

A complete characterization of group-strategyproof mechanisms of cost-sharing

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Abstract. We study the problem of designing group-strategyproof cost-sharing mechanisms. The players report their bids for getting serviced and the mechanism decides a set of players that are going to be serviced and how much each one of them is going to pay. We determine three conditions: *Fence Monotonicity*, *Stability* of the allocation and *Validity* of the tie-breaking rule that are necessary and sufficient for group-strategyproofness, regardless of the cost function. Consequently Fence Monotonicity characterizes group-strategyproof cost-sharing schemes closing an important open problem. Finally, we use our results to prove that there exist families of cost functions, where any group-strategyproof mechanism has unbounded approximation ratio.

1 Introduction

Algorithmic Mechanism Design [1] is a field of Game Theory, that tries to construct algorithms for allocating resources, that give to the players incentives to report their true interest in receiving a good, a service, or in participating in a given collective activity. The pivotal constraint when designing a mechanism for any problem is that it is truthful. Truthfulness also known as strategy-proofness requires that no player can strictly improve her utility by lying. In many settings this single requirement for an algorithm to be truthful restricts the repertoire of possible algorithms dramatically [2–5].

In settings where the repertoire of possible algorithms is not restricted too much by truthfulness, like for example in Cost-sharing problems it is desirable to construct mechanisms that are also resistant to manipulation by groups of players. Group-strategyproofness naturally generalizes truthfulness by requiring that no group of players can improve their utility by lying, when the values of the other players are fixed.

In this paper we study the following problem: We want to determine a set of n customers/players, who are going to receive a service. Each player reports her willingness to pay for getting serviced and the mechanism decides which players are going to be serviced and the price, that each one of them will pay, that is we consider direct revelation mechanisms. We want to characterize all possible mechanisms that satisfy group-strategyproofness via identifying necessary and sufficient conditions for the payment functions that give rise to these mechanisms.

We provide a complete characterization of group-strategyproof mechanisms, closing an open question posed by Immorlica et. al. [6, 7], by extending the property of semi-cross-monotonicity they identified in [6] to a new property, which we call Fence Monotonicity. Semi-cross-monotonicity is a local property that should be satisfied by the payment part (cost-sharing scheme) of the mechanism. Fence Monotonicity generalizes it, it is also referred only to the payment part of the mechanism, and it is *necessary and sufficient* for group-strategyproofness. Given any payment rule that satisfies Fence Monotonicity we show that every allocation that satisfies *Validity of the tie-breaking rule* and *Stability*, yields a group-strategyproof mechanism, we call any such mechanism a Fencing mechanism. Our characterization of group-strategyproofness is regardless of the cost function and without any additional constraints (like tie-breaking rules used in [6, 8]).

Our results provide a new, as general as it can be, framework for designing group-strategyproof mechanisms and thus open new perspectives to the study of a very important problem, the Cost-Sharing Problem, whose study was initiated in [9], where we additionally have a cost function C , such that for each subset of players S the cost for providing service to all the players in S is $C(S)$, however the strength of our results is that they apply to any cost function, since throughout our proof we do not make any assumptions at all about this cost function. We believe that our work here can be the starting point for constructing new interesting classes of mechanisms for specific cost-sharing problems.

As our characterization is the first characterization of group-strategyproofness it is interesting to make a parallelization between the many known characterizations of strategyproofness [10, 11, 2–5] and our characterization of group-strategyproofness, not however in the technical level, but in order to be able to determine the quality and usefulness of our characterization. Fence Monotonicity is a condition rather similar to the condition Cycle Monotonicity [10], in the sense that both are conditions that should be satisfied by all possible restrictions of the output space of the mechanism. A great virtue for a characterization of a mechanism that uses money is to be able to separate the payment from the allocation part. Cycle Monotonicity is a condition that should be satisfied by the allocation part of the mechanism. If it is satisfied, we know a way to define truthful payments [10]. Fence Monotonicity refers only to the payment part of a mechanism. If it is satisfied, then the Fencing Mechanisms we introduce are a way to determine the allocation part of the mechanism. Managing to separate the payments from the allocation part of the mechanism and avoiding to add any additional restrictions in the characterization we propose are undoubtedly its great virtues.

We believe that Fence Monotonicity can also be a starting point in the quest for alternative characterizations, as Cycle monotonicity has done so far [11, 2–5]. Unfortunately even though Fence Monotonicity is succinct in its description, it is much more complicated than Cycle Monotonicity. Nonetheless group-strategyproofness is a notion much more complicated than strategyproofness and we believe that Fence Monotonicity is not only important and unavoidable but

also useful. We demonstrate the latter by proving the first of its kind very simple lower bound on the budget balance of any group-strategyproof mechanism.

Our results and related work The design of group-strategyproof mechanisms for cost-sharing was first discussed by Moulin and Shenker [9, 12]. Moulin defined a condition on the payments called cross-monotonicity, which states that the payment of a serviced player should decrease as the set of serviced players grows. Any mechanism whose payments satisfy cross-monotonicity can be easily turned to a simple mechanism called after Moulin. A Moulin mechanism first checks if it can provide service to all players, so that each one has positive utility and if not it gradually diminishes the set of players that are candidates to be serviced, by throwing away at each step a player that cannot pay to get serviced (and who because of cross-monotonicity still doesn't want to pay if the set of players that receive service becomes smaller). In fact if the cost function is sub-modular and 1-budget balanced then the only possible group-strategyproof mechanisms are Moulin mechanisms [12]. The great majority of cost-sharing mechanisms proposed are Moulin mechanisms.

However recent results showed that for several important cost-sharing games Moulin mechanisms can only achieve a very bad budget balance factor [6, 13, 14]. Some alternative, very interesting and much more complicated in their description mechanisms that are group-strategyproof but not Moulin have been proposed in [6, 15], however these do not exhaust the class of group-strategyproof mechanisms. In this work we introduce *Fencing Mechanisms*, a new general framework for designing group-strategyproof mechanisms, that generalizes Moulin mechanisms [9].

