Short Tutorial on Integer Programming

- Formulating Problems as Integer Programs
  - Knapsack Problem
  - Assignment or Bipartite Matching
  - Traveling Salesman
- Solving via Branch-&-Bound
- Cutting Planes
  - Generic Cutting Planes
  - Cutting Planes for Special Problems

Reference: http://mat.gsia.cmu.edu/orclass/integer/integer.html

The Knapsack Problem

Given

- a set I of items, each item $i \in I$ with a value c_i and a weight w_i.
- a maximal weight W of your knapsack,

pack items in the knapsack, such that

- the total weight is at most W
- the value is maximized

Integer Programming Formulation

$$
\begin{array}{rrcl}
\max & \sum_{i \in I} c_i x_i & & \\
s.t. & \sum_{i \in I} w_i x_i & \leq & W \\
 & x_i & \in & \{0,1\}
\end{array}
$$

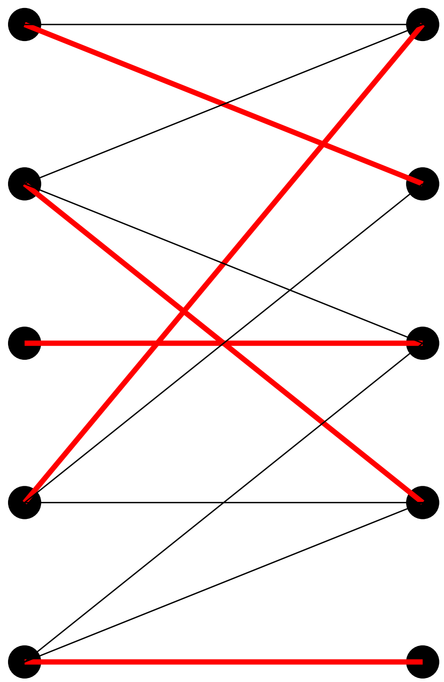
Remarks:

- $x_i \in \{0,1\}$ is equivalent to $0 \leq x_i \leq 1$ and x_i integer
- Christian Gross will present algorithms to solve this problem (without Integer Programming)

Assignment or Bipartite Matching

Given a bipartite graph G and edge cost $c_e, e \in E$

find a matching with maximal weight.



Integer Programming Formulation

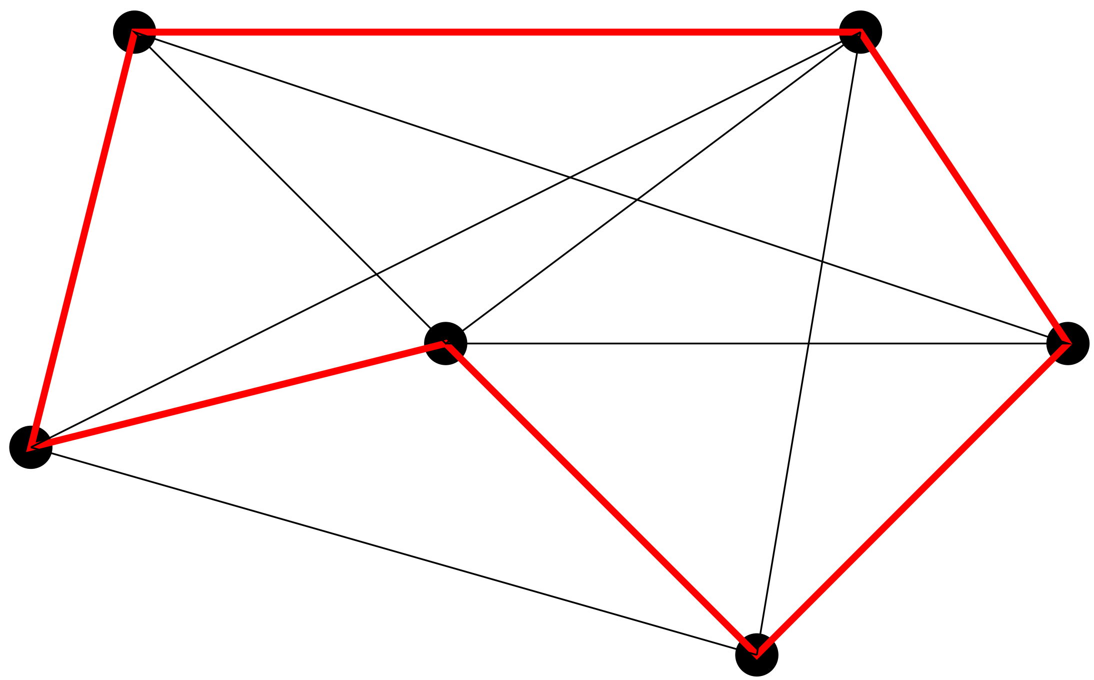
$$\begin{array}{llcll} \max & \sum_{e \in E} c_e x_e & & & \\ s.t. & \sum_{e \in \delta(v)} x_e & \leq & 1 & \forall v \in V \\ & x_e & \in & \{0,1\} & \forall e \in E \end{array}$$

