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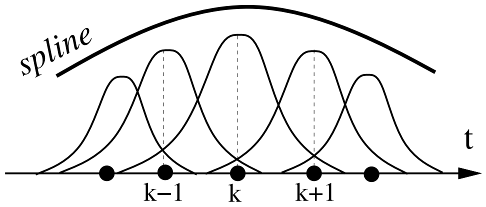
Uniform B-splines. Subdivision.

Consider control points $\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_n$, $\mathbf{p}_i = (x_i, y_i, z_i)$. The weighting (blending) functions $\{w_k(t)\}$ define a curve

$$\mathbf{r}(t) = \mathbf{p}_0 w_0(t) + \mathbf{p}_1 w_1(t) + \ldots + \mathbf{p}_n w_n(t) \qquad (1)$$

whose shape is determined by the control polygon $\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_n$.

It is very convenient if for each k the graph of $w_k(t)$ has a bell-shaped form.



Convolution.

The convolution of two functions $f(t)$ and $g(t)$ is given by

$$m(t) = (f \otimes g)(t) = \int f(s) g(t-s)\, ds.$$

The convolution satisfies the following properties:

$$\begin{array}{rl} f(t) \otimes (g(t) + h(t)) = f(t) \otimes g(t) + f(t) \otimes h(t) & \text{linearity} \\ f(t-i) \otimes g(t-k) = m(t-i-k) & \text{time shift} \\ f(2t) \otimes g(2t) = \frac{1}{2} m(2t) & \text{time scaling} \end{array}$$

Box function.

A piecewise-constant function can be represented by

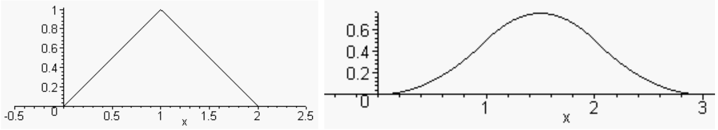
$$f(t) = \sum c_k B_0(t-k) \quad \text{where} \quad B_0(t) = \begin{cases} 1 & \text{if } \ 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$

The function $B_0(t)$ is called the **box function**.

If $f(t)$ is C^k-continuous, then $f(t) \otimes B_0(t)$ is C^{k+1}-continuous.

Let us define $B_n(x) = B_{n-1}(x) \otimes B_0(t)$. Thus $B_n(t)$ is C^n-continuous.

The figure below shows the graphs of $B_1(x)$ (left image) and $B_2(x)$ (right image).



One can see that

$$\mathbf{r}(t) = \sum c_k B_0(t-k)$$

is a piecewise-constant function, while

$$\mathbf{r}(t) = \sum c_k B_1(t-k)$$

defines a piecewise-linear function.

B-splines: dilation property.

A polygonal curve can be represented by

$$\mathbf{r}(t) = \sum c_k B_1(t-k), \quad B_1(t) = B_0 \otimes B_0.$$

The box function $B_0(t)$ satisfies the following **dilation equation**

$$B_0(t) = B_0(2t) + B_0(2t-1).$$

It implies

$$B_1(t) = \frac{1}{2}\big(B_1(2t) + 2B_1(2t-1) + B_1(2t-2)\big)$$

$$B_2(t) = \frac{1}{4}\big(B_2(2t) + 3B_2(2t-1) + 3B_2(2t-2) + B_2(2t-3)\big)$$

$$B_n(t) = \frac{1}{2^n} \sum_{k=0}^{n+1} \binom{n+1}{k} B_n(2t-k),$$

where $\binom{n+1}{k}$ are binomial coefficients.

The **linear B-spline** $\mathbf{r}(t) = \sum_k \mathbf{p}_k B_1(t-k)$ can be rewritten as

$$\sum_k \mathbf{p}_k \left(\frac{1}{2} B_1(2t-2k) + B_1(2t-2k-1) + \frac{1}{2} B_1(2t-2k-2) \right).$$

It means we have doubled the number of control points by adding the midpoints of the segments $\mathbf{p}_k \mathbf{p}_{k+1}$. Re-parameterization $(2t-1) \to t$ gives

$$\mathbf{p}_{2k}^{\text{new}} = \mathbf{p}_k^{\text{old}}, \quad \mathbf{p}_{2k+1}^{\text{new}} = \frac{1}{2}\left(\mathbf{p}_k^{\text{old}} + \mathbf{p}_{k+1}^{\text{old}}\right).$$

The **quadratic B-spline** $\mathbf{r}(t) = \sum_k \mathbf{p}_k B_2(t-k)$ can be rewritten as

$$\sum_k \mathbf{p}_k \left(\frac{1}{4} B_2(2t-2k) + \frac{3}{4} B_2(2t-2k-1) + \frac{3}{4} B_2(2t-2k-2) + \frac{1}{4} B_2(2t-2k-3) \right).$$

It means we have doubled the number of control points and the new control points are given as linear combination of old ones. Re-parameterization $(2t-1) \to t$ gives