Recently Mehta et. al. [16] proposed the notion of weak group-strategyproofness, that relaxes group-strategyproofness. It regards a formation of a coalition, as successful, when *each* player who participates in the coalition *strictly* increases her personal utility. They also introduce acyclic mechanisms, a general framework for designing weakly group-strategyproof mechanisms, however the question of determining all possible weakly group-strategyproof mechanisms is an important question that remains open. Another alternative notion was proposed by Bleischwitz et. al. in [17].

The problem we solve here, that of finding a complete characterization of the cost-sharing schemes that give rise to a group-strategyproof mechanism was a major open problem posed in [6],[7, Chapter 15]. The same question was posed in [16] for weak group-strategyproof cost-sharing schemes and still remains open. Many interesting results arose in the attempt to find such a characterization [15, 8]. In contrast to previous characterization attempts that characterized mechanisms satisfying some additional boundary constraints [6, 8] our characterization is complete and succinct. The only complete characterization that was known was for the case of two players [15, 8]. It remains open how can our characterization help for constructing new efficient mechanisms for specific cost-sharing problems. We believe that it can significantly enrich the repertoire of mechanisms with good approximation guarantees for specific problems.

In the notion of group-strategyproofness it is important to understand that ties play a very important role. This is in contrast to strategyproofness, where ties can be in most cases broken arbitrarily (see for example [18]). An intuitive way to understand this is that a mechanism designer of a group-strategyproof mechanism expects a player to tell a lie in order to help the other players increase their utility, even when she would not gain any profit for herself. This player is at a tie but decides strategically if she should lie or not. Consequently a characterization that assumes a priori a tie-breaking rule, and thus greatly restricts the repertoire of possible mechanisms, like the one in [6, 19] might be useful for specific problems and easier in its statement, but can never capture the very notion group-strategyproofness.

Our proofs are involved and based on set-theoretic arguments and the repeated use of induction. The main difficulty of our work was to identify some necessary and sufficient conditions for group-strategyproof payments that are also succinct to describe and add to our understanding of the notion of group-strategyproofness. In proving the necessity of Fence Monotonicity we first have to prove Lemmas that also reveal interesting properties of the allocation part of the mechanism. A novel tool that we introduce is the *harm relation* that generalizes the notion of negative elements defined in [6]. Proving that Fencing Mechanisms, i.e. mechanisms whose payments satisfy Fence Monotonicity and whose allocation satisfies Stability and Validity of the tie-breaking rule, are group-strategyproof turns out to be rather complicated. It is likely that our techniques can be extended for characterizing weakly-group-strategyproof mechanisms.

2 Defining the model

The Mechanism. Suppose that $\mathcal{A} = \{1, 2, \dots, n\}$ is a set of players interested in receiving a service. Each of the players has a private type v_i , which is her valuation for receiving the service.

A cost sharing mechanism (O, p) consists of a pair of functions, $O : \mathbb{R}^n \rightarrow 2^{\mathcal{A}}$ that associates with each bid vector b the set of serviced players and $p : \mathbb{R}^n \rightarrow \mathbb{R}^n$ that associates with each bid vector b a vector $p(b) = (p_1(b), \dots, p_n(b))$, where the i -th coordinate is the payment of player i .

Each player wants to maximize her utility, which assuming quasi-linear utilities is $v_i a_i - p_i(b)$ where $a_i = 1$ if $i \in O(b)$ and $a_i = 0$ if $i \notin O(b)$.

As is common in the literature and in order not to come up with absurd mechanisms, we concentrate on mechanisms that satisfy the following very simple conditions [9, 12, 6]: *Voluntary Participation (VP)*: A player that is not serviced is not charged ($i \notin O(b) \Rightarrow p_i(b) = 0$) and a served player is never charged more than his bid ($i \in O(b) \Rightarrow p_i(b) \leq b_i$). *No Positive Transfer (NPT)*: The payment of each player i is positive ($p_i(b) \geq 0$ for all i). *Consumer Sovereignty (CS)*: For each player i there exists a value $b_i^* \in \mathbb{R}^1$ such that if he bids b_i^* then it is guaranteed that i will receive the service no matter what the other players

¹ From VP it holds that $b_i^* \geq \max_{S \subseteq \mathcal{A}} \xi(i, S)$. Its easy to verify that strategyproofness implies that any value greater than b_i^* satisfies the CS property for player i . Thus, when we refer to this crucial value, we will wlog assume that its inequality is strict.

bid. Note that this assumption together with CS ensures us that the mechanism produces all possible allocations.

In accordance with [9, 6] we will concentrate on mechanisms, where the players are additionally able to deny receiving service, no matter what the other players bid. To ensure this we just assume that players can report negative bids. Then VP and NPT imply that if a player announces a negative amount, then she will not receive the service. While, negative bids may not seem realistic, they can model the denial of revealing any information to the mechanism.

To put things simply, if a player i bids -1 , then she does not receive service and if she bids b_i^* then she receives service, no matter what the other players bid. Also notice that the value b_i^* is greater than any possible payment player i is charged, i.e. $b_i^* > p_i(b)$, for all $b \in \mathbb{R}^n$.

Group-strategyproofness. We are interested in mechanisms that are *group-strategyproof* (GSP). A mechanism is group-strategyproof if for every two valuation vectors v, v' and every coalition of players $S \subseteq \mathcal{A}$, satisfying $v_i = v'_i$ for all $i \notin S$, one of the following is true: (a) There is some $i \in S$, such that $v'_i a_i - p_i(v') < v_i a_i - p_i(v)$, or (b) for all $i \in S$, it holds that $v_i a'_i - p_i(v') = v_i a_i - p_i(v)$.

