Alexander Belyaev (belyaev@mpi-sb.mpg.de, http://www.mpi-sb.mpg.de/∼belyaev)

Topology of Surfaces and Critical Points of Functions

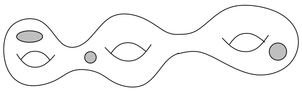
Problems

1. Consider a triangulation of a compact connected closed surface S. Show that
$$3F = 2E, \quad 2E \le V(V-1), \quad V \ge \frac{7+\sqrt{49-24\chi}}{2}.$$
If S is a sphere, then $V \ge 4$, $E \ge 6$, $F \ge 4$.

2. There are 20 points inside a square. They are connected by non-intersecting segments with each other and with the vertices of the square, in such a way that the square is dissected into triangles. How many triangles are there?

3. Consider a triangulation of a sphere. Suppose the triangulation has only two types of vertices: where 5 edges meet and where 6 edges meet. Let V_5 be the number of vertices of the first type (at which exactly 5 edges meet) and V_6 be the number of vertices of the second type (at which exactly 6 edges meet). Determine V_5.

4. Find the Euler characteristics of the following surface



5. Show that for any triangulation of a torus
$V \ge 7$, $E \ge 21$, $F \ge 14$.

6. Consider a regular triangulation of a torus: the same number of triangles, say a, meet at each vertex. Determine a.

7. Draw a vector filed on the torus
   - (a) without critical points
   - (b) with two centers and two saddles
   - (c) with one center and one saddle point

8. Draw a vector filed on the sphere with
   - (a) two nodes
   - (b) two centers
   - (c) only one critical point
   - (d) three critical points

9. Consider an island.
   - (a) Suppose the island has no lakes. Show that
   $$\#\, piks - \#\, passes + \#\, pits = 1.$$
   - (b) Suppose now that the island has lakes. Prove that
   $$\#\, piks - \#\, passes + \#\, pits = 1 - \#\, lakes.$$

Critical points.

Let $f : \mathbf{R}^n \to \mathbf{R}$ be a smooth function. A point $u = (u_1, \ldots, u_n)$ is a **critical point** of f if
$$\left.\frac{\partial f}{\partial x_1}\right|_{x=u} = \ldots = \left.\frac{\partial f}{\partial x_1}\right|_{x=u} = 0.$$
The value $f(u)$ at a critical point u is called a **critical value** of f.

Critical points occur when the graph of f has a horizontal tangent. If $n = 1$ these are traditionally classified as **maxima**, **minima**, and **inflections**, see Fig. 1.

Note that maxima and minima are stable with respect to small perturbations of f and the inflections are unstable.

It turns out that stability of a critical point depends on the **Hessian matrix**
$$H_f = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]_{x=u}$$
of f at the critical point: the critical point is stable if and only if $\det H_f \neq 0$.

For a function of two variables, the stable critical points are maxima, minima and saddles, see Fig. 2. Some unstable critical points are demonstrated at Fig. 3.

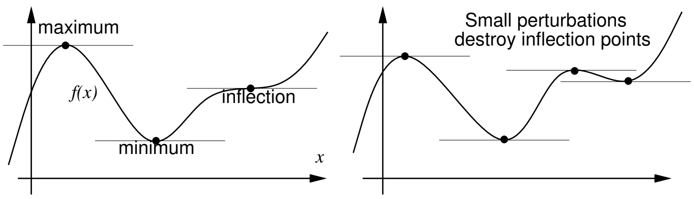


Figure 1: Critical points of a function of one variable.

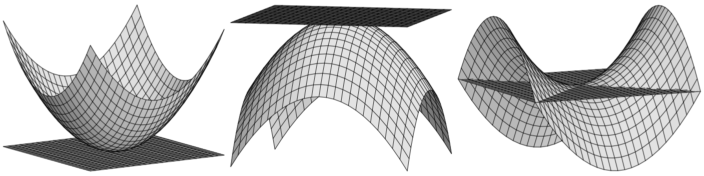


Figure 2: Stable critical points: a minimum, a maximum, a saddle.

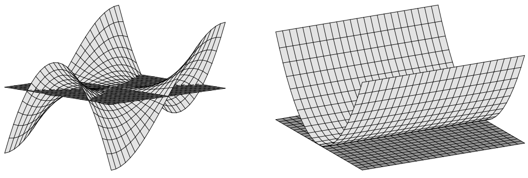


Figure 3: Unstable critical points: a monkey saddle, a pig-trough.

Critical Point Configuration Graph.

