Bézier curve via subdivision

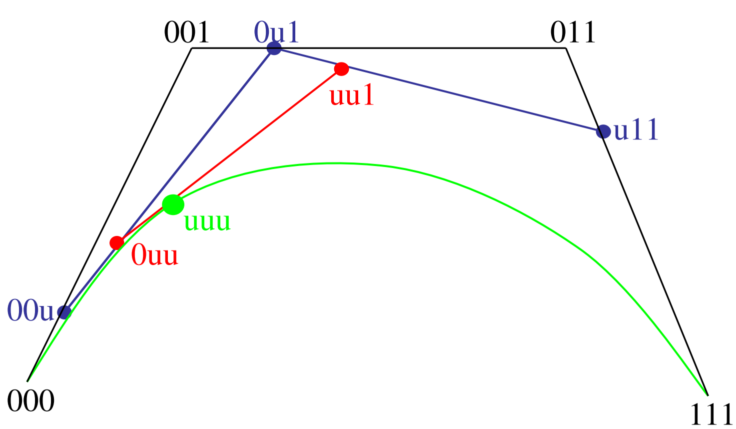
Every polynomial curve $P(u)$ of degree $\leq n$ can be associated with a unique n-variate symmetric polynomial $p[u_1, \ldots, u_n]$ such that $p[u, \ldots, u] = P(u)$

The polynomial is referred to as the **polar form** or **blossom** of $P(u)$

Bézier curve via subdivision

Any polynomial curve $P(u)$ defined over interval [a,b] can be considered as a Bézier curve with control points

$$P_k = p[\underbrace{a, \ldots a}_{n-k \text{ times}}, \underbrace{b, \ldots, b}_{k \text{ times}}]$$



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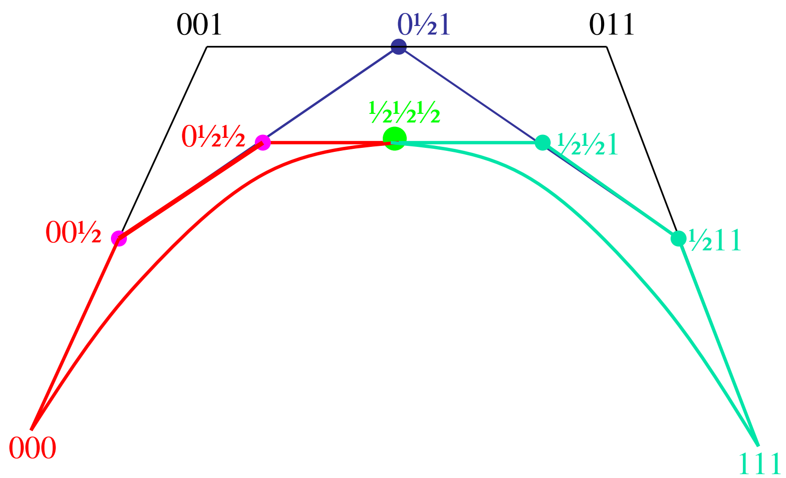

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Theorem

$$\left\| p[\underbrace{a,\ldots,a}_{n-i},\underbrace{b,\ldots,b}_{i}] - P\left(\frac{n-i}{n}a + \frac{i}{n}b\right) \right\| = O\left((b-a)^2\right)$$

Proof of the theorem: it is enough to show that

$$\left\| p[\underbrace{0,\ldots,0}_{n-i},\underbrace{h,\ldots,h}_{i}] - P\left(\frac{i}{n}h\right) \right\| = O\left(h^2\right)$$

Denote $Q_i(h) = p[\underbrace{0,\ldots,0}_{n-i},\underbrace{h,\ldots,h}_{i}]$ and $R_i(h) = P\left(\frac{i}{n}h\right)$

We have to demonstrate that $\left\|Q_i(h) - R_i(h)\right\| = O(h^2)$

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Blossom of $Q_i(h)$ is $q_i\left[h_1,\ldots,h_i\right] = p[\underbrace{0,\ldots,0}_{n-i},\underbrace{h_1,\ldots,h_i}_{i}]$

Its Bezier rep: $Q_i(h) = \sum_{j=0}^{i} q_i[\underbrace{0,\ldots,0}_{i-j},\underbrace{1,\ldots,1}_{j}]B_j^i(h)$

$$= \sum_{j=0}^{i} p[\underbrace{0,\ldots,0}_{n-j},\underbrace{1,\ldots,1}_{j}]B_j^i(h)$$

Similarly, since $R_i(h) = P\left(\frac{i}{n}h\right)$

$$R_i(h) = \sum_{j=0}^{n} p[\underbrace{0,\ldots,0}_{n-j},\underbrace{1,\ldots,1}_{j}]B_j^n\left(\frac{i}{n}h\right)$$

Bézier curve via subdivision

$$Q_i(0) = p[\underbrace{0,\ldots,0}_{n-i},\underbrace{h,\ldots,h}_{i}]\Big|_{\mathrm{h=0}} = \mathrm{p}(0,\ldots,0) = R_i(0)$$

$$Q_i'(h) = \sum_{j=0}^{i-1} i\left(p[\underbrace{0,\ldots,0}_{n-(j+1)},\underbrace{1,\ldots,1}_{j+1}] - p[\underbrace{0,\ldots,0}_{n-j},\underbrace{1,\ldots,1}_{j}] \right)B_j^{i-1}(h)$$

$$Q_i'(0) = i\big(p(0,\ldots,0,1) - p(0,\ldots,0,0)\big)$$

Bézier curve via subdivision

$$R_i'(h) = \sum_{j=0}^{n-1} n \left(p[\underbrace{0,\ldots,0}_{n-(j+1)},\underbrace{1,\ldots,1}_{j+1}] - p[\underbrace{0,\ldots,0}_{n-j},\underbrace{1,\ldots,1}_{j}] \right) B_j^{n-1}\left(\frac{i}{n}h\right)\frac{i}{n}$$

$$R_i'(0) = n\big(p(0,\ldots,0,1) - p(0,\ldots,0,0)\big)\frac{i}{n} = Q_i'(0)$$

$$Q_i(h) - R_i(h) = \big(Q_i(0) - R_i(0)\big) + \big(Q_i'(0) - R_i'(0)\big)\cdot h + O(h^2)$$

This completes the proof.