Polar forms

Every polynomial curve $P(u)$ of degree $\leq n$ can be associated with a unique n-variate symmetric polynomial $p[u_1,\ldots,u_n]$ such that $p[u,\ldots,u]=P(u)$

The polynomial is referred to as the **polar form** or **blossom** of $P(u)$

Polar forms: properties

- $P[u_1,\ldots,u_n]$ agrees with $p(u)$ on its diagonal

$$P[u,\ldots,u]=p(u)$$

- $P[u_1,\ldots,u_n]$ is symmetric in its variables which means that for any permutation $(u_{i_1},\ldots,u_{i_n})$ of $(u_1,\ldots,u_n)$

$$p[u_1,\ldots,u_n]=p[u_{i_1},\ldots,u_{i_n}]$$

- $p[u_1,\ldots,u_n]$ is affine in each variable:

$$p[\alpha u+\beta u,u_2,\ldots,u_n]=$$
$$\alpha\, p[u,u_2,\ldots,u_n]+\beta\, p[u,u_2,\ldots,u_n]$$

Polar forms: examples I

$$n=1,\quad P(u)=1,\quad p(u)=1$$

$$n=1,\quad P(u)=u,\quad p(u)=u$$

$$n=2,\quad P(u)=1,\quad p(u_1,u_2)=1$$

$$n=2,\quad P(u)=u,\quad p(u_1,u_2)=(u_1+u_2)/2$$

$$n=2,\quad P(u)=u^2,\quad p(u_1,u_2)=u_1u_2$$

$$n=2,\quad P(u)=u^2,\quad p(u_1,u_2)=u_1u_2$$

$$n=2,\quad P(u)=a_0+a_1u+a_2u^2,$$
$$p(u_1,u_2)=a_0+a_1(u_1+u_2)/2+a_2u_1u_2$$

Polar forms: examples II

$$n=3,\quad P(u)=1,\quad p(u_1,u_2,u_3)=1$$

$$n=3,\quad P(u)=u,\quad p(u_1,u_2,u_3)=(u_1+u_2+u_3)/3$$

$$n=3,\quad P(u)=u^2,\quad p(u_1,u_2,u_3)=(u_1u_2+u_2u_3+u_3u_1)/3$$

$$n=3,\quad P(u)=u^3,\quad p(u_1,u_2)=u_1u_2u_3$$

$$p(u)=u^3+6u^2+3u+1$$

$$p[u_1,u_2,u_3]=1+3\frac{u_1+u_2+u_3}{3}+6\frac{u_1u_2+u_2u_3+u_3u_1}{3}+u_1u_2u_3$$

Polar forms: examples III

$$P(u) = u^k,$$

$$p(u_1, \ldots, u_n) = \frac{1}{\binom{n}{k}} \sum_{1 \le j_1 < \ldots < j_k \le n} u_{j_1} \ldots u_{j_k}$$

Bézier Curves over arbitrary interval

$$\Delta = [a, b], \quad \varphi : [a, b] \to [0,1], \quad \varphi(u) = \frac{u-a}{b-a}$$

$$B_k^{\Delta,n}(u) := B_k^n(\varphi(u)) = \binom{n}{k} \varphi(u)^k (1 - \varphi(u))^{n-k}$$

$$= \binom{n}{k} \left(\frac{u-a}{b-a}\right)^k \left(1 - \frac{u-a}{b-a}\right)^{n-k} = \binom{n}{k} \left(\frac{u-a}{b-a}\right)^k \left(\frac{b-u}{b-a}\right)^{n-k}$$

