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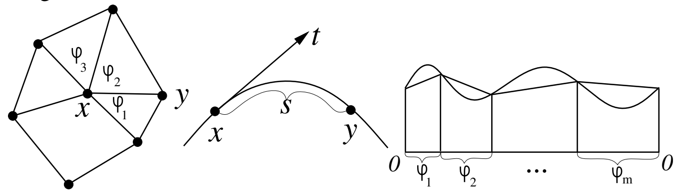
Curvature Estimation

Triangle mesh

Integral estimation of the mean curvature. Note that

$$\frac{1}{2\pi}\int_0^{2\pi} k_n(\varphi)\,d\varphi = \frac{k_{\max}+k_{\min}}{2} = H$$

This observation leads to a method to estimate the mean curvature of a triangulated surface.



$$\mathbf{r}(s) = \mathbf{r}(0) + s\mathbf{r}'(0) + \frac{s^2}{2}\mathbf{r}''(0) + \ldots, \quad \mathbf{y} = \mathbf{x} + s\mathbf{t} + \frac{s^2}{2}k\mathbf{n} + \ldots,$$

$$\mathbf{n}(\mathbf{y}-\mathbf{x}) \approx k\frac{s^2}{2}, \quad s \approx \|\mathbf{y}-\mathbf{x}\|, \quad k \approx \frac{2\mathbf{n}(\mathbf{y}-\mathbf{x})}{\|\mathbf{y}-\mathbf{x}\|^2}$$

$$\int_0^{2\pi} k_n(\varphi)\,d\varphi \approx k_1\left(\frac{\varphi_1+\varphi_m}{2}\right) + k_2\left(\frac{\varphi_2+\varphi_3}{2}\right) + \ldots$$

Integral estimation of the Gaussian curvature. Note that

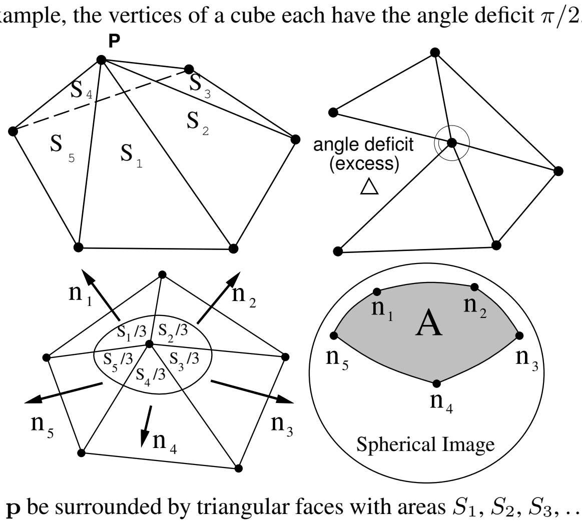
$$\frac{1}{2\pi}\int_0^{2\pi} k_n(\varphi)^2\,d\varphi = \frac{3}{2}H^2 - \frac{1}{2}K.$$

It allows us to estimate the Gaussian curvature of a triangulated surface similarly to the above estimation of the mean curvature.

Gaussian curvature estimation via Gauss mapping. The angle deficit $\Delta(\mathbf{p})$ of a vertex $\mathbf{p}$ of a polygonal surface is defined as the vertex angle deficit (excess)

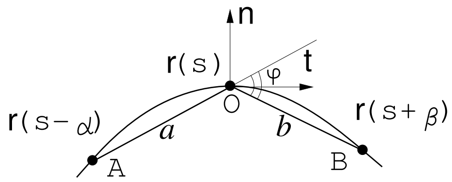
$$\Delta(\mathbf{p}) = 2\pi - \theta(\mathbf{p}) = 2\pi - \sum_{i=1}^{m}\theta_i(\mathbf{p})$$

For example, the vertices of a cube each have the angle deficit $\pi/2$.



Let $\mathbf{p}$ be surrounded by triangular faces with areas $S_1, S_2, S_3, \ldots$ and unit normals $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3, \ldots$. The spherical image of the polygonal surface is a set of points on the unit sphere (the heads of unit vectors parallel to $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3, \ldots$). Let us connect these points by arcs of great circles to form a spherical polygon on the unit sphere. The area A of the spherical polygon is equal to the angle deficit of $\mathbf{p}$. The area of each triangular face adjacent to $\mathbf{p}$ can be portioned into three equal parts corresponding to the vertices of the face. So the total area related to $\mathbf{p}$ is $\sum S_k/3$. Thus the Gaussian curvature at $\mathbf{p}$ can be approximated by

$$K(\mathbf{p}) = 3\Delta(\mathbf{p}) \Big/ \sum S_k$$



Curvature vector estimation for 2D polylines.

Let $\boldsymbol{t} = \boldsymbol{r}'$, $\boldsymbol{n} = \boldsymbol{t}^{\perp}$ be the Frenet basis at O. According to the Frenet formulas

$$\boldsymbol{t}' = \boldsymbol{k} = k\,\boldsymbol{n}, \quad \boldsymbol{n}' = -k\,\boldsymbol{t},$$

where k is the curvature and $\boldsymbol{k}$ is the curvature vector.