A *cost-sharing scheme* is a function $\xi : \mathcal{A} \times 2^{\mathcal{A}} \rightarrow \mathbb{R}^+ \cup \{0\}$, such that, for every $S \subset \mathcal{A}$ and every $i \notin S$ we have $\xi(i, S) = 0$. In fact the value $\xi(i, S)$ will be exactly the payment of player i when the set of players that receive service is S . In [6] it was shown that in any group-strategyproof mechanism for our setting, the payment of a player depends only on the allocation produced by the mechanism for a particular bid vector and does not depend directly on the bids of the players. Consequently the payments of any GSP mechanism are given by a cost-sharing scheme ξ .

The cost function and budget balance. The cost of providing service is given by a cost function $C : 2^{\mathcal{A}} \rightarrow \mathbb{R}^+ \cup \{0\}$, where $C(S)$ specifies the cost of providing service to all players in S .

A desirable property of cost-sharing mechanisms is budget balance. We say that a mechanism is α -*budget balanced*, where $0 \leq \alpha \leq 1$, if for all bid vectors b it holds, that $\alpha \cdot C(O(b)) \leq \sum_{i \in S} \xi(i, S) \leq C(O(b))$.

We chose to define the cost function last in order to stress that our results are completely independent of the cost function and apply to any cost-sharing problem.

Cross- and Semi-cross-monotonicity. We say that a cost-sharing scheme is *cross-monotonic* [12] if $\xi(i, S) \geq \xi(i, T)$ for every $S \subset T \subseteq \mathcal{A}$ and every player $i \in S$. This means that the payment of a player decreases as the number of players that receive service increases.

In the attempt to provide a characterization of GSP cost-sharing schemes Immorlica et. al. [6] provided a partial characterization and identified semi-cross monotonicity an important condition that should be satisfied by any GSP cost-sharing scheme. A cost sharing scheme ξ is *semi-cross monotonic* if for every $S \subseteq \mathcal{A}$, and every player $i \in S$, either $\xi(j, S \setminus \{i\}) \leq \xi(j, S)$ for all $j \in S \setminus \{i\}$, or $\xi(j, S \setminus \{i\}) \geq \xi(j, S)$ for all $j \in S \setminus \{i\}$.

As we later show in Proposition 1 (a) semi-cross-monotonicity can be almost directly derived from the condition of Fence Monotonicity we define in this work and more specifically from part (a) of Fence Monotonicity.

3 Our Characterization

3.1 Fence Monotonicity and Fencing Mechanisms

Fence Monotonicity considers each time a restriction of the mechanism that can only output as the serviced set, subsets of U that contain all players in L . To be more formal consider all possible subsets of the players L, U such that $L \subseteq U \subseteq \mathcal{A}$. Fixing a pair $L \subseteq U$ Fence Monotonicity considers only sets of players S with $L \subseteq S \subseteq U$. Each one of the conditions of Fence monotonicity should be satisfied for any such restriction of the mechanism.

We denote by $\xi^*(i, L, U)$ be the minimum payment of player i , for getting serviced when the output of the mechanism is restricted by L and U , i.e. $\xi^*(i, L, U) := \min_{\{L \subseteq S \subseteq U, i \in S\}} \xi(i, S)$.

Definition 1 (Fence Monotonicity). *We will say that a cost-sharing scheme satisfies Fence Monotonicity, if for every $L \subseteq U \subseteq \mathcal{A}$, it satisfies the following three conditions:*

- (a) *There exists at least one set S with $L \subseteq S \subseteq U$, such that for all $i \in S$ we have $\xi(i, S) = \xi^*(i, L, U)$.*
- (b) *For each player $i \in U \setminus L$ there exists at least one set S_i , with $i \in S_i$ and $L \subseteq S_i \subseteq U$, such that for all $j \in S_i \setminus L$, we have $\xi(j, S_i) = \xi^*(j, L, U)$. (Note that $i \in S_i \setminus L$ and thus $\xi(i, S_i) = \xi^*(i, L, U)$. Also note that we might have $S_i \neq S_j$ for $i \neq j$.)*
- (c) *If there exists a set $C_i \subset U$ where $i \in C_i$, such that for some player i we have $\xi(i, C_i) < \xi^*(i, L, U)$ (obviously $L \not\subseteq C_i$), then there exists at least one set $T \neq \emptyset$, with $T \subseteq L \setminus C_i$ such that for all $j \in T$, $\xi(j, C_i \cup T) = \xi^*(j, L, U)$.*

The first condition says that if we necessarily want to provide service to all players in the set L , then by providing service to a superset S of the players we initially wanted to service, all of the players in S can achieve their lowest possible payment $\xi^*(i, L, U)$. As we show in Proposition 1(a), this condition generalizes the condition of semi-cross-monotonicity, and in this sense our work completes the partial characterization obtained in [6].

The second condition says then for each player $i \in U \setminus L$ there exists an outcome S_i , such that $i \in S_i$, and such that all players in $S_i \setminus L$ are served with their lowest possible payment. Loosely speaking this gives a way to enlarge L by adding to it more players in a way that is *optimal for all players that we add*. Note that the cost of the players already in L might however increase.²

² If we further demand the second condition to hold for every $j \in S_i$, i.e. if we can always enlarge the set L in a way that is optimal for all players that receive service, and not only for the ones we added, then our cost-sharing scheme would immediately satisfy Cross-Monotonicity. In this sense Cross-Monotonicity can be obtained by restricting part (b) of Fence Monotonicity.