Remarks:

- In fact $x_e \in \{0,1\}$ can be replaced by $0 \leq x_e \leq 1$.
- Deepak Ajwani will show this (and a generalization to matchings in general graphs)
- Januzaj Visar will give an algorithm to solve the matching problem.

The Traveling Salesman Problem

Given a graph G with edge-weight $c_e, e \in E$, find the shortest tour visiting all nodes of the graph.



Integer Programming Formulation

$$
\begin{array}{llll}
\min & \sum_{e \in E} c_e x_e & & \\
s.t. & \sum_{e \in \delta(v)} x_e = & 2 & \forall v \in V \\
 & \sum_{e \in \delta(S)} x_e \geq & 2 & \forall S \subset V \\
 & x_e \in & \{0,1\} & \forall e \in E
\end{array}
$$

Remark:

- This LP has about 2^n inequalities, thus it can not be solved directly.
- Dominik Schultes will present an approximation scheme in the Euclidean case.

Branch and Bound

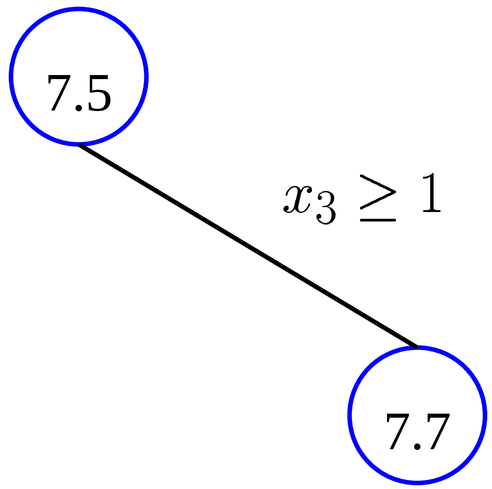
Look at the IP: $\min_{\substack{Ax \leq b \\ x \text{ integer}}} c^T x$

- Relax the integer condition, i.e. solve the LP $\min_{Ax \leq b} c^T x$.
- If the value is as least as large as the best known solution, we terminate.
- If (by chance) the optimal solution is integral, we have found a solution (and terminate).
- Otherwise, we choose a variable x_i with a fractional value d and recursively solve $\min_{\substack{Ax \leq b \\ x_i \leq \lfloor d \rfloor \\ x \text{ integer}}} c^T x$ and $\min_{\substack{Ax \leq b \\ x_i \geq \lceil d \rceil \\ x \text{ integer}}} c^T x$

