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Elements of Computational Geometry

Convex Hull

A **convex combination** of points $\mathbf{p}_1, \ldots, \mathbf{p}_n$ is

$$\{\alpha_1 \mathbf{p}_1 + \ldots + \alpha_n \mathbf{p}_n \;:\; \alpha_k \geq 0,\, \alpha_1 + \ldots + \alpha_n = 1\}$$

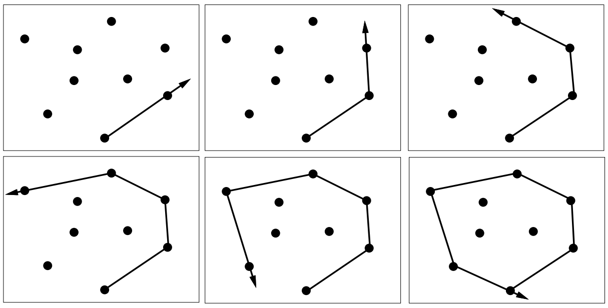
For example, a triangle consists of all convex combinations of its four vertices.

The **convex hull** of a set of points P is the set of all convex combinations of points of P.

Computing the convex hull of a set of points in two or more dimensions in one of the basic problems in computational geometry. Below we consider several popular convex hull algorithms.

2D Gift Wrapping. A set of n points on the plane is given.

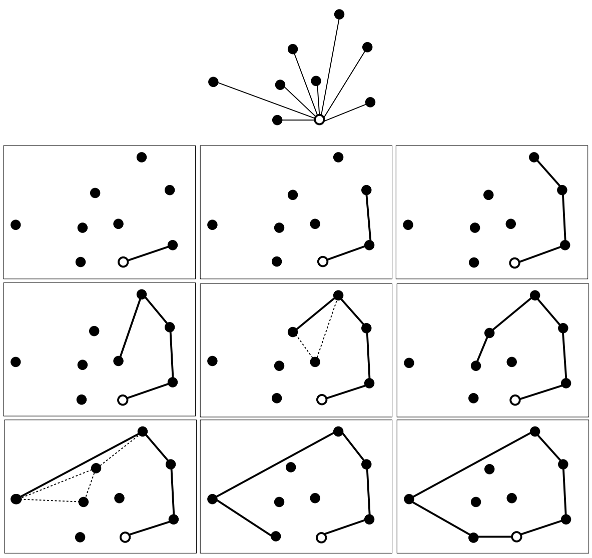
> Find the lowest point $\mathbf{p}_0$ (smallest y-coordinate)
> **repeat**
>   find $\mathbf{p}_{k+1}$ such that all points lie to the left of $\mathbf{p}_k\mathbf{p}_{k+1}$ by scanning through all the points
> **until** $\mathbf{p}_{k+1} = \mathbf{p}_0$.



If the number of sides of the hull is h then the complexity of the algorithm is nh.

Graham Scan. A set of n points on the plane is given.

> 1. Find rightmost lowest point $\mathbf{p}_0$
> 2. Shift the origin of coordinates to $\mathbf{p}_0$ and sort all other points according to their polar angles. If two points have the same polar angle, the point closest to $\mathbf{p}_0$ goes first.
> 3. Build the hull, by marching around all the points, adding edges when a left turn is made, and back-tracking when a right turn is made (use stack to do back-tracking).



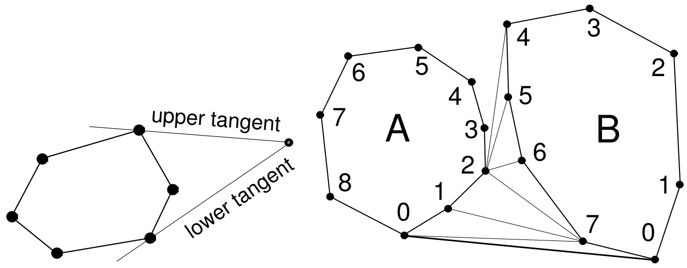
The most expensive part of the Graham scan algorithm is sorting. So the the complexity of the algorithm is $O(n \log n)$.

Divide-and-Conquer. A set of n points on the plane is given. The divide-and-conquer algorithm consists of the following steps

1. Sort the points by x coordinate.
2. Divide the points into two sets A and B, where A contains the left $\lceil n/2 \rceil$ points and B contains the right $\lfloor n/2 \rfloor$ points.
3. Compute the convex hulls of the sets A and B.
4. Merge the convex hulls of A and B into the convex hull of $A \cup B$.

The merging step consists of finding the lower and upper tangents of the convex hulls of A and B. The algorithm to compute the lower tangent is given below.

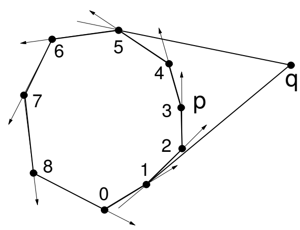
> Enumerate the vertices of the convex hulls of A and B in the counter-clockwise order starting from the lowest points.
> $a \leftarrow$ rightmost point of A.
> $b \leftarrow$ leftmost point of A.
> **while** $T = ab$ is not lower tangent to both A and B **do**
>   **while** T is not lower tangent to A **do**
> $a \leftarrow a - 1$
>   **while** T is not lower tangent to B **do**
> $b \leftarrow b + 1$



The the algorithm complexity is $O(n \log n)$.

