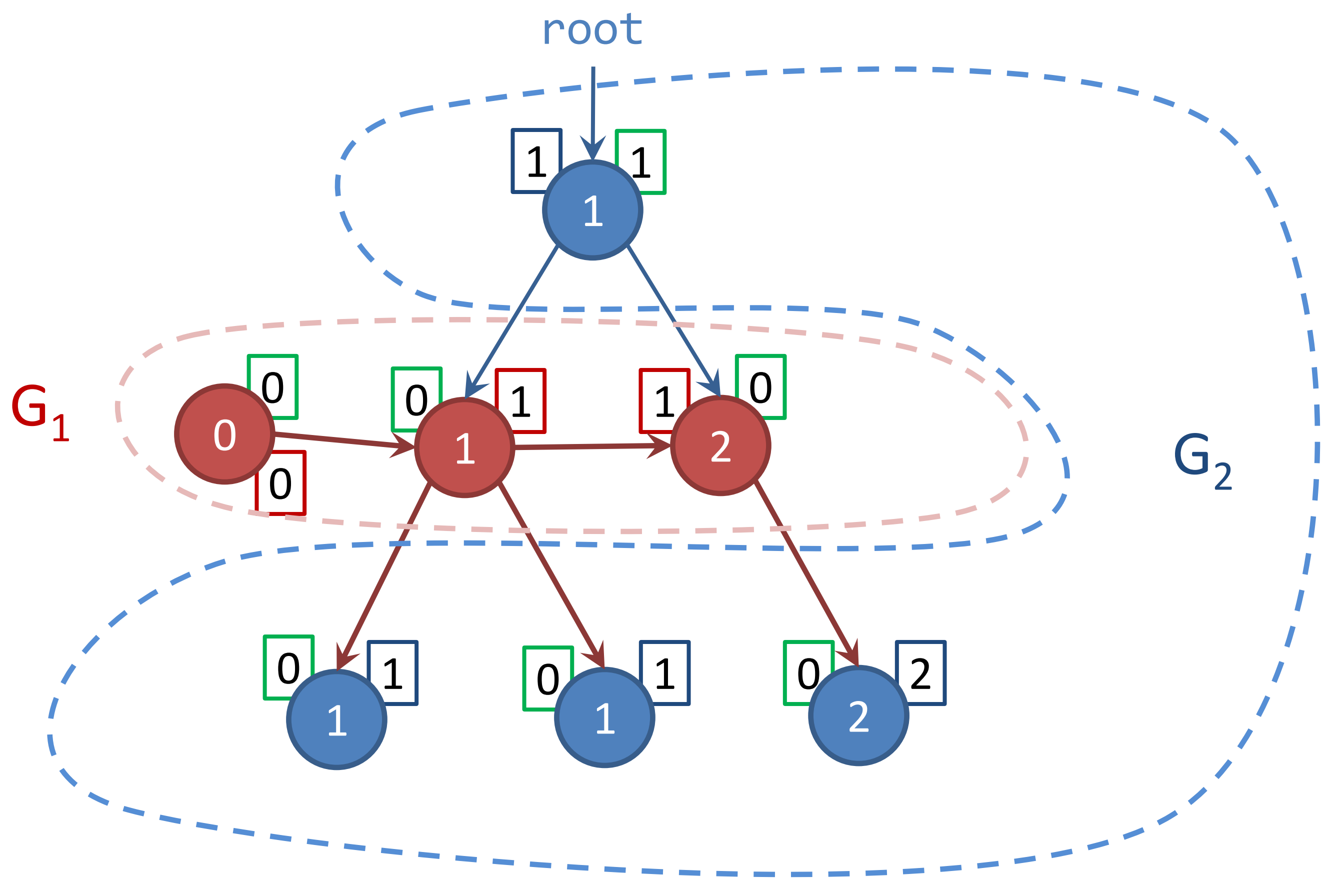


Flow Interface Graph Composition

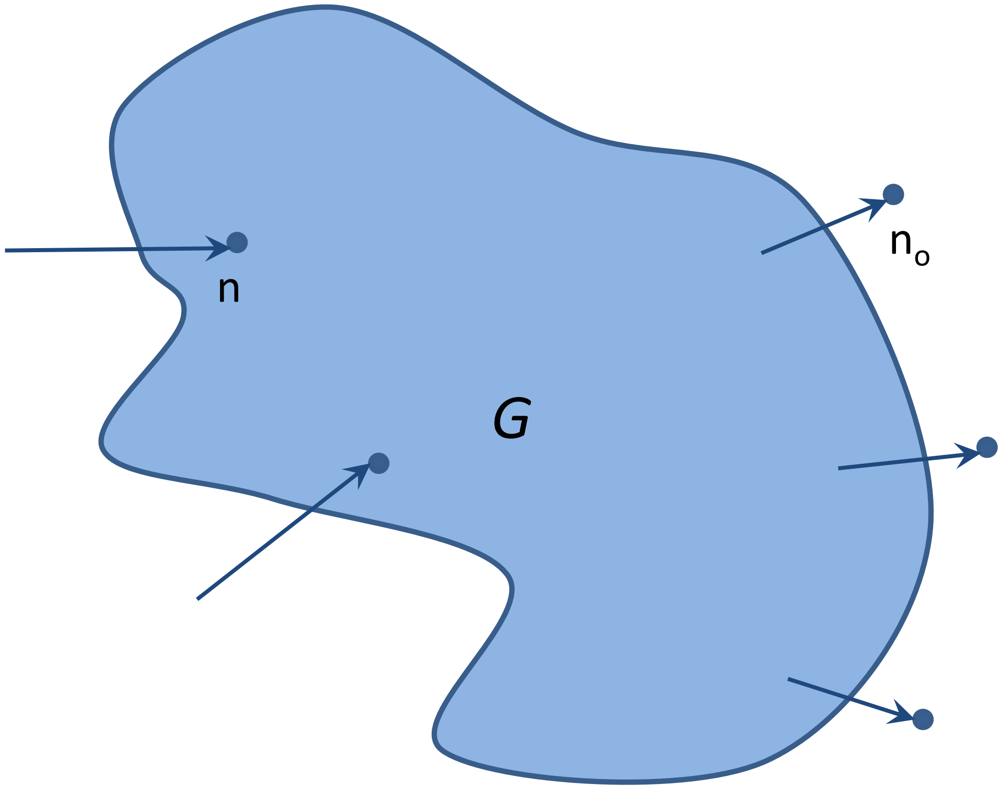
- $H_1 \bullet H_2$ is
  - commutative: $H_1 \bullet H_2 = H_2 \bullet H_1$
  - associative : $(H_1 \bullet H_2) \bullet H_3 = H_1 \bullet (H_2 \bullet H_3)$
  - cancelative: $\mathrm{H} \bullet H_1 = \mathrm{H} \bullet H_2 \Rightarrow H_1 = H_2$

$\Rightarrow$ Flow interface graphs form a *separation algebra*.