As one moves higher, contours bounding a local maximum, called a **peak**, become smaller and smaller, ultimately becoming a point. Likewise, as one moves lower, contours bounding local minimum, called a **pit**, become smaller and smaller, ultimately becoming a point as well. However, in some cases, as one smoothly changes elevation, two contours meet at a single point, forming a single self-intersecting contour topologically equivalent to the figure 8. That point, called a **a pass** or a saddle, is nether a local minimum nor a local maximum.

In general peaks, pits, and passes are isolated from each other. It therefore is meaningful to consider paths between them. A curve drawn so that it crosses at right angles every contour it meets is called a **slope line**. At every point on a slope line there are two possible directions of travel, one ascending, one descending. These two directions are the directions of steepest ascent and steepest descent, respectively, from any point on the slope line. Therefore, if one travels along a slope line in the ascending direction, one must eventually reach a peak or a pass; traveling in the opposite direction, one must eventually reach a pit, a pass, or the island coastline. From any point of a slope line, the portion traversed by traveling in the ascending direction is called the **ascending slope line** from that point, while the other portion is called the **descending slope line** from that point.

Mathematically, a stable critical point is

- a peak (local maximum)
  if $\det(H_f) > 0$ and $\mathrm{tr}(H_f) < 0$;
- a pit (local minimum)
  if $\det(H_f) > 0$ and $\mathrm{tr}(H_f) > 0$;
- a pass (saddle) if $\det(H_f) < 0$.

Here $H_f = \left[\partial^2 f / \partial x_i \partial x_j\right]$ is the Hessian matrix.

The slope line are the integral paths of the gradient vector field associated with the height function $z = f(x, y)$
$$V(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

and, hence, are the solutions of the system of differential equations.

$$\left\{ \begin{array}{lcl} dx/dt & = & \partial f/\partial x \\ dy/dt & = & \partial f/\partial y \end{array} \right.$$

Note that the peaks, pits, and passes are the critical points of the system. Moreover, the peaks and pits are nodes and the passes are saddles.

Each pass is associated with four slope lines meeting in it: two ascending and two descending. Each ascending (descending) slope line emanating from a pass either intersects the coastline, reaches a peak (pit), or reaches another pass.

We can now define a **Critical Point Configuration Graph**. Let us mark pits, passes, and peaks by ∇, $+$, and Δ, respectively. The edges are the slope lines emanating from the passes. arrows on the edges indicate the ascending direction.

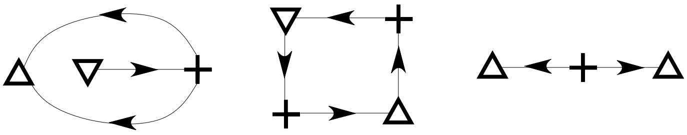
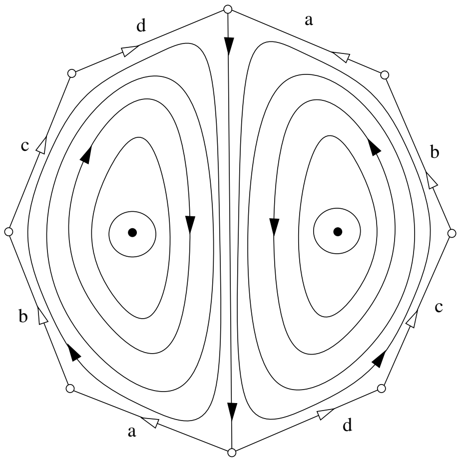


Figure 4: Critical Point Configuration Graphs.

Problem. Draw topographic maps (set of contours) associated with Critical Point Configuration Graphs exposed at Fig. 4

Critical points of functions on closed surfaces.

On a sphere with handles there exists a smooth function f with only three critical points: the minimum, the maximum, and a degenerate saddle. See the level sets of a function with only three critical points on a sphere with two handles



Problem. Show that the function constructed above cannot be realized as a height function.

Problem. Let $\mathcal{C}$ be a simple closed planar curve and let $f_{\mathcal{C}}$ denote the function along the curve $\mathcal{C}$. The set of points on the curve at which the gradient vector of f is not tangent to the curve and is directed into its interior is called the *entrant portal* of the region enclosed by the curve. Let $n_{\bigtriangledown}$, n_{+} and $n_{\bigtriangleup}$ denote, respectively, the number of pits, passes, and peaks of f in the region enclosed by $\mathcal{C}$, and let $n_{\downarrow}$ and $n_{\uparrow}$ denote, respectively, the number of of minima and maxima of $f_{\mathcal{C}}$ that occur on the entrant portal. Suppose that there are no critical points of $f(x, y)$ on $\mathcal{C}$ and the extrema of $f_{\mathcal{C}}$ are isolated. Prove that

$$n_{\bigtriangledown} + n_{\downarrow} - n_{+} - n_{\uparrow} + n_{\bigtriangleup} = 1$$