Polar forms and Bézier curves I

$$u = \left(\frac{u-a}{b-a}\right)b + \left(\frac{b-u}{b-a}\right)a$$

$$P(u) = p(u,\ldots,u) = \left(\frac{u-a}{b-a}\right)p(b,u,\ldots,u) + \left(\frac{b-u}{b-a}\right)p(a,u,\ldots,u)$$

$$= \left(\frac{u-a}{b-a}\right)^2 p(b,b,u,\ldots,u) + 2\left(\frac{u-a}{b-a}\right)\left(\frac{b-u}{b-a}\right)p(b,a,u,\ldots,u)$$

$$+ \left(\frac{u-a}{b-a}\right)^2 p(a,a,u,\ldots,u) = \sum_{k=0}^{n}\binom{n}{k}\left(\frac{u-a}{b-a}\right)^k\left(\frac{b-u}{b-a}\right)^{n-k} p(\underbrace{a,\ldots a}_{n-k},\underbrace{b,\ldots b}_{k})$$

$$= \sum_{k=0}^{n} B_k^{\Delta,n}(u)\, p(\underbrace{a,\ldots a}_{n-k},\underbrace{b,\ldots b}_{k})$$

Polar forms and Bézier curves II

Any polynomial curve $P(u)$ defined over interval [0,1] can be considered as a Bézier curve with control points

$$P_k = p[\underbrace{0,\ldots 0}_{n-k \text{ times}}, \underbrace{1,\ldots,1}_{k \text{ times}}]$$

For example:

$$P(u) = u^3 + 6u^2 + 3u + 1, \quad 0 \le u \le 1$$

$$p(u_1,u_2,u_3) = u_1u_2u_3 + 2(u_1u_2 + u_2u_3 + u_1u_3) + (u_1 + u_2 + u_3) + 1$$

$$P_0 = p[0,0,0] = 1, \quad P_1 = p[0,0,1] = 2,$$

$$P_2 = p[0,1,1] = 5, \quad P_3 = p[1,1,1] = 11$$

Polar forms and Bézier curves III

Any polynomial curve $p(u)$ defined over interval [a,b] can be considered as a Bézier curve with control points

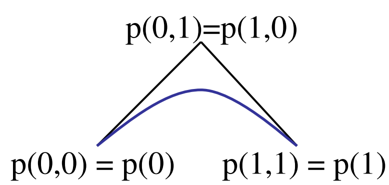
$$P_k = p[\underbrace{a,\ldots a}_{n-k \text{ times}}, \underbrace{b,\ldots,b}_{k \text{ times}}]$$

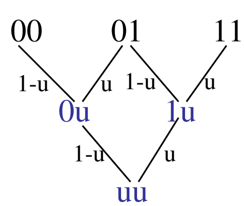
For example:

$$P(u) = \begin{bmatrix} u \\ u^2+1 \end{bmatrix}, \quad u \in [-1,1], \qquad p(u_1,u_2) = \begin{bmatrix} (u_1+u_2)/2 \\ u_1u_2+1 \end{bmatrix},$$

$$P_0 = p(-1,-1) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad P_1 = p(-1,1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad P_2 = p(1,1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