Incremental Algorithm. It is easy to invent an incremental algorithm to construct the convex hull of a set of points.

Consider a convex hull (polygon) whose vertices $\mathbf{p}_1, \ldots, \mathbf{p}_n$ are counted in the counter-clockwise direction and a new point $\mathbf{q}$. If the new point $\mathbf{q}$ is inside the hull there is nothing to do. Otherwise, let us delete all the edges that the new point can see. Add two edges to connect the new point to the remainder of the old hull.



An edge $p_k p_{k+1}$ is seen from $\mathbf{q}$ if the signed area of the triangle $\mathbf{q}\mathbf{p}_k\mathbf{p}_{k+1}$ is negative. If $\mathbf{p}_k = (x_k, y_k)$, $\mathbf{p}_{k+1} = (x_{k+1}, y_{k+1})$, and $\mathbf{q} = (x_0, y_0)$ then the signed area of the triangle $\mathbf{q}\mathbf{p}_k\mathbf{p}_{k+1}$ is given by

$$A = \frac{1}{2} \begin{vmatrix} x_0 & y_0 & 1 \\ x_k & y_k & 1 \\ x_{k+1} & y_{k+1} & 1 \end{vmatrix}.$$

Voronoi Diagram

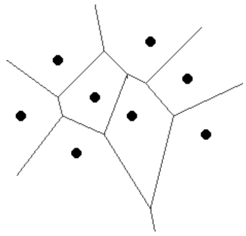
Given a set of points on the plane, the Voronoi diagram of the set is a partition of the plane into cells (Voronoi cells), each of which consists of the points closer to one particular point of the set than to any others.

More precisely, let $P = \{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n\}$ be a set of points on the plane. The points are called the **sites**. The **Voronoi region** $V(\mathbf{p}_i)$ of a site $\mathbf{p}_i$ is the set of all points on the plane that are closer to $\mathbf{p}_i$ than they are to any other site:

$$V(\mathbf{p}_i) = \{\mathbf{x} \;:\; |\mathbf{x} - \mathbf{p}_i| \leq |\mathbf{x} - \mathbf{p}_j| \quad \text{for any} \quad j \neq i\}.$$

The set of all points that have more than one nearest site form the **Voronoi diagram** for the set of sites P.

The edges of the Voronoi regions are called **Voronoi edges**, and the vertices are called **Voronoi vertices**. Note that a point on the interior of a Voronoi edge has two nearest sites, and a Voronoi vertex has at least three nearest sites.



The Voronoi diagram possesses several interesting properties:

V1. Each Voronoi region $V(\mathbf{p}_i)$ is convex.

V2. $V(\mathbf{p}_i)$ is unbounded if and only if $\mathbf{p}_i$ lies on the convex hull of P.

V3. If $\mathbf{v}$ is a Voronoi vertex at the junction of $V(\mathbf{p}_1)$, $V(\mathbf{p}_2)$, and $V(\mathbf{p}_3)$, then v is the center of the circle $C(\mathbf{v})$ determined by $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$. The interior of $C(\mathbf{v})$ contains no sites.

Delaunay Triangulation

Consider $P = \{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n\}$ be a set of points on the plane and let $\mathrm{Vor}(P)$ be its Voronoi diagram. If two Voronoi regions $V(\mathbf{p}_i)$ and $V(\mathbf{p}_j)$ have a common Voronoi edge, let us draw the segment connecting the Voronoi sites $\mathbf{p}_i$ and $\mathbf{p}_j$. The segments form the so-called **Delaunay triangulation** $\mathrm{DT}(P)$ of P. See Fig. 1.

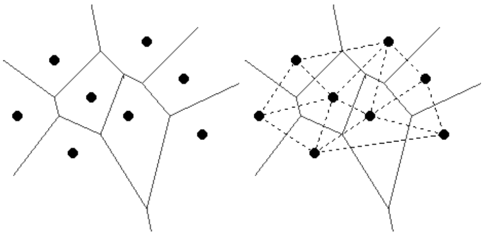


Fig. 1: Left: a set of points and its Voronoi diagram. Right: Dealunay triangulation is added.

The Delaunay triangulation possesses many interesting properties:

D1. The boundary of $\mathrm{DT}(P)$ is the convex hull of the sites.

D2. There no sites inside each triangle of $\mathrm{DT}(P)$.

D3. If there is some circle through $\mathbf{p}_i$ and $\mathbf{p}_j$ which contains no other sites, then $(\mathbf{p}_i, \mathbf{p}_j)$ is an edge of $\mathrm{DT}(P)$. The reverse is also true: for every Delaunay edge, there is some empty circle.

Proof of **D3**. If $\mathbf{ab} \in \mathrm{DT}(P)$, then $V(\mathbf{a})$ and $V(\mathbf{b})$ share an edge $e \in \mathrm{Vor}(P)$. Consider a circle $C(\mathbf{x})$ with center $\mathbf{x}$ on the interior of e with radius equal to the distance to $\mathbf{a}$ or $\mathbf{b}$. If there is a site $\mathbf{c} \in \mathrm{Vor}(P)$ on or in the circle, then $\mathbf{x} \in V(\mathbf{c})$ that gives a contradiction.