The third condition is the most important and involved. It compares the minimum possible non-zero payment of a player in this restriction of the mechanism, with his minimum possible non-zero payment when the outcome can be any subset of U (i.e. the the output should not necessarily contain the players in L). If player i is better off when the outcome is some set C_i , where $L \not\subseteq C_i$, (i.e. loosely speaking when some of the players in $L \setminus C_i$ “harm” player i by their presence in the outcome) then there exists some non-empty set of players T , with $T \subseteq L \setminus C_i$, which is indifferent between its payment in $C_i \cup T$ and its minimum payment when the outcome of the mechanism is in the restriction of the mechanism defined by L and U .

Proposition 1. *Let ξ be a cost sharing scheme*

(i) *If it satisfies condition (a) of Fence Monotonicity for all L, U with $|U \setminus L| = 1$ then it satisfies Semi-cross-monotonicity.*

(ii) *If it satisfies conditions (b) of Fence Monotonicity for all L, U with $|U \setminus L| = 2$ and (c) of Fence Monotonicity for all L, U with $|U \setminus L| = 1$, then it satisfies the following property: For all $S \subseteq \mathcal{A}$ and all distinct $i, j \in S$, if $\xi(j, S \setminus \{i\}) < \xi(j, S)$ then $\xi(i, S \setminus \{j\}) = \xi(i, S)$.*

Proposition 1 shows exactly how one can derive the two conditions, necessary for group-strategyproofness, which were identified by Immorlica et. al., who proposed the problem of characterizing group-strategyproof mechanisms in [6]: Part (i) of this Proposition shows how we can derive Semi-cross-monotonicity and part (ii) how we can derive the condition identified in Remark B.1 from [6] (Remark B.1 is a special case of the condition we identify here, namely they only showed the same fact, only when it also holds that $\xi(i, S \setminus \{j\}) \leq \xi(i, S)$). The two conditions they identified are obtained if apply Fence Monotonicity for cases when the cardinality of $U \setminus L$ is either 1 or 2, i.e. when we restrict the outcome space to only two or four different possible outcomes.³

Theorem 1. *A cost sharing scheme gives rise to a group-strategyproof mechanism if and only if it satisfies Fence Monotonicity.*

If you are given a cost sharing scheme that satisfies Fence Monotonicity it is not straightforward how to construct a GSP mechanism. Fence Monotonicity should be coupled with an allocation rule that satisfies a simple property, which we call *Stability* and a *Valid tie-breaking rule* in order for the mechanisms to be GSP.

Given any cost sharing scheme ξ that satisfies Fence Monotonicity we provide a general method for constructing a mechanism, that uses ξ as payment function and satisfies group-strategyproofness. We call the underlying mechanisms of this framework Fencing mechanisms.

The mechanism takes as input the bids of the players and determines a pair of sets L, U where $L \subseteq U$, where L is the set of players, that are going to be

³ Parallelizing with characterizations of strategyproofness the Monotonicity Property coincides with Cycle-monotonicity [10] when we only consider cycles of length 2, i.e. cycles that consider a restriction of the outcome space to only 2 outcomes.

serviced by the mechanism with strictly positive utility. On the other hand $U \setminus L$ is the set of players that are indifferent between getting serviced or not, because their bid equals their payment, and we use a tie-breaking policy to determine which of these players will get serviced. The existence of a tie-breaking policy that does not violate group-strategyproofness is guaranteed by part (a) of Fence Monotonicity. The intuition behind the tie-breaking rule is that it is optimal for the players in L , in the sense that from all subsets of U we choose to serve the one where the players in L achieve their minimum payments provided that they all get serviced.

The mechanisms we design can be put in the following general framework: Given a bid vector as input, we search for a certain pair of sets L, U , where $L \subseteq U \subseteq A$ that meet the criteria of stability we define below and then we choose one of the allocations that service the players in L according to a valid tie-breaking rule. If the search is exhaustive the resulting algorithm is exponential time and we do not know any polynomial-time algorithm. However if we restrict our attention to payments that satisfy certain conditions like for example cross-monotonicity we can come up with a polynomial-time algorithm for finding a stable pair.

Definition 2 (Stability). *A pair L, U is stable at b if the following conditions are true: 1. For all $i \in L$, $b_i > \xi^*(i, L, U)$, 2. for all $i \in U \setminus L$, $b_i = \xi^*(i, L, U)$ and 3. for all $R \subseteq A \setminus U$, there is some $i \in R$, such that $b_i < \xi^*(i, L, U \cup R)$.*

Assume that ξ is cross-monotonic and let S be the output of Moulin mechanism for some bid vector b . Then, the pair L, U , where $L = \{i \in S \mid b_i > \xi(i, S)\}$ and $U = S$, is the unique stable pair at b .

After identifying a stable pair these mechanisms output a set S , where $L \subseteq S \subseteq U$ given by a tie-breaking function.

The mapping $\sigma : 2^A \times 2^A \times \mathbb{R}^n \rightarrow 2^A$ is a *valid* tie-breaking rule for ξ , if for all b and $L \subseteq U \subseteq A$, such that L, U is stable at b , for the set $S = \sigma(L, U, b)$ it holds that $L \subseteq S \subseteq U$ and for all $i \in S$, $\xi(i, S) = \xi^*(i, L, U)$. Part (a) of Fence Monotonicity guarantees that there exists at least one output S with this property and each different such output gives a different tie-breaking rule.