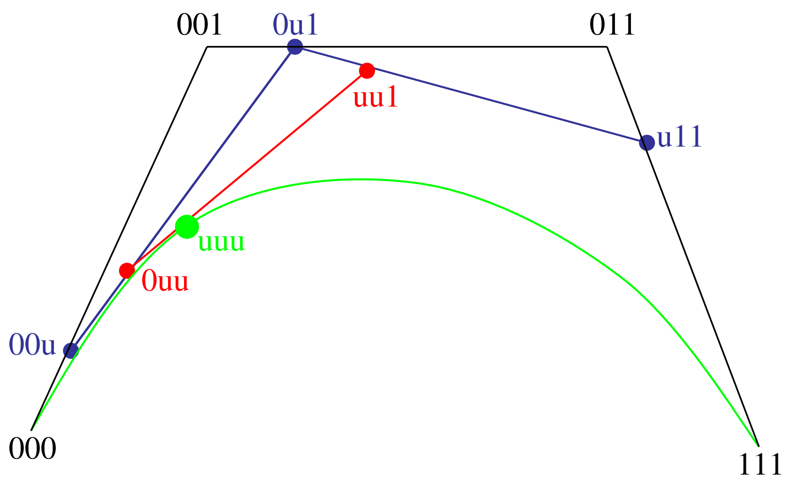
Polar Forms in de Casteljau Algorithm I





p(u) = p(u,u) = $(1\text{-}u)^2$ P0 + 2u(1-u) P1 + u^2 P2

Polar Forms in de Casteljau Algorithm II



Degree Elevation

$$p[u_1,\ldots,u_n] \qquad p(u) = p[u,\ldots,u]$$

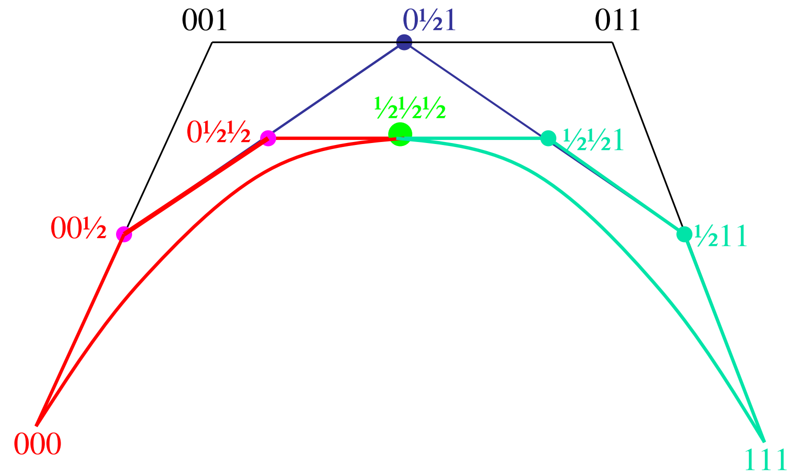
$$q[u_0,\ldots,u_n] = \frac{1}{n+1}\sum_{k=0}^{n} p[u_0,\ldots,u_k^*,\ldots,u_n]$$

$$Q_k = q[\underbrace{a,\ldots,a}_{n+1-k},\underbrace{b,\ldots,b}_{k}]$$

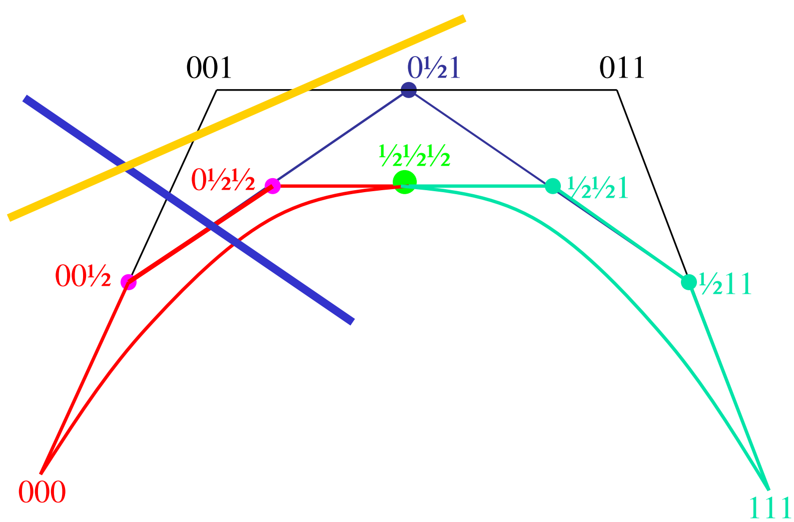
$$= \frac{k}{n+1} p[\underbrace{a,\ldots,a}_{n+1-k},\underbrace{b,\ldots,b}_{k-1}] + \frac{n+1-k}{n+1} p[\underbrace{a,\ldots,a}_{n-k},\underbrace{b,\ldots,b}_{k}]$$