$\Rightarrow$ We can use them to give semantics to SL assertions.

- How do we abstract flow interface graphs?

Flow Map of a Flow Interface Graph



$\mathrm{fm}(G)(n, n_o) = \sum \{ \, \mathit{pathproduct}(p) \mid p \text{ path from } n \text{ to } n_o \text{ in } G \}$

$\mathrm{flow}(in, G)(n_o) = \sum \{ \, \mathrm{in}(\mathrm{n}) \cdot \mathrm{fm}(G)(n, n_o) \mid \mathrm{n} \in \mathrm{G} \}$

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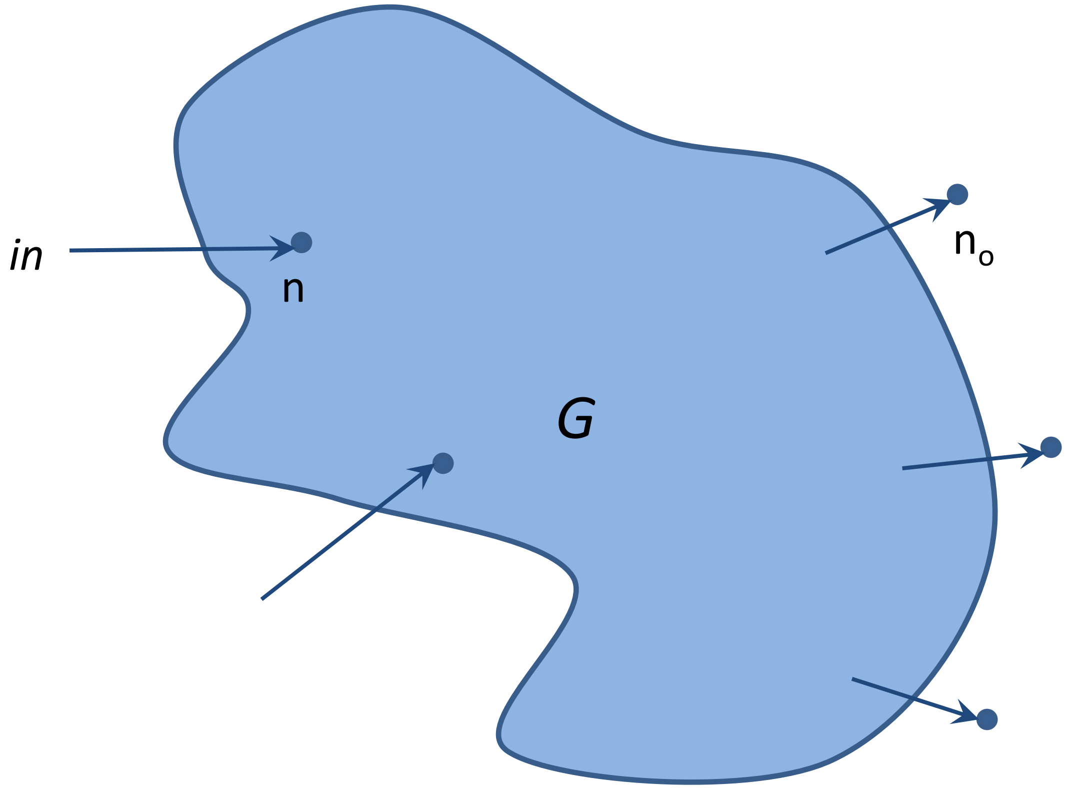