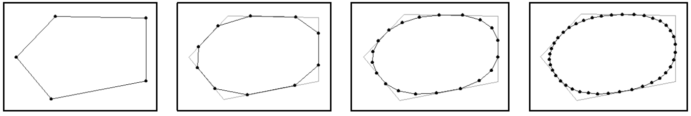
$$\mathbf{p}_{2k}^{\text{new}} = \frac{3}{4}\mathbf{p}_k^{\text{old}} + \frac{1}{4}\mathbf{p}_{k-1}^{\text{old}}, \quad \mathbf{p}_{2k+1}^{\text{new}} = \frac{3}{4}\mathbf{p}_k^{\text{old}} + \frac{1}{4}\mathbf{p}_{k+1}^{\text{old}}.$$

We can rewrite the above as follows

$$\mathbf{q}_{2k} = \mathbf{p}_k, \quad \mathbf{q}_{2k+1} = \frac{1}{2}\left(\mathbf{p}_k^{\text{old}} + \mathbf{p}_{k+1}^{\text{old}}\right), \qquad \mathbf{p}_k^{\text{new}} = \sum_i a_i \mathbf{q}_{k+i},$$

where $(a_{-1}, a_0) = \frac{1}{2}(1,1)$ and all other a_i are zeros.

It turns out that if we keep repeating the above corner-cutting procedure, the resulting polyline will converge to a smooth curve which is the quadratic B-spline approximating the initial control points. See the figure below.



The **cubic B-spline** $\mathbf{r}(t) = \sum_k \mathbf{p}_k B_3(t-k)$ can be rewritten as

$$\sum_k \mathbf{p}_k \left(\frac{1}{8} B_3(2t-2k) + \frac{4}{8} B_3(2t-2k-1) + \frac{6}{8} B_3(2t-2k-2) + \frac{4}{8} B_2(2t-2k-3) + \frac{1}{8} B_3(2t-2k-4) \right).$$

Re-parameterization $(2t-2) \to t$ gives

$$\mathbf{p}_{2k}^{\text{new}} = \frac{1}{8}\mathbf{p}_{k-1}^{\text{old}} + \frac{6}{8}\mathbf{p}_k^{\text{old}} + \frac{1}{8}\mathbf{p}_{k+1}^{\text{old}}, \quad \mathbf{p}_{2k+1}^{\text{new}} = \frac{1}{2}\mathbf{p}_k^{\text{old}} + \frac{1}{2}\mathbf{p}_{k+1}^{\text{old}}.$$

It is convenient to rewrite the above as follows

$$\mathbf{q}_{2k} = \mathbf{p}_k, \quad \mathbf{q}_{2k+1} = \frac{1}{2}\left(\mathbf{p}_k^{\text{old}} + \mathbf{p}_{k+1}^{\text{old}}\right), \qquad \mathbf{p}_k^{\text{new}} = \sum_i a_i \mathbf{q}_{k+i},$$

where $(a_{-1}, a_0, a_1) = \frac{1}{4}(1,2,1)$ and all other a_i are zeros.

Cubic B-splines via subdivision

A common approach to free form shape design consists of approximating smooth curves (surfaces) by piecewise polynomial curves (surfaces).

The subdivision approach is based on modeling smooth curves (surfaces) as a limit of successively refined polygonal curves (surfaces).

Consider a polyline $(\ldots, \mathbf{p}_1^j, \mathbf{p}_0^j, \mathbf{p}_{-1}^j, \ldots)$. The subdivision procedure consists of two steps: the **splitting step** and the **averaging step**. The splitting step inserts the midpoints in the segments of the polyline

$$\mathbf{q}_{2k}^{j+1} = \mathbf{p}_k^j, \qquad \mathbf{q}_{2k+1}^{j+1} = \frac{1}{2}\left(\mathbf{p}_k^j + \mathbf{p}_{k+1}^j\right)$$

The averaging step computes a weighted average of several successive points

$$\mathbf{p}_k^{j+1} = \sum_i a_i \mathbf{q}_{k+i}^{j+1}$$

For example, if we consider the averaging mask

$$(a_{-1}, a_0, a_1) = \frac{1}{4}(1, 2, 1)$$

the polyline $(\ldots, \mathbf{p}_1^j, \mathbf{p}_0^j, \mathbf{p}_{-1}^j, \ldots)$ converges to a cubic B-spline as $j \to \infty$.