Let us also set

$$\begin{aligned} \boldsymbol{a} &= \boldsymbol{r}(O) - \boldsymbol{r}(A) = -\overrightarrow{OA}, \\ \boldsymbol{b} &= \boldsymbol{r}(B) - \boldsymbol{r}(O) = \overrightarrow{OB}. \end{aligned}$$

The second derivative of $\boldsymbol{r}(s)$ at O can be approximated by

$$\begin{aligned} \boldsymbol{r}'' = \boldsymbol{t}' = \boldsymbol{k} = k\boldsymbol{n} \approx \\ \approx \frac{2\boldsymbol{r}(A)}{a(a+b)} - \frac{2\boldsymbol{r}(O)}{ab} + \frac{2\boldsymbol{r}(B)}{(a+b)b}. \end{aligned} \qquad (1)$$

Expanding the right-hand side of the above finite-difference approximation into Taylor series with respect to a and b and using the Frenet formulas we arrive at the following expansion of the curvature vector

$$\begin{aligned} \boldsymbol{k} = k\,\boldsymbol{n} &\approx \frac{2\boldsymbol{r}(A)}{a(a+b)} - \frac{2\boldsymbol{r}(O)}{ab} + \frac{2\boldsymbol{r}(B)}{(a+b)b} = \\ &= \frac{2}{a+b}\left[\frac{\boldsymbol{r}(B)-\boldsymbol{r}(O)}{b} + \frac{\boldsymbol{r}(A)-\boldsymbol{r}(O)}{a}\right] = \\ &= \frac{2}{a+b}\left[\frac{\boldsymbol{b}}{b} - \frac{\boldsymbol{a}}{a}\right] = \\ &= \left(\frac{a-b}{4}k^2 + O(a,b)^2\right)\boldsymbol{t} + \\ &+ \left(k + \frac{b-a}{3}k' + O(a,b)^2\right)\boldsymbol{n}. \end{aligned}$$

Note that (1) can be considered as the 2D version of the mean curvature normal estimation scheme proposed in [**?**] since

$$-\frac{2}{a+b}\nabla(a+b) = \frac{2}{a+b}\left[\frac{\overrightarrow{OA}}{a} + \frac{\overrightarrow{OB}}{b}\right],$$

where ∇ is the gradient with respect to the position of O. Note also that (1) defines the angle bisector direction at O.

Mean curvature vector estimation via area variation. Desbrun *et al.* [1] proposed recently an accurate and robust discrete approximation of the mean curvature vector at a mesh vertex P:

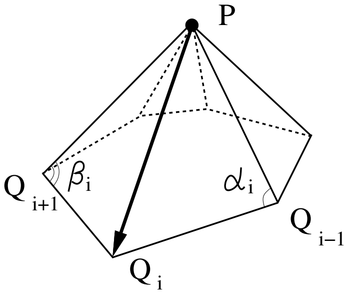
$$H\boldsymbol{n} = -\frac{\nabla A}{2A},$$

where $A = \sum A_i$ is the sum of the areas of the triangles surrounding P. Calculations show that

$$H\boldsymbol{n}(P) = \frac{1}{4A}\sum_i(\cot\alpha_i + \cot\beta_i)(Q_i - P), \qquad (2)$$

where $\{Q_i\}$ are the neighbors of P, α_i and β_i are the two angles opposite to the edge Q_iP, as seen in the figure below.

[1] M. Desburn, M. Meyer, P. Schröder, and A. H. Barr, “Implicit Fairing of Irregular Meshes using Diffusion and Curvature Flow”, *SIGGRAPH’99*.

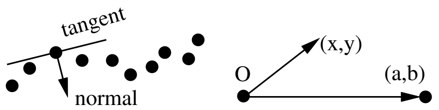


Least Squares Estimation

Least squares for normal and tangent plane estimation. Consider a set of 3D points located along a smooth surface. Our aim is to estimate the surface tangent plane and normal at each point.

First let us consider the simplest case of two 2D points. The unit normal vector (x, y) delivers minimum in

$$\min_{x^2+y^2=1} (ax + by)^2 \qquad (3)$$



To solve (3) via the method of Lagrange multipliers let us form the function

$$L(x, y, \lambda) = (ax + by)^2 - \lambda(x^2 + y^2 - 1)$$

$$\frac{\partial L}{\partial x} = 0 = \frac{\partial L}{\partial y} = 0 = \frac{\partial L}{\partial \lambda}, \quad \begin{pmatrix} a^2 & ab \\ ab & b^2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

Thus $(x, y) = (-b, a)/\sqrt{a^2 + b^2}$.

Now let us consider a set of 3D points $X = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ assumed to be on or near an unknown surface. Let $\mathbf{x} \in X$ be a point where we want to estimate the tangent plane and normal. The scatter matrix is defined as a symmetric matrix

$$M(\mathbf{x}) = \sum_{\mathbf{y} \in Nbhd(\mathbf{x})} (\mathbf{y} - \mathbf{x}) \otimes (\mathbf{y} - \mathbf{x})$$

where $\otimes$ denotes the outer product vector operation (if $\mathbf{a}$ and $\mathbf{b}$ have components a_i and b_j respectively, then the matrix $\mathbf{a} \otimes \mathbf{b}$ has $a_i b_j$ as its ij-entry). Here $Nbhd(\mathbf{x})$ can be defined, for example, as

$$Nbhd(\mathbf{x}) = \{\mathbf{y} \in X \,:\, \|\mathbf{x} - \mathbf{y}\| \leq \rho\}$$

where ρ is a given positive parameter.