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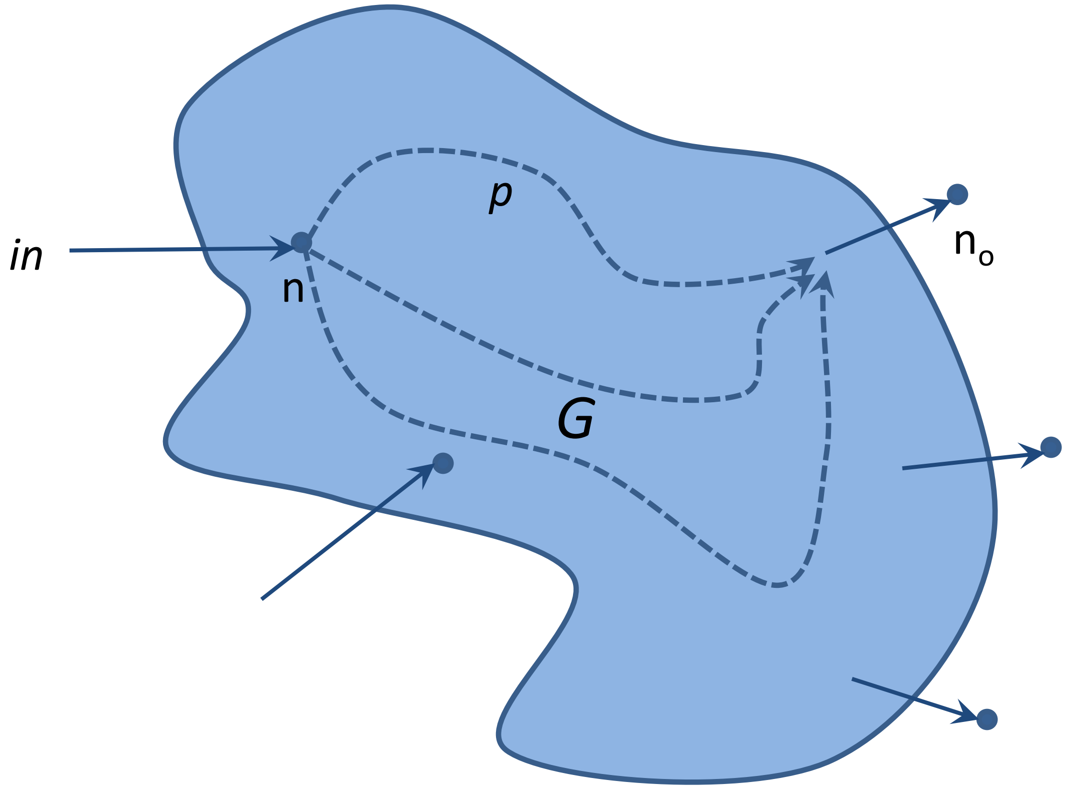

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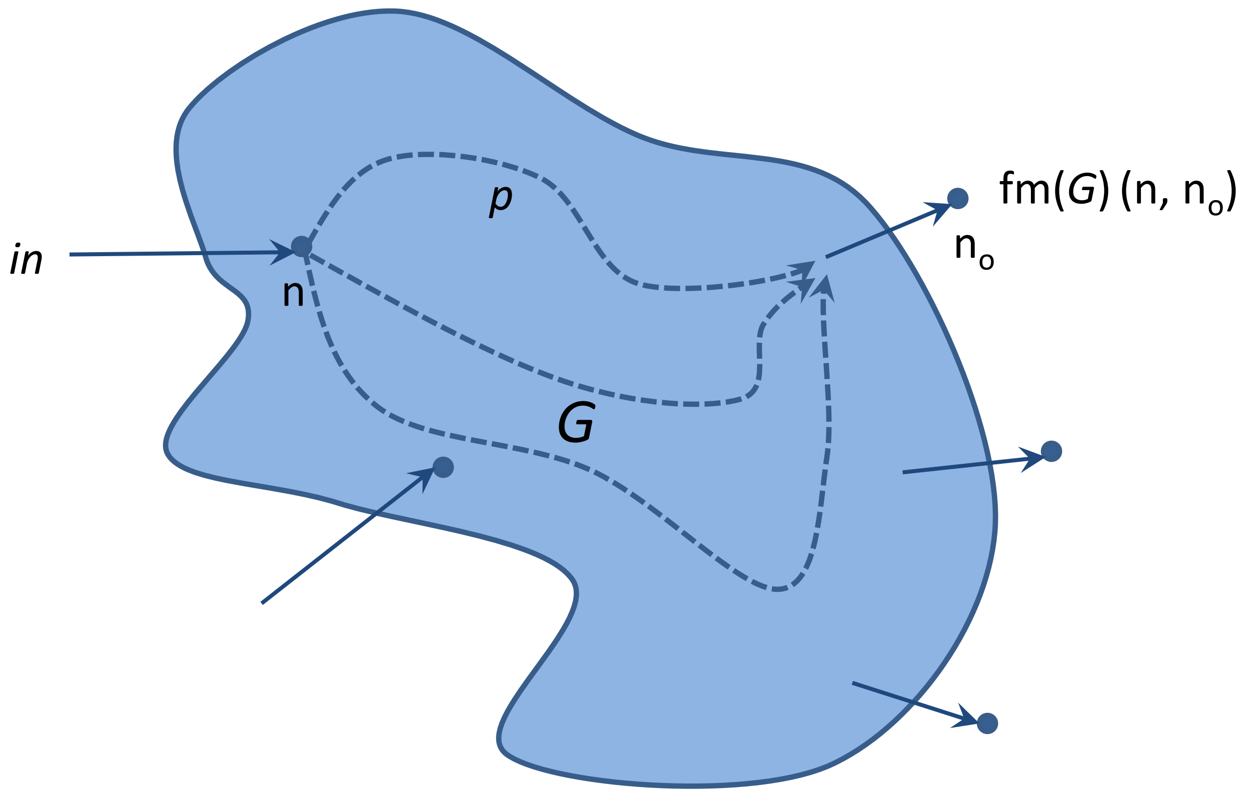