Definition 3. *We will say that a mechanism is a Fencing Mechanism if at input b it finds a stable pair L, U at b , outputs $\sigma(L, U, b)$, where σ is a valid tie-breaking rule and charges each player i , the value $\xi(i, \sigma(L, U, b))$.*

It is easy to verify that every Fencing Mechanism satisfies VP, because the players that get serviced belong to the set U and as the mechanism satisfies stability these players have non-negative utility, and CS, because if a player bids higher than any of his payments, then again by stability he belongs to the set L and gets serviced.⁴

Theorem 2. *A mechanism is group-strategyproof if and only if it is a Fencing Mechanism.*

⁴ Moulin mechanisms are GSP and consequently they can be viewed as special case of the general framework of Fencing Mechanisms. In Moulin mechanisms ξ is cross-monotonic and consequently the bigger the set of serviced players, the lower is the

3.2 Every GSP Mechanism is a Fencing Mechanism

Necessity of Fence Monotonicity Here we prove that the payment function of every GSP mechanism satisfies Fence Monotonicity. For this, we select an arbitrary GSP mechanism and consider some $U \subseteq \mathcal{A}$. We show that for every $L \subseteq U$ the cost sharing scheme satisfies each one of the three conditions of Fence Monotonicity using induction on $|U \setminus L|$. We first define the notion of a *harm relation* and we prove that it is a strict partial order.

Lemma 1. *If $U \subseteq U_1$ and $L_1 \subseteq L$, then for all $i \in U$, $\xi^*(i, L, U) \geq \xi^*(i, L_1, U_1)$.*

Proposition 2 (Harm relation). *Fix two sets $L \subseteq U$ and suppose that $i, j \in U$. We say that i harms j , if $\xi^*(j, L, U) < \xi^*(j, L \cup \{i\}, U)$. If i does not harm j then $\xi^*(j, L, U) = \xi^*(j, L \cup \{i\}, U)$ (we get this equality applying Lemma 1).*

(i) *The harm relation satisfies anti-symmetry and transitivity and consequently it is strict partial order and the induced sub-graph $G[U \setminus L]$ is a directed acyclic graph.*

(ii) *For every $j \in L$ one of the following holds: either every $i \in U \setminus L$ harms j , or there exists some sink of the subgraph $G[U \setminus L]$ that does not harm j .*

(iii) *For every $j \in U \setminus L$ one of the following holds: either j is a sink of the sub-graph $G[U \setminus L]$, or there exists some a sink k of the sub-graph $G[U \setminus L]$ that is harmed by j .*

We continue by revealing several important allocation properties for GSP mechanisms and use them to prove the induction step for each property of Fence Monotonicity.

Condition (a) of Fence Monotonicity For the proof of condition (a), we also want the players in $U \setminus L$ to be indifferent between being serviced and being excluded from the outcome, i.e. they have zero utility. Thus, we assume that every player in $U \setminus L$ has bidden exactly her minimum payment ξ^* at L, U .

Lemma 2. *At the bid vector b , where for all $i \in L$, $b_i = b_i^*$, for all $i \in U \setminus L$, $b_i = \xi^*(i, L, U)$ and for all $i \notin U$, $b_i = -1$, it holds that $L \subseteq O(b) \subseteq U$ and that for all $i \in O(b)$, $\xi(j, O(b)) = \xi^*(j, L, U)$.*

Setting $S = O(b)$, condition (a) of Fence Monotonicity is satisfied at L, U .

Condition (b) of Fence Monotonicity If j is a sink it is implied by the induction hypothesis that j satisfies Fence Monotonicity (b). Otherwise from Proposition 2(ii) there exists a sink k that does not harm j .

We have to show that at any bid vector b^j with $b_i = b_i^*$ for $i \in L$, $b_j = \xi^*(j, L, U) + \epsilon$, $b_k = \xi^*(k, L, U) + \epsilon$, $b_i = \xi^*(i, L, U)$ for all $i \in U \setminus (L \cup \{j\})$, and

cost for each one of them. Thus it holds that for all $L \subseteq U \subseteq \mathcal{A}$, that for all $i \in U$, $\xi(i, U) = \xi^*(i, L, U)$ and the mapping $\sigma(L, U, b) = U$ for all $L \subseteq U \subseteq \mathcal{A}$ and all b such that L, U is stable, is a valid tie-breaking rule. This simplifies the algorithm substantially, as we just have to find a set U of players that can be serviced with utility greater or equal to zero. Moreover as the payment of each player increases when the serviced players becomes smaller we need the set U with maximal cardinality.

$b_i = -1$ for all $i \notin U$, it holds that $k \notin O(b^j)$, $j \in O(b^j)$, $L \subseteq O(b^j) \subseteq U$ and every player $i \in O(b^j) \setminus L$, is charged $\xi^*(i, L, U)$. Setting $S_j = O(b^j)$ we get that Fence Monotonicity (b) is satisfied for every $j \in U \setminus L$ that is not a sink as well. (See the Appendix for proofs and Figure 1 for gaining more intuition.)

Condition (c) of Fence Monotonicity To show that the cost-sharing scheme satisfies the third property of Fence Monotonicity, at L, U , we need the induction hypothesis only for showing (as we have already done) that condition (a) of Fence Monotonicity is satisfied at this pair and specifically only the allocation properties of Lemma 2. Now we consider inputs for which we don't get anymore directly from CS, that the players in L surely receive service. The idea is to gradually generalize the bid vectors whose allocation we characterize. Notice that every family of inputs we consider is a subset of the previous one.

| | L | $U \setminus L$ | $\notin U$ | allocation of a GSP mechanism |
|--------|--------------------|------------------|------------|---|
| Lem.2 | b_i^* | $\xi^*(i, L, U)$ | -1 | $\forall i \in O(b)[\xi(i, O(b)) = \xi^*(i, L, U)] \& L \subseteq O(b) \subseteq U$ |
| Lem.11 | $> \xi^*(i, L, U)$ | $\xi^*(i, L, U)$ | -1 | $\forall i \in O(b)[\xi(i, O(b)) = \xi^*(i, L, U)] \& L \subseteq O(b) \subseteq U$ |
| Lem.12 | $> \xi^*(i, L, U)$ | $\in \mathbb{R}$ | -1 | $\forall i \in O(b), \xi(i, O(b)) \geq \xi^*(i, L, U)$ |

Lemma 12 sheds light on an important property of the allocation of (non-Moulin) GSP mechanisms. It reveals the following fact: If the bids of all players in a set $L \subseteq U$ have surpassed their respective minimum payments at L, U , (and the players in $\mathcal{A} \setminus U$ don't want to participate), then a GSP mechanism never excludes a subset of L from the outcome in favor of another serviced player, who would pay less if they were not present in the outcome. Loosely speaking the players in L rule out any outcome $C \subset U$ such that there is some $j \in C$ such that $\xi(j, C) < \xi^*(j, L, U)$ (see also Figure 1). Since this property must be satisfied, even for arbitrary bids of the players in $U \setminus L$, it follows that every GSP cost-sharing scheme must satisfy condition (c) of Fence Monotonicity, as shown below.