Let us find $\mathbf{p}^\infty$. The key observation in determining $\mathbf{p}^\infty$ is that in each step of subdivision, the position of $\mathbf{p}^{j+1}$ and its immediate neighbors can be determined from $\mathbf{p}^j$ and its neighbors. If $\mathbf{p}_+$ and $\mathbf{p}_-$ denote the left and right neighbors of $\mathbf{p}$, the splitting and averaging steps can be combined to give

$$\begin{aligned}
\mathbf{p}_-^{j+1} &= \frac{1}{2}\left(\mathbf{p}_-^j + \mathbf{p}^j\right) \\
\mathbf{p}^{j+1} &= \frac{1}{8}\left(\mathbf{p}_-^j + 6\mathbf{p}^j + \mathbf{p}_+^j\right) \\
\mathbf{p}_+^{j+1} &= \frac{1}{2}\left(\mathbf{p}^j + \mathbf{p}_+^j\right)
\end{aligned}$$

or, in matrix form,

$$\begin{bmatrix} \mathbf{p}_-^{j+1} \\ \mathbf{p}^{j+1} \\ \mathbf{p}_+^{j+1} \end{bmatrix} = \mathbf{S} \begin{bmatrix} \mathbf{p}_-^{j} \\ \mathbf{p}^{j} \\ \mathbf{p}_+^{j} \end{bmatrix}, \quad \text{where} \quad \mathbf{S} = \frac{1}{8}\begin{bmatrix} 4 & 4 & 0 \\ 1 & 6 & 1 \\ 0 & 4 & 4 \end{bmatrix}$$

Thus

$$\begin{bmatrix} \mathbf{p}_-^{\infty} \\ \mathbf{p}^{\infty} \\ \mathbf{p}_+^{\infty} \end{bmatrix} = \lim_{j\to\infty} (\mathbf{S})^j \begin{bmatrix} \mathbf{p}_-^{0} \\ \mathbf{p}^{0} \\ \mathbf{p}_+^{0} \end{bmatrix}$$

The matrix $\mathbf{S}$ has eigenvalues $\lambda_1 = 1$, $\lambda_2 = 1/2$, $\lambda_3 = 1/4$ associated with the right eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

Decomposing with respect to the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ gives

$$\begin{bmatrix} \mathbf{p}_-^{0} \\ \mathbf{p}^{0} \\ \mathbf{p}_+^{0} \end{bmatrix} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3,$$

$$\begin{bmatrix} \mathbf{p}_-^{\infty} \\ \mathbf{p}^{\infty} \\ \mathbf{p}_+^{\infty} \end{bmatrix} = \lim_{j\to\infty} (\mathbf{S})^j \begin{bmatrix} \mathbf{p}_-^{0} \\ \mathbf{p}^{0} \\ \mathbf{p}_+^{0} \end{bmatrix} =$$

$$= \lim_{j\to\infty} \left(c_1(\lambda_1)^j\mathbf{v}_1 + c_2(\lambda_2)^j\mathbf{v}_1 + c_3(\lambda_3)^j\mathbf{v}_1\right) = c_1\mathbf{v}_1$$

Thus, the limit position $\mathbf{p}^\infty$ is the middle entry of $c_1\mathbf{v}_1$. Since the middle entry of $\mathbf{v}_1$ is 1, the limit position $\mathbf{p}^\infty$ is c_1. Computations show that

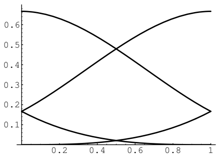
$$\mathbf{p}^\infty = \frac{1}{6}\left(\mathbf{p}_-^0 + 4\mathbf{p}^0 + \mathbf{p}_+^0\right).$$

One can demonstrate that $\mathbf{p}^\infty$ lies on the cubic B-spline. Indeed, the B-spline segment generated by the control points $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ is given by

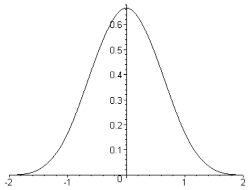
$$\mathbf{r}(t) = \sum_{k=0}^{3} b_k(t)\mathbf{p}_k$$

with

$$\begin{aligned}
b_0(t) &= (1-t)^3/6 \\
b_1(t) &= (4 - 6t^2 + 3t^3)/6, \\
b_2(t) &= (1 + 3t + 3t^2 - 3t^3)/6 \\
b_3(t) &= t^3/6
\end{aligned}$$



$$B_3(t) = \begin{cases} 0 & t < -2 \\ b_3(t+2) & -2 \le t < -1 \\ b_2(t+1) & -1 \le t < 0 \\ b_1(t) & 0 \le t < 1 \\ b_0(t-1) & 1 \le t < 2 \\ 0 & t \ge 2 \end{cases}$$

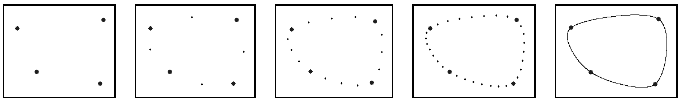


Thus $\mathbf{r}(1) = \dfrac{1}{6}\left(\mathbf{p}_1 + 4\mathbf{p}_2 + \mathbf{p}_3\right)$ and $\mathbf{p}^\infty$ belongs to the cubic B-spline.

Interpolatory subdivision

One can define an interpolatory subdivision scheme as follows

$$\mathbf{p}_{2k}^{j+1} = \mathbf{p}_k^j, \qquad \mathbf{p}_{2k+1}^{j+1} = \frac{1}{16}\left(-\mathbf{p}_{k-1}^j + 9\mathbf{p}_k^j + 9\mathbf{p}_{k+1}^j - \mathbf{p}_{k+2}^j\right)$$



Here we use the mask $\dfrac{1}{16}(-1, 9, 9, -1)$.