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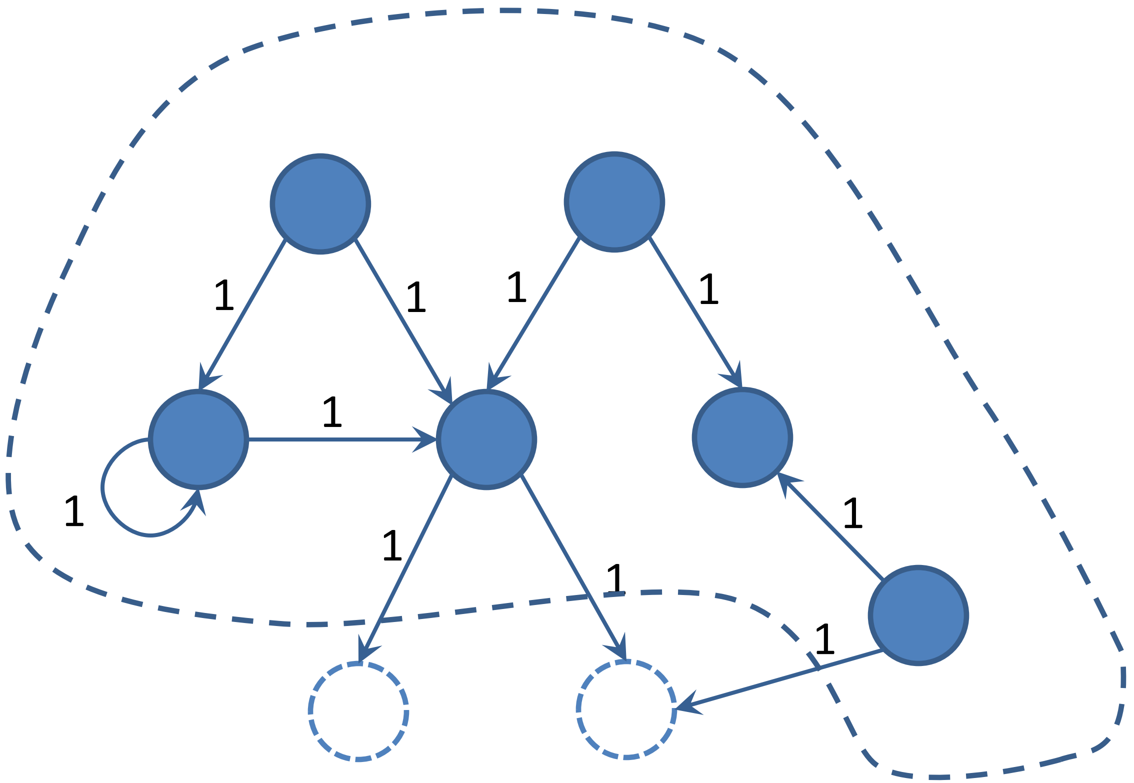
Flow Map: Example