Lemma 3. *For each set C_j with $C_j \subset U$ that satisfies $\xi(j, C_j) < \xi^*(j, L, U)$ we construct the bid vector b^c as follows: For all $i \in C_j$, $b_i^c = b_i^*$, for all $i \in L \setminus C_j$, $b_i^c = \xi^*(i, L, U) + \epsilon$ and for all $i \notin C_j \cup L$, $b_i^c = -1$. For the bid vector b^c we have that (a) For all $i \in O(b^c)$ it holds that $\xi(i, O(b^c)) \geq \xi^*(i, L, U)$, (b) $C \subset O(b^c) \subseteq L \cup C_j$, (c) For all $i \in O(b^c) \setminus C_j$, it holds that $\xi(i, O(b^c)) = \xi^*(i, L, U)$.*

Finally setting $T = O(b^c) \setminus C_j$ we can conclude that every group-strategyproof mechanism satisfies condition (c) of Fence Monotonicity.

Necessity of Stability and Valid tie-breaking. Now we show that the allocation of every GSP mechanism satisfies Stability and uses a Valid tie-breaking rule. We already know that the payment function satisfies Fence Monotonicity and thus we can prove the following generalization of Lemma 11.

Lemma 4. *Let $L \subseteq U \subseteq \mathcal{A}$. For every bid vector b , such that L, U is stable, it holds that $L \subseteq O(b) \subseteq U$ and for all $i \in O(b)$, $\xi(i, O(b)) = \xi^*(i, L, U)$.*

Lemma 7 guarantees that there is a stable pair at every input. Thus, given a bid vector, we may apply Lemma 4 with L, U being the corresponding stable pair at this input, and get that $L \subseteq O(b) \subseteq U$ (Stability) and for all $i \in O(b)$, $\xi(i, O(b)) = \xi^*(i, L, U)$ (Validity of the tie-breaking rule).

3.3 The classes of GSP and Fencing Mechanisms coincide

Here, we complete our characterization by proving that Fencing Mechanisms are group-strategyproof.

Lemma 5. *For every bid vector b and set L with $L \subseteq \mathcal{A}$, there exists a unique maximal set U , $U \supseteq L$, such that for all $i \in U \setminus L$ we have $b_i \geq \xi^*(i, L, U)$ and any other set with the same property is a subset of U . Moreover, the pair L, U satisfies condition 3. of Stability.*

The next two Lemmas prove our statement.

Lemma 6. *For the inputs where there exists a stable pair it, it is unique and every Fencing Mechanism is GSP.*

Lemma 7. *For every bid vector b there exists a unique stable pair. Consequently, Fencing Mechanisms are meaningful and GSP.*

There is a specific reason behind this ordering of these Lemmas. As to prove that there exists a unique stable pair at every bid vector, we parametrize the bid space with a natural number and use induction on it. To prove the induction step we deal with bid vectors where a stable pair exists and the implications of Lemma 6 turn to simplify our proof substantially.

4 Conclusion and future directions

We demonstrate the use of our characterization by showing that even in the case of three players group-strategyproof mechanisms fail in having a constant budget balance.

Theorem 3. *There are cost function families where every GSP mechanism has unbounded approximation ratio.*

We believe that the most interesting directions of our research are the following: How can our characterization be applied for obtaining cost-sharing mechanism with better approximation and budget-balance guarantees or lower bounds for specific problems?

Does there exist a polynomial-time algorithm for finding the allocation of a cost-sharing scheme that satisfies Fence Monotonicity or maybe we can show that the problem of finding a stable pair is computationally hard?

A natural question that arises in this context is whether, it is computationally more efficient to find the appropriate outcome than identifying the stable pair. Suppose that you have an algorithm that computes the outcome of a GSP mechanism.

Theorem 4. *Suppose that we are given the outcome of a group-strategyproof mechanism at b . Given that we have already computed $\xi^*(i, L, U)$ for all $L \subseteq U \subseteq \mathcal{A}$ and all $i \in U$, there is a polynomial time algorithm for identifying its stable pair.*

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A Examples of Mechanisms that violate just one part of Fence Monotonicity and are not GSP

We will give three representative examples to illustrate, why a cost sharing scheme, which does not satisfy Fence Monotonicity, cannot give rise to a group-

strategyproof mechanism. We chose our examples in a way that only one condition of Fence Monotonicity is violated and only at a specific pair L, U . (in fact in condition (c), the violation is present at two pairs, however it can be shown that this is unavoidable.)

Example 1 (a).