Let $\lambda_1 \geq \lambda_2 \geq \lambda_3$ denote the eigenvalues of $M(\mathbf{x})$ associated with unit eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, respectively. The unit normal vector at $\mathbf{x}$, $\boldsymbol{n}(\mathbf{x})$, can be chosen either $\mathbf{v}_3$ or $-\mathbf{v}_3$. The eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ corresponding to the two largest eigenvalues form a basis in the tangent plane at $\mathbf{x}$.

Least squares for curvature estimation. Once we decide the normal and two tangent directions, we use them as a local reference frame. The biquadratic fit involves finding the five coefficients of the polynomial

$$z = f(x, y) = ax^2 + bxy + cy^2 + dx + ey$$

where the coordinates (x, y) are measured along the tangent directions, and the coordinate z is measured along the normal direction. If there are at least four neighboring points, say $n - 1$ points, the least-square estimate of the five coefficients can be done:

$$F(a, b, c, d, e) \longrightarrow \min$$

where

$$F(a, b, c, d, e) = \sum \left(ax_k^2 + bx_k y_k + cy_k^2 + dx_k + ey_k - z_k\right)^2$$

In matrix form, the coefficient vector $\vec{c} = (a, b, c, d, e)^T$ which minimize the fit error is

$$A^T A \vec{c} = A^T \vec{Z}$$

where A is a $5 \times n$ matrix with kth row being $[x_k^2, x_k y_k, y_k^2, x_k, y_k]$, $\vec{Z}$ is a column vector of the z_k values, and the superscript T stands for transposition. Once the coefficients a, b, c, d, e are found, curvature computations are straightforward.

Curvatures Features of Implicit Surfaces

Let us consider a surface given in implicit form

$$F(x^1, x^2, x^3) = 0$$

We use upper indices for vector components, subindices for partial derivatives with respect to x^i, $i = 1, 2, 3$. Let us consider the matrix $-\nabla\mathbf{n}$ where the unit normal vector $\mathbf{n}$ is given by

$$\mathbf{n} = -\nabla F / |\nabla F|.$$

Calculations show that

$$-\nabla\mathbf{n} = \frac{1}{|\nabla F|} \left(\mathbf{I} - \mathbf{n} \cdot \mathbf{n}^T\right) \mathbf{Hes}(F),$$

where $\mathbf{I}$ is the identity matrix, $\mathbf{Hes}(F)$ is the Hessian of F, $[\mathbf{Hes}(F)]_{ij} = F_{ij}$, and $\left[\mathbf{n} \cdot \mathbf{n}^T\right]_{ij} = F_i F_j / |\nabla F|^2$. The matrix $-\nabla\mathbf{n}$ has the eigenvalues $\lambda_1 = k_{\max}$, $\lambda_2 = k_{\min}$, and $\lambda_3 = 0$ associated with the eigenvectors $\mathbf{v}_1 = \mathbf{t}_{\max}$, $\mathbf{v}_2 = \mathbf{t}_{\min}$, and $\mathbf{v}_3 = \mathbf{n}$, respectively. Thus the characteristic polynomial $\det[\nabla\mathbf{n} + \lambda\mathbf{I}]$ of $-\nabla\mathbf{n}$ has the form $\lambda(\lambda^2 - 2H\lambda + K)$. It allows us to represent the Gaussian and mean curvatures K and H in the following elegant form

$$K = -\frac{1}{|\nabla F|^4} \det \begin{vmatrix} F_{11} & F_{12} & F_{13} & F_1 \\ F_{21} & F_{22} & F_{23} & F_2 \\ F_{31} & F_{32} & F_{33} & F_3 \\ F_1 & F_2 & F_3 & 0 \end{vmatrix}$$

$$H = \frac{1}{2|\nabla F|^3} \left(|\nabla F|^2 \Delta F - \sum_{i=1}^{3} \sum_{j=1}^{3} F_i F_j F_{ij} \right),$$

where Δ is the Laplace operator, the sum of second-order derivatives with respect to x^1, x^2, x^3. The principal curvatures are given by

$$k_{\max} = H + \sqrt{H^2 - K}, \qquad k_{\min} = H - \sqrt{H^2 - K}.$$

In order to find the principal direction $\mathbf{t}_{\max}$ let us consider the vectors obtained as the pairwise vector products of the rows of the matrix $[\nabla\mathbf{n} + k_{\max}\mathbf{I}]$. Now let us choose among them a vector of maximal norm and normalize it. We get $\mathbf{t}_{\max}$. The principal direction $\mathbf{t}_{\min}$ is found as the vector product of $\mathbf{t}_{\max}$ and $\mathbf{n}$.