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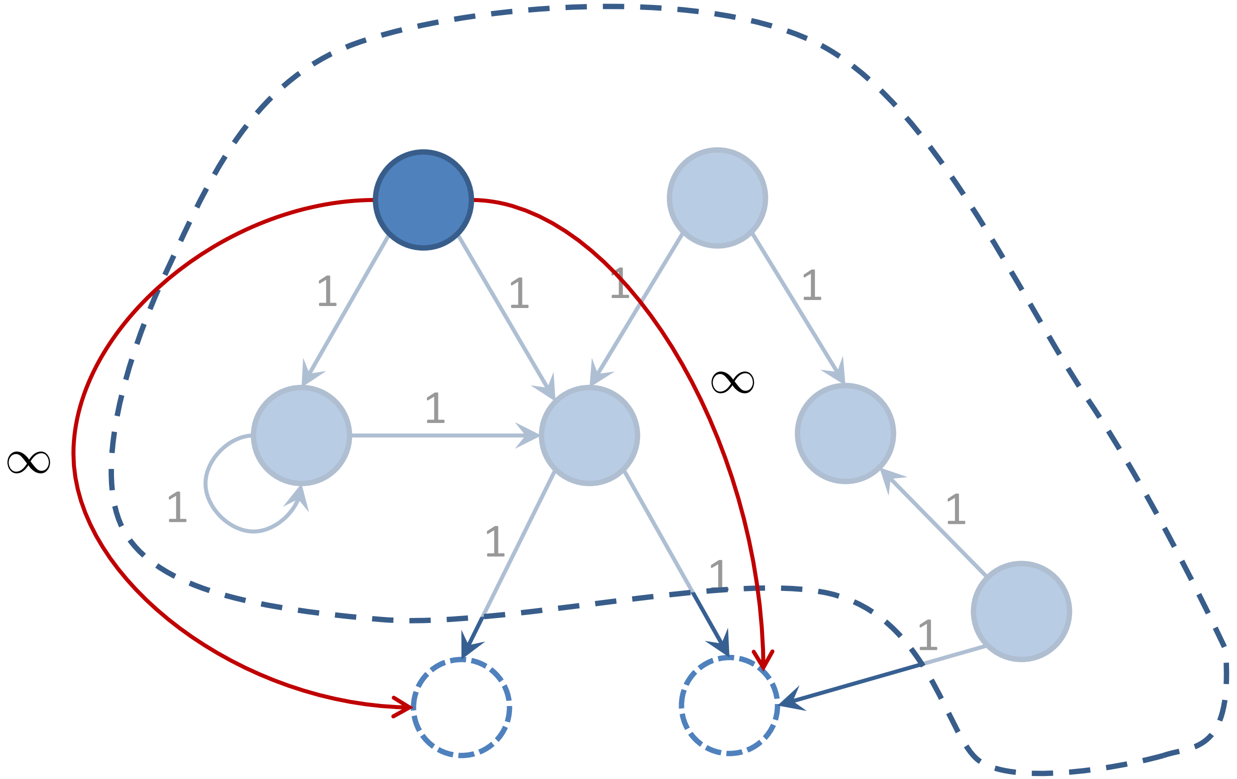
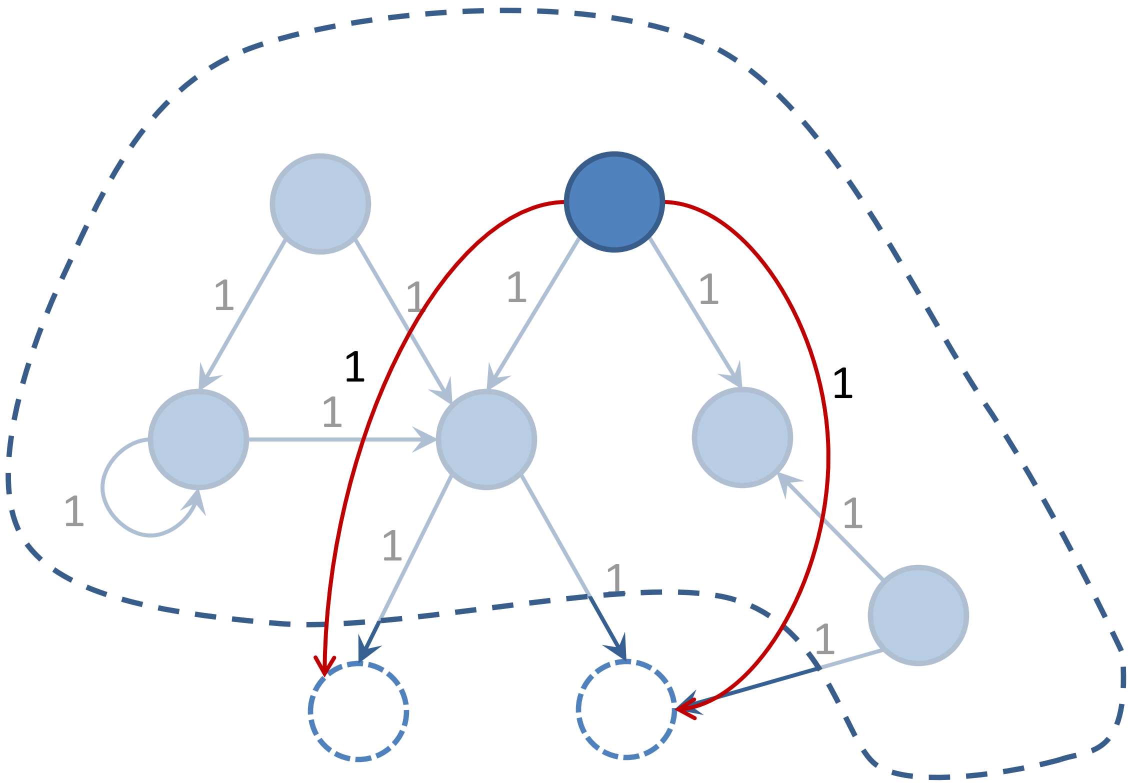