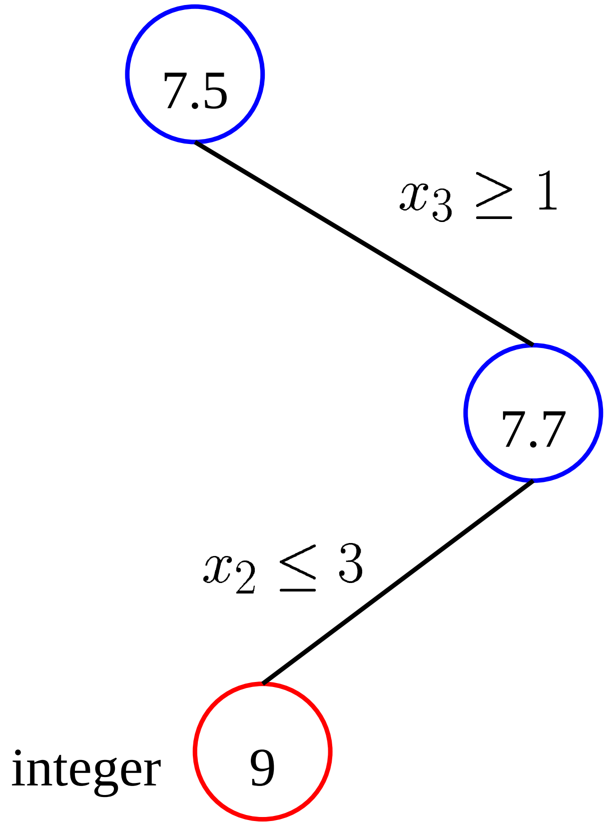
Visualization

7.5

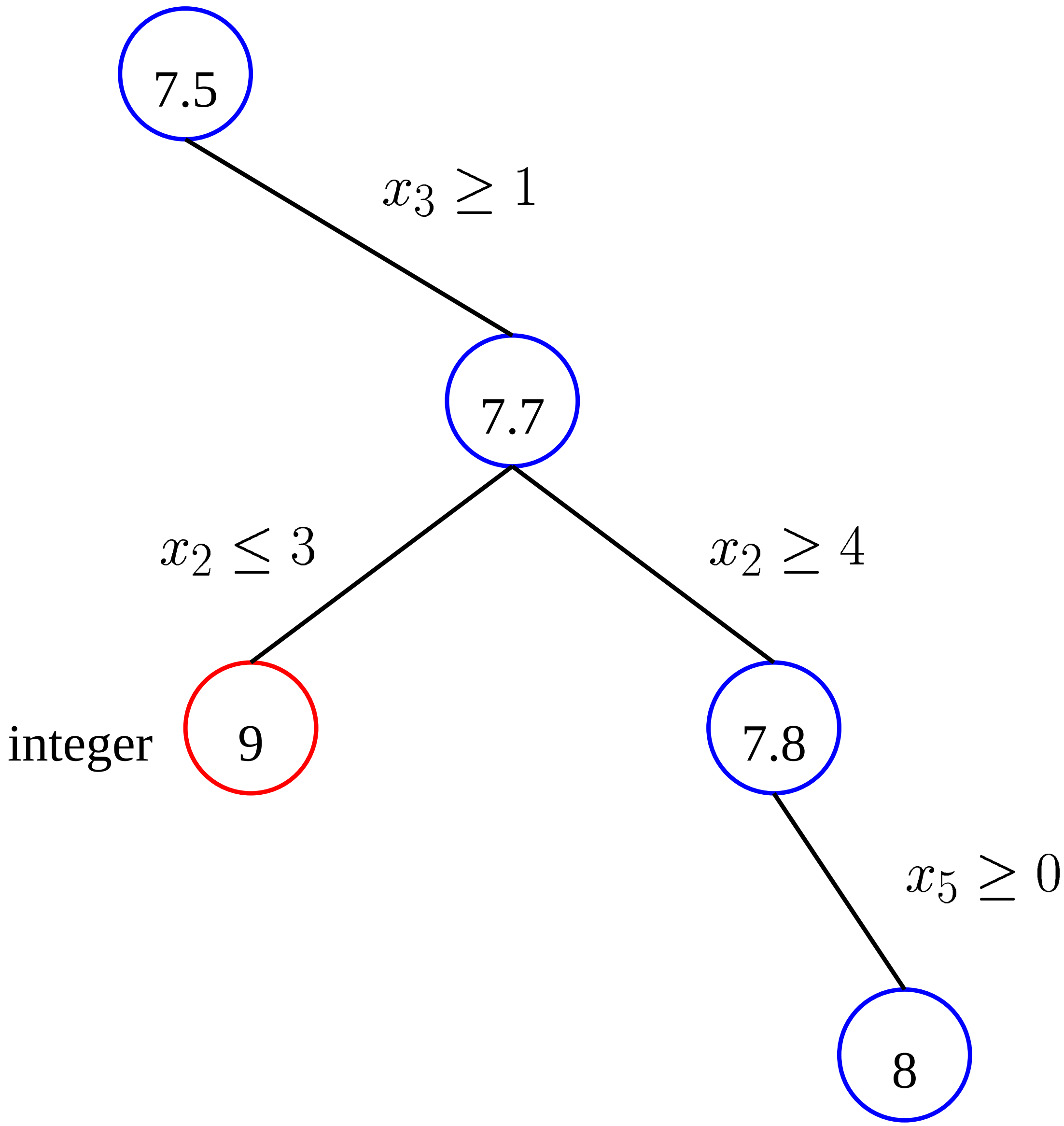
Visualization



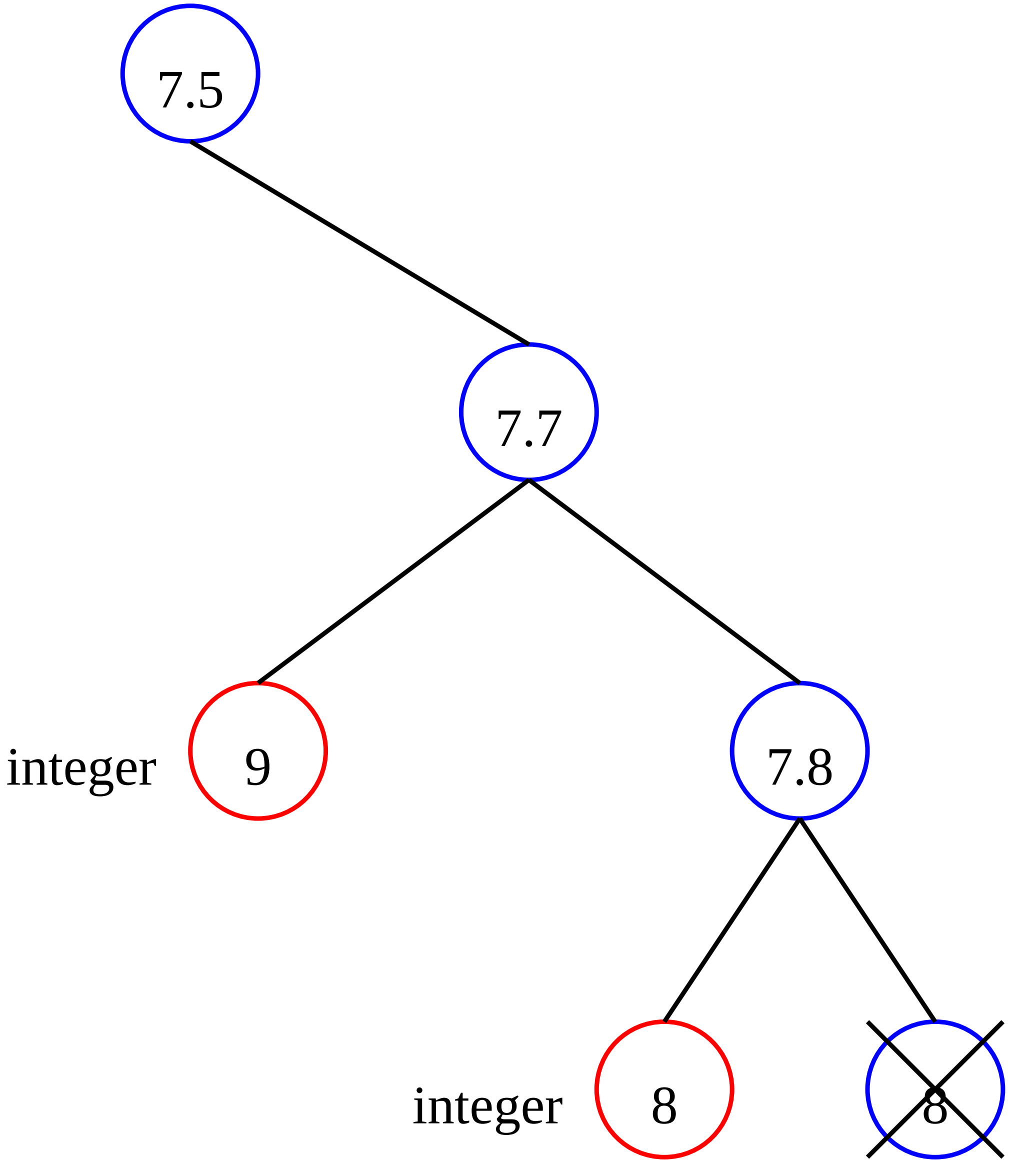
Visualization



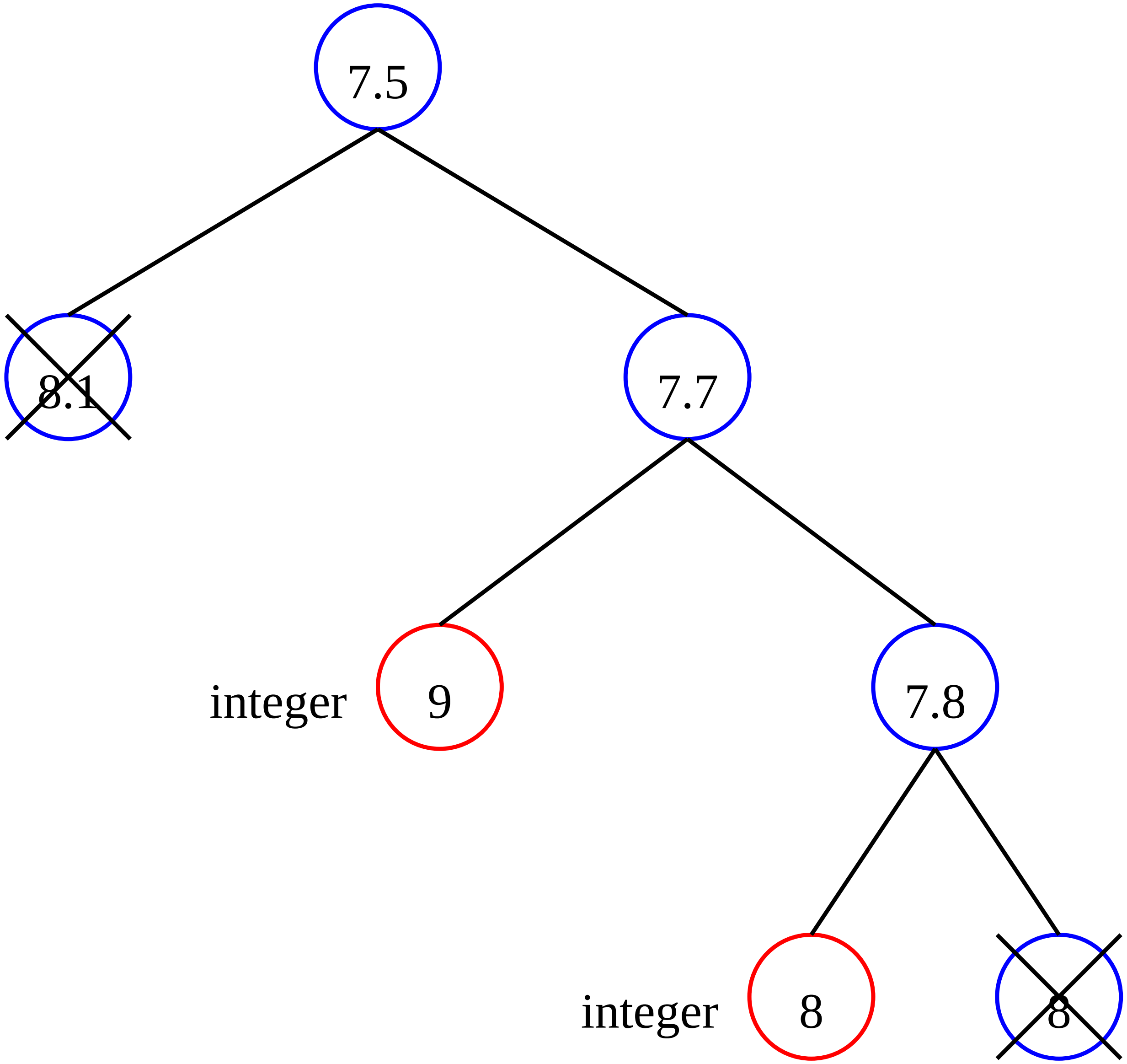
Visualization



Visualization



Visualization



Remarks

- The number of enumerated nodes depends on:
  - The quality of the LP-Bound (very important, see below)
  - How fast good solutions are found
  - On which variables we split
  - and many more things
- One can use any lower bound (not necessarily the LP-Bound)
  - One famous is “Lagrangian Relaxation” (Junming Yin)

Remarks - Continued

- If the LP value coincides with the ILP-value, no enumeration is necessary
  - Matchings
  - Matroids (Evghenia Stegantova)
- If one can prove something on the quality of the LP, we can derive approximation algorithms
  - Multicommodity Flows (Imran Rauf)
  - Jain's Algorithm (Adrian)

Cutting Planes

We can solve LPs with a large number of inequalities with the following idea

- Select a small number of initial inequalities
- Solve this LP. Gives solution x^*
- Look for an inequality that violates x^*.
- If there is such an inequality, add it to the LP and iterate
- Otherwise x^* is the solution of the LP over the large set of inequalities

Remarks

- For the Traveling Salesman problem, violated inequalities can be found via a mincut algorithm (Lijun)
- To improve the value of the relaxation, one looks for further valid inequalities
  - Either derived directly from the given LP (Gennady Shmonin)
  - Or for special problems (Muhammad Kamran Azam for the TSP)
  - Usually cuts for special problems are preferable.