Let $\mathcal{A} = \{1, 2, 3, 4\}$. We construct a cost sharing scheme, such that condition (a) of Fence Monotonicity is not satisfied at $L = \{1, 2\}$ and $U = \{1, 2, 3, 4\}$, as follows.

| ξ | 1 | 2 | 3 | 4 | ξ | 1 | 2 | 3 | 4 | ξ | 1 | 2 | 3 | 4 |
|------------------|-----------|-----------|-----------|-----------|------------|-----------|-----------|-----------|-----------|------------|-----------|-----------|-----------|-----------|
| $\{1, 2, 3, 4\}$ | 30 | 30 | 30 | 30 | $\{1, 2\}$ | 30 | 30 | – | – | $\{3, 4\}$ | – | – | 30 | 30 |
| $\{1, 2, 3\}$ | 20 | 30 | 30 | – | $\{1, 3\}$ | 20 | – | 30 | – | $\{1\}$ | 30 | – | – | – |
| $\{1, 2, 4\}$ | 30 | 20 | – | 30 | $\{1, 4\}$ | 30 | 30 | – | – | $\{2\}$ | – | 30 | – | – |
| $\{1, 3, 4\}$ | 30 | – | 20 | 30 | $\{2, 3\}$ | – | 30 | 30 | – | $\{3\}$ | – | – | 30 | – |
| $\{2, 3, 4\}$ | – | 30 | 30 | 20 | $\{2, 4\}$ | – | 20 | – | 30 | $\{4\}$ | – | – | – | 30 |

Consider the bid vector $b := (b^*1, b^*2, 30, 30)$. Notice that players 3 and 4 are indifferent to being serviced or not, as the single value they may be charged as payment equals their bid. Moreover, notice that either player 1 or player 2 (or both) must pay 30 strictly over their minimum payment 20 under this restriction. Without loss of generality assume that $\xi(1, O(b)) = 30$. Consider the bid vector $b' := (b^*_1, b^*_2, b^*_3, -1)$. By VP and CS it holds that $O(b') = \{1, 2, 3\}$ and thus $\xi(1, O(b')) < \xi(1, O(b))$. Notice that the utilities of players 3 and 4 remain zero, and consequently $\{1, 3, 4\}$ form a successful coalition. In a similar manner we prove the existence of successful coalition when $\xi(2, O(b)) = 30$.

Example 2 (b).

Let $\mathcal{A} = \{1, 2, 3, 4\}$. We construct a cost sharing scheme, such that condition (b) of Fence Monotonicity is not satisfied at $L = \{1, 2\}$ and $U = \{1, 2, 3, 4\}$ for player 3, as follows.

| ξ | 1 | 2 | 3 | 4 | ξ | 1 | 2 | 3 | 4 | ξ | 1 | 2 | 3 | 4 |
|------------------|-----------|-----------|-----------|-----------|------------|-----------|-----------|-----------|-----------|------------|-----------|-----------|-----------|-----------|
| $\{1, 2, 3, 4\}$ | 30 | 30 | 30 | 30 | $\{1, 2\}$ | 30 | 30 | – | – | $\{3, 4\}$ | – | – | 30 | 30 |
| $\{1, 2, 3\}$ | 30 | 30 | 40 | – | $\{1, 3\}$ | 30 | – | 30 | – | $\{1\}$ | 30 | – | – | – |
| $\{1, 2, 4\}$ | 30 | 30 | – | 20 | $\{1, 4\}$ | 30 | – | – | 30 | $\{2\}$ | – | 30 | – | – |
| $\{1, 3, 4\}$ | 30 | – | 30 | 30 | $\{2, 3\}$ | – | 30 | 30 | – | $\{3\}$ | – | – | 30 | – |
| $\{2, 3, 4\}$ | – | 30 | 30 | 30 | $\{2, 4\}$ | – | 30 | – | 30 | $\{4\}$ | – | – | – | 30 |

Consider the bid vector $b^{3,4} := (b^*_1, b^*_2, 35, b^*_4)$. Strategyproofness implies that $3 \in O(b^{3,4})$, since otherwise if she is not serviced (zero utility), she can misreport b^*_3 changing the outcome to $\{1, 2, 3, 4\}$ and increasing her utility to $35 - 30 > 0$.

Next, consider the bid vector $b^4 := (b^*_1, b^*_2, 30, 25)$. Assume that $4 \notin O(b^4)$. Moreover, notice that by VP it is impossible that $3 \in O(b^4)$. Thus, $\{3, 4\}$ can form a successful coalition bidding $b' = (b^*_1, b^*_2, -1, b^*_4)$, changing the outcome to

$\{1, 2, 4\}$ increasing the utility of player 4 to $25 - 20 > 0$, while keeping the utility of player 3 at zero.

Finally, consider the bid vector $b^3 := (b_1^*, b_2^*, 35, 25)$. Notice that the b^3 differs with b^4 and $b^{3,4}$ in the coordinates that correspond to players 3 and 4 respectively. Like in the case of b^4 the only possible outcomes by VP and CS at b^3 are $\{1, 2, 4\}$ and $\{1, 2\}$.

Assume that player 4 is serviced at b^3 , which implies that $\xi(4, O(b^3)) < \xi(4, O(b^{3,4}))$. This contradicts strategyproofness, since when the true values are $b^{3,4}$, player 4 can bid according to b^3 in order to decrease her payment and still being serviced.

Now, assume that player 4 is not serviced at b^3 . Then $\{3, 4\}$ can form a successful coalition when true values are b^3 bidding b^4 , increasing the utility of player 4 to $25 - 20 > 0$, while keeping the utility of player 3 at zero.

Example 3 (c).