Flow map abstracts from internal structure of the graph

Flow Map: Example



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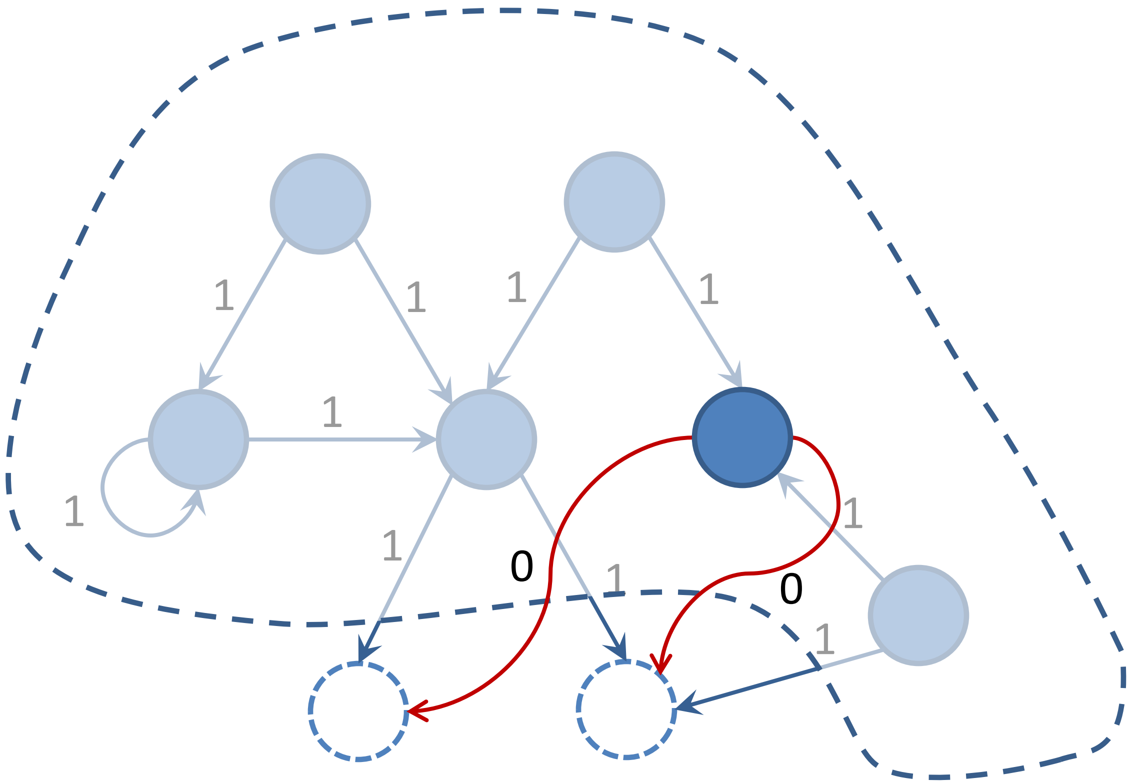


Flow map abstracts from internal structure of the graph

Flow Interfaces

- I = (*in*, *f*) is a *flow interface* if
  - in: N $\to$ D is an inflow
  - f: N $\times$ N_o $\to$ D is a flow map

- $[\![(in, f)]\!]_{good}$ denotes all flow interface graphs (in, G) s.t.
  - fm(G) = f
  - for all n $\in$ N *good*(*in*(n), $G|_n$) holds
- where *good* is some *good node* condition
  - e.g. *good*(i, _) = i $\leq$ 1

Flow Interfaces with Node Abstraction

- $I = (in, \alpha, f)$ is a *flow interface* if
  - in: $N \to D$ is an inflow
  - f: $N \times N_o \to D$ is a flow map
  - $\alpha \in A$ is a node label
- $[\![(in, \alpha, f)]\!]_{\text{good}}$ denotes all flow interface graphs (in, G) s.t.
  - $\text{fm}(G) = f$
  - $\alpha = \sqcup \{ \lambda_G(n) \mid n \in N \}$
  - for all $n \in N$ $good(in(n), G|_n)$ holds
- where *good* is some *good node* condition
  - e.g. $good(i, \_) = i \leq 1$

Flow Interface Composition

Composition of flow interface graphs can be lifted to flow interfaces:

- $I \in I_1 \oplus I_2$ iff $\exists$ H, H_1, H_2 such that
  - $H \in [\![I]\!]$, $H_1 \in [\![I_1]\!]$, and $H_2 \in [\![I_2]\!]$
  - $H = H_1 \bullet H_2$