Suppose there is an empty circle $C(\mathbf{x})$ through $\mathbf{a}$ and $\mathbf{b}$, with center x. We want to show that $\mathbf{ab} \in \mathrm{DT}(P)$. Note that $\mathbf{x} \in V(\mathbf{a}) \cap V(\mathbf{b})$. Since there no other cites on the boundary of $C(\mathbf{x})$, we can move $\mathbf{x}$ a bit along the bisector between $\mathbf{a}$ and $\mathbf{b}$ and maintain emptiness. Therefore $\mathbf{x}$ lies on a Voronoi edge (a subset of the bisector) shared between $V(\mathbf{a})$ and $V(\mathbf{b})$ and $\mathbf{ab} \in \mathrm{DT}(P)$.

Delaunay Triangulation and Convex Hull in 3D

Consider the paraboloid $z = x^2 + y^2$ and a set of points $P = \{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n\}$ on the xy-plane. With every point $\mathbf{p}_k = (x_k, y_k)$ let us associate a 3D point $(x_k, y_k, x_k^2 + y_k^2)$ and consider the convex hull of the obtained set of 3D points. The convex hull is a convex polyhedron with triangular faces.

Theorem. The Delaunay triangulation $\mathrm{DT}(P)$, where $P = \{(x_k, y_k)\}$, is the projection to the xy-plane of the lower part of the convex hull of the set of 3D points $\{(x_k, y_k, x_k^2 + y_k^2)\}$.

Proof. The plane tangent to the paraboloid at the point $(a, b, a^2 + b^2)$ is

$$z = 2ax + 2by - (a^2 + b^2)$$

Let us shift the plane upward by some positive number r^2. The shifted plane has the equation

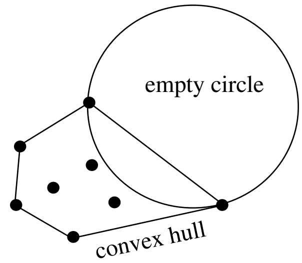
$$z = 2ax + 2by - (a^2 + b^2) + r^2$$

and intersects the paraboloid along a curve that projects to the circle

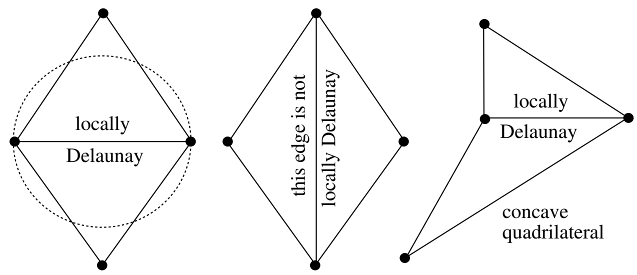
$$(x - a)^2 + (y - b)^2 = r^2.$$

Flip Algorithm for Delaunay Triangulation

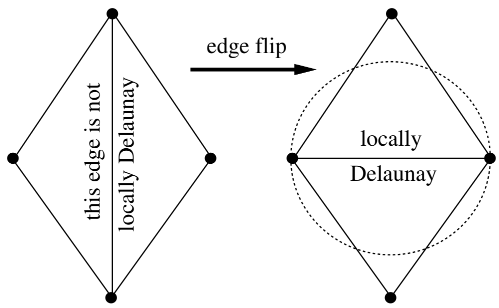
Given a set of points P on the plane, a circle is called **empty** if it contains no point of V. Consider two points $\mathbf{a} \in P$ and $\mathbf{b} \in P$. The edge $\mathbf{ab}$ belongs to the Dalaunay $\mathrm{DT}(P)$ if and only if there exists an empty circle that passes through $\mathbf{a}$ and $\mathbf{b}$. The edge $\mathbf{ab}$ is said to be **Delaunay**. Now it is clear that for any edge belonging to the convex hull of P it is easy to find an empty circle.



An edge is is said to be **locally Delaunay** if and only if its a chord of a circle that does not contain the vertices opposite to the edge. Note that if the triangles adjacent to an edge form a concave quadrilateral, the edge is locally Delaunay.



The **flip algorithm** begins with an arbitrary triangulation, and searches for an edge that is not locally Delaunay. All edges on the boundary (convex hull) of the triangulation are considered to be locally Delaunay. Whenever the flip algorithm identifies an edge that is not locally Delaunay, the edge is flipped.



Proposition 1. Consider a triangulation T of a set of points and let $e \in T$ be an edge of the triangulation. Either e is locally Delaunay, or e is flippable and the edge created by flipping e is locally Delaunay.

Proposition 2. Consider a consider a set of points and its triangulation T which contain the convex hull of the set and whose edges are locally Delaunay. Then T is the Delaunay triangulation.

Theorem. Given a set n points and its triangulation containing the convex hull of the set, the flip algorithm terminates after $O(n^2)$ edge flips, yielding the Delaunay triangulation.