$$= \frac{k}{n+1} P_{k-1} + \left(1-\frac{k}{n+1}\right) P_k$$

Subdividing Bézier Curve

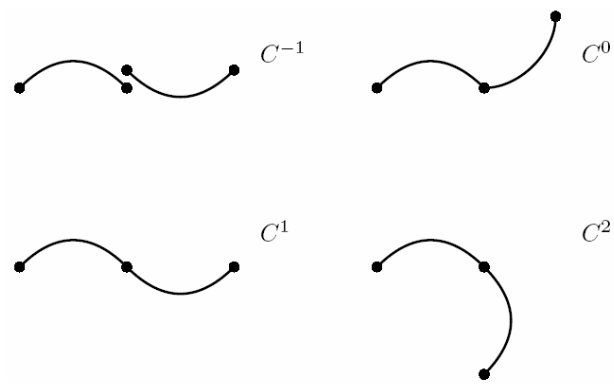


Variation Diminishing property



C^k-Continuity of two Bézier Curves I

C^0-continuity = continuous, no jumps
C^1-continuity = C^0-continuity + same tangent vector
C^2-continuity = C^1-continuity + same osculating circle



C^k-Continuity of two Bézier Curves II

$$\Delta_1 = [a,b], \quad \Delta_2 = [b,c], \quad a < b < c$$

$$P(u) = \sum_{i=0}^{n} \mathbf{P}_i B_i^{\Delta_1,n}(u) \quad Q(u) = \sum_{i=0}^{n} \mathbf{Q}_i B_i^{\Delta_2,n}(u)$$

$$P(u) = \sum_{i=0}^{n} B_i^{\Delta_1,n}(u)\, p[\underbrace{a,\ldots,a}_{n-i},\underbrace{b,\ldots,b}_{i}]$$

$$P'(u) = \frac{n}{b-a} \sum_{i=0}^{n-1} B_i^{\Delta_1,n-1}(u) \left(p[\underbrace{a,\ldots,a}_{n-i-1},\underbrace{b,\ldots,b}_{i+1}] - p[\underbrace{a,\ldots,a}_{n-i},\underbrace{b,\ldots,b}_{i}] \right)$$

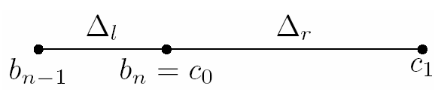
$$\vdots$$

$$P^{(k)}(u) = \frac{n!}{(b-a)^k (n-k)!} \sum_{i=0}^{n-k} B_i^{\Delta_1,n-k}(u) \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j}\, p[\underbrace{a,\ldots,a}_{n-i-j},\underbrace{b,\ldots,b}_{i+j}]$$

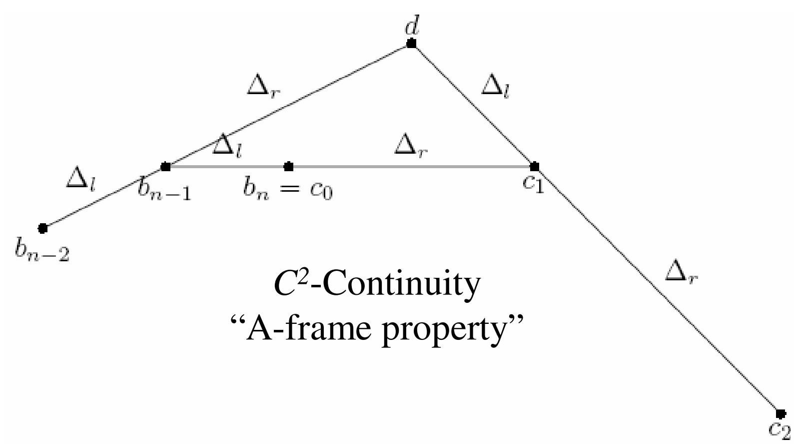
C^k-Continuity of two Bézier Curves III

$$p[\underbrace{b,\ldots,b}_{n-k},\underbrace{c,\ldots,c}_{k}] = q[\underbrace{b,\ldots,b}_{n-k},\underbrace{c,\ldots,c}_{k}]$$

C^k-Continuity of two Bézier Curves IV



C^1-Continuity



C^2-Continuity
"A-frame property"