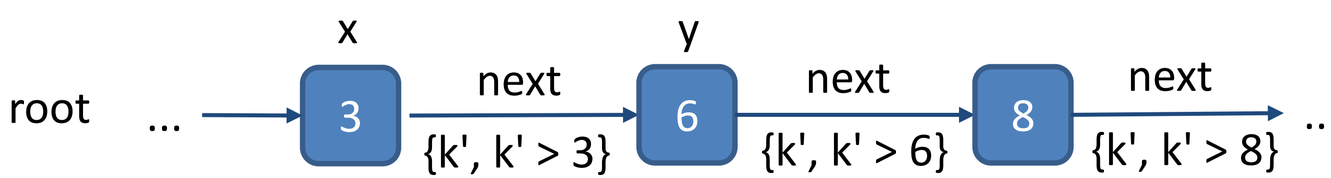
Some nice properties of $\oplus$

- $\oplus$ is associative and commutative
- $[\![I_1]\!] \bullet [\![I_2]\!] \subseteq [\![I_1 \oplus I_2]\!]$
- if $I \in I_1 \oplus I_2$, then for all $H_1 \in [\![I_1]\!]$, $H_2 \in [\![I_2]\!]$, $H_1 \bullet H_2$ defined
- …

Separation Logic with Flow Interfaces

- Good graph predicate $\mathrm{Gr}_\gamma(\mathtt{I})$
  - γ: SL predicate that defines good node condition and abstraction of heap onto nodes of flow graph
  - $\mathtt{I}$: flow interface term
- Good node predicate $\mathrm{N}_\gamma(\mathrm{x}, \mathtt{I})$
  - like Gr but denotes a single node
  - definable in terms of Gr
- Dirty region predicate $[\mathrm{P}]_{\gamma,\mathtt{I}}$
  - P: SL predicate
  - denotes heap region that is expected to satisfy interface $\mathtt{I}$ but may currently not

Graph Predicate: Linked List



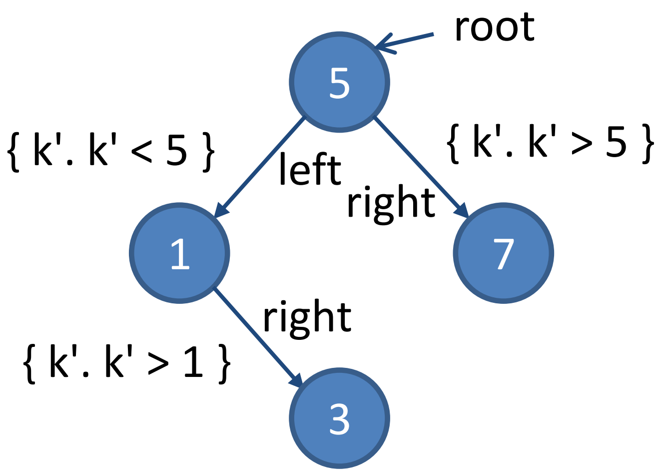
- Abstraction of linked list node

$$\begin{aligned}\gamma(x, in, C, f) = \exists k, y.\ x \mapsto (\text{data: } k, \text{next: } y) \land\ & \\ C = \{k\} \land k \in in \land\ & \\ f = \text{ITE}(\text{y = null}, \epsilon, \{ (x,y) \mapsto \{k'.\ k' > k\} \})&\end{aligned}$$

- Invariant

$$\exists \mathtt{I} :: \text{Gr}_\gamma(\mathtt{I}) \land \mathtt{I}^{in} = \{\text{root} \mapsto \text{KS}\}.\mathbf{0} \land \mathtt{I}^f = \epsilon$$

Graph Predicate: Binary Search Tree



- Abstraction of BST node

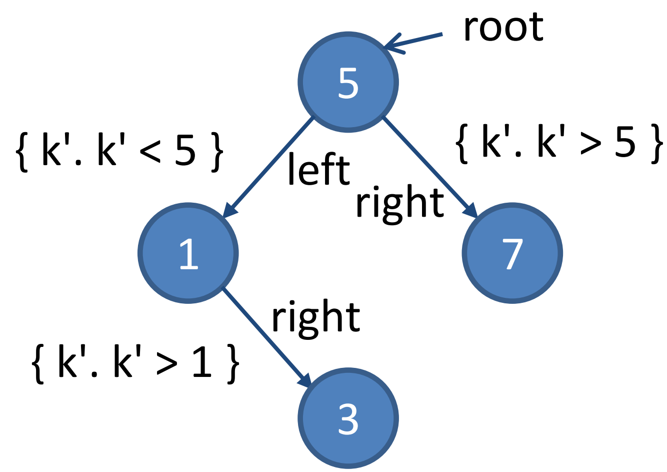
$$\gamma(x, in, C, f) = \exists k, y, z.\ x \mapsto (\text{data: } k, \text{left: } y, \text{right: } z) \wedge$$
$$C = \{k\} \wedge k \in in \wedge$$
$$f = \mathrm{ITE}(y = \text{null}, \epsilon, \{ (x,y) \mapsto \{k'.\ k' < k\} \}.$$
$$\mathrm{ITE}(z = \text{null}, \epsilon, \{ (x,z) \mapsto \{k'.\ k' > k\} \}$$

- Invariant

$$\exists \mathtt{I} :: \mathrm{Gr}_\gamma(\mathtt{I}) \wedge \mathtt{I}^{in} = \{\text{root} \mapsto \mathrm{KS}\}.\mathbf{0} \wedge \mathtt{I}^{f} = \epsilon$$

Graph Predicate: Binary Search Tree

Need tree invariant?



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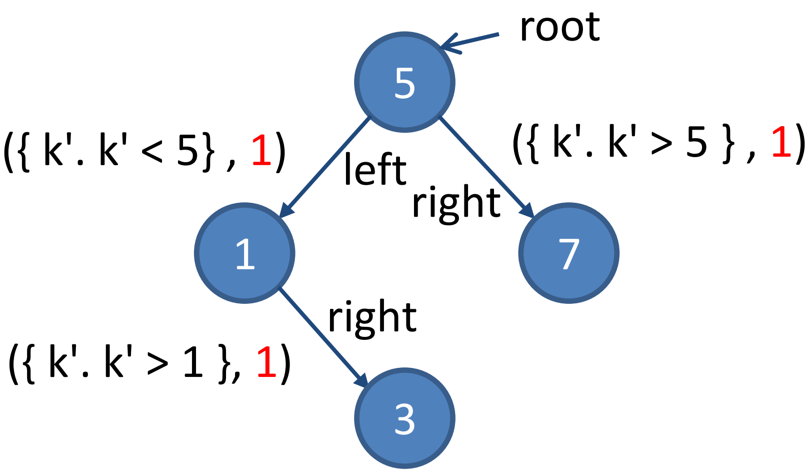
- Invariant

$$\exists \mathrm{I} :: \mathrm{Gr}_\gamma(\mathrm{I}) \wedge \mathrm{I}^{in} = \{\text{root} \mapsto \text{KS}\}.\mathbf{0} \wedge \mathrm{I}^{f} = \epsilon$$

Graph Predicate: Binary Search Tree

Need tree invariant?

No problem!



- Abstraction of BST node

$$\begin{aligned}\gamma(x, (in, pc), C, f) = \exists k, y, z.\ & x \mapsto (\text{data: } k, \text{left: } y, \text{right: } z) \wedge \\ & C = \{k\} \wedge k \in in \wedge\ pc = 1 \wedge \\ & f = \text{ITE}(y = \text{null}, \epsilon, \{ (x,y) \mapsto (\{k'.\ k' < k\}, 1) \}. \\ & \quad \text{ITE}(z = \text{null}, \epsilon, \{ (x,z) \mapsto (\{k'.\ k' > k\}, 1) \}\end{aligned}$$

- Invariant

$$\exists \mathtt{I} :: \mathrm{Gr}_\gamma(\mathtt{I}) \wedge \mathtt{I}^{in} = \{\text{root} \mapsto (\text{KS}, 1)\}.\mathbf{0} \wedge \mathtt{I}^{f} = \epsilon$$

Data-Structure-Agnostic Proof Rules

Decomposition

$$\frac{\mathrm{Gr}(\mathtt{I}) \wedge x \in \mathtt{I}^{\mathrm{in}}}{\mathrm{N}(x, \mathtt{I}_1) * \mathrm{Gr}(\mathtt{I}_2) \wedge \mathtt{I} \in \mathtt{I}_1 \oplus \mathtt{I}_2}$$

Abstraction

$$\frac{\mathrm{Gr}(\mathtt{I}_1) * \mathrm{Gr}(\mathtt{I}_2) \wedge \mathtt{I} \in \mathtt{I}_1 \oplus \mathtt{I}_2}{\mathrm{Gr}(\mathtt{I}) \wedge \mathtt{I} \in \mathtt{I}_1 \oplus \mathtt{I}_2}$$

Replacement

$$\frac{\mathtt{I} \in \mathtt{I}_1 \oplus \mathtt{I}_2 \wedge \mathtt{I}_1 \precsim \mathtt{J}_1}{\mathtt{J} \in \mathtt{J}_1 \oplus \mathtt{I}_2 \wedge \mathtt{I} \precsim \mathtt{J}}$$

Generic R/G Actions

- Lock node $N(x, (in, 0, f)) \rightarrow N(x, (in, T, f))$
- Unlock node $N(x, (in, T, f)) \rightarrow N(x, (in, 0, f))$
- Dirty $[\mathit{true}]_{\mathtt{I}} \wedge \mathtt{I}^{\alpha} = t \rightarrow [\mathit{true}]_{\mathtt{I}}$
- Sync $[\mathit{true}]_{\mathtt{I}} \wedge \mathtt{I}^{\alpha} = t \rightarrow Gr(\mathtt{I}') \wedge \mathtt{I} \precsim \mathtt{I}'$

Conclusion

- Radically new approach for building compositional abstractions of data structures.
- Fits in existing (concurrent) separation logics.
- Enables simple correctness proofs of concurrent data structure algorithms
- Proofs abstract from the details of the specific data structure implementation.