Let $\mathcal{A} = \{1, 2, 3, 4\}$. This time we construct a cost sharing scheme, such that part (c) is not satisfied for $L = \{1, 2\}$ (or $\{1, 2, 3\}$) and $U = \{1, 2, 3, 4\}$ and specifically for $C_i = \{3, 4\}$ and $i = 3$.

| ξ | 1 | 2 | 3 | 4 | ξ | 1 | 2 | 3 | 4 | ξ | 1 | 2 | 3 | 4 |
|------------------|----|----|----|----|------------|----|----|----|----|------------|----|----|----|----|
| $\{1, 2, 3, 4\}$ | 30 | 30 | 30 | 30 | $\{1, 2\}$ | 20 | 20 | - | - | $\{3, 4\}$ | - | - | 20 | 30 |
| $\{1, 2, 3\}$ | 20 | 20 | 30 | - | $\{1, 3\}$ | 20 | - | 20 | - | $\{1\}$ | 30 | - | - | - |
| $\{1, 2, 4\}$ | 30 | 30 | - | 30 | $\{1, 4\}$ | 30 | 30 | - | - | $\{2\}$ | - | 30 | - | - |
| $\{1, 3, 4\}$ | 30 | - | 30 | 30 | $\{2, 3\}$ | - | 20 | 20 | - | $\{3\}$ | - | - | 30 | - |
| $\{2, 3, 4\}$ | - | 30 | 30 | 30 | $\{2, 4\}$ | - | 30 | - | 30 | $\{4\}$ | - | - | - | 30 |

Suppose that the values are $b := (25, 25, b_3^*, 30)$. The feasible by VP outcomes are $\{1, 2, 3\}$, $\{1, 3\}$, $\{2, 3\}$, $\{3, 4\}$ and $\{3\}$. Notice that player 4 has zero utility regardless of the outcome.

Assume that $O(b) \neq \{1, 2, 3\}$ and w.l.o.g that $1 \notin O(b)$. Then $\{1, 2, 4\}$ can form a successful coalition bidding $b' := (b_1^*, b_2^*, b_3^*, -1)$ increasing the utility of player 1 to $25 > 0$ without decreasing the utility of player 2 (either $O(b) = \{2, 3\}$ and $\xi(2, O(b')) = \xi(2, O(b))$ and her utility remains the same utility or $2 \notin O(b)$ and her utility increases to $25 > 20$, like in the case of player 1) and keeping the utility of player 4 at zero.

Finally, assume that $O(b) = \{1, 2, 3\}$. Then $\{3, 4\}$ can form a successful coalition bidding $b'' := (25, 25, b_3^*, b_4^*)$. Obviously $\{3, 4\} \subseteq O(b'')$. Notice that VP excludes each of the following outcomes: $\{1, 3, 4\}$, $\{2, 3, 4\}$ and $\{1, 2, 3, 4\}$. As a result, $O(b'') = \{3, 4\}$ implying that the utility of player 3 increases, as $\xi(3, O(b)) > \xi(3, O(b''))$, while player 4 keeps her utility at zero.

B Missing proofs from Section 3

Missing proofs from Subection 3.1

Proof of Proposition 1 (i) We apply condition (a) of Fence Monotonicity, for $L = S \setminus \{i\}$ and $U = S$. It follows that all players $k \in S \setminus \{i\}$ achieve their

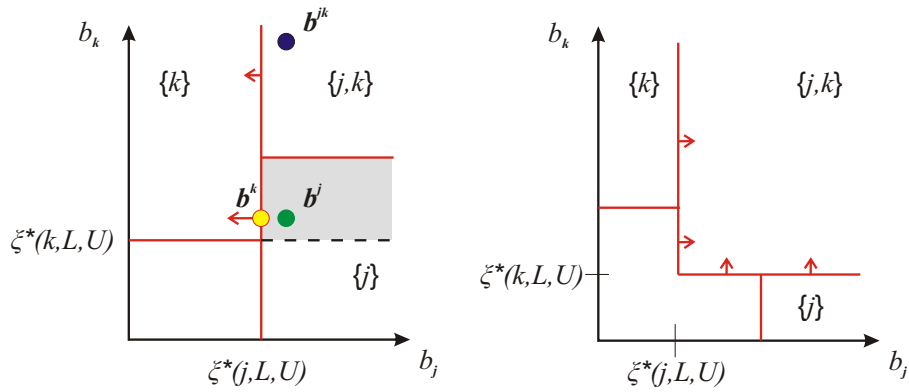


Fig. 1. In the first figure there are only two players, j harms k and k is a sink ($L = \emptyset, U = \{1, 2\}$). The second figure is a Moulin Mechanism. These are the only two possible mechanisms for the case of two players. Unfortunately the shapes for the case of three players become too many and too complicated. However the example of two players makes it much easier to understand the allocation properties that we need in order to prove that condition (b) of Fence Monotonicity is necessary for group-strategyproofness. The allocation at the bid vector b^k , which is on the boundary of the mechanism is $\{k\}$. Notice also the gray area. This is the most interesting area of the mechanism. If you are given the cost-sharing scheme and you want to determine the allocation, you realize that there are two candidate outcomes: $\{k\}$ and $\{j\}$, in the sense that both players have bidden above their ξ^* . However a group-strategyproof mechanism always produces the allocation $\{j\}$. This follows from Lemma 12: a GSP mechanism would never exclude $\{j\}$ from the outcome, in order that $\{k\}$ pays less. However at b^k which is on the boundary of the gray area, the outcome of a GSP mechanism should be $\{k\}$.

minimum payment $\xi^*(k, S \setminus \{i\}, S)$ at the same set of serviced players, either all when the serviced set is S , or all when it is $S \setminus \{i\}$.

There are two cases: If there exists some player $j \in S$ such that $\xi(j, S \setminus \{i\}) < \xi(j, S)$, then this minimum payment is achieved when the set of players that receive service is $S \setminus \{i\}$ and consequently $\xi(k, S \setminus \{i\}) \leq \xi(k, S)$ for all $k \in S \setminus \{i\}$. Otherwise if $\xi(j, S \setminus \{i\}) > \xi(j, S)$, then this minimum payment is achieved when the set of players that receive service is S and consequently $\xi(k, S \setminus \{i\}) \geq \xi(k, S)$ for all $k \in S \setminus \{i\}$.

(ii) We will first use condition (b) of Fence Monotonicity, to show that $\xi(i, S \setminus \{j\}) \leq \xi(i, S)$. Let $L = S \setminus \{i, j\}$ and $U = S$. As we assumed that $\xi(j, S \setminus \{i\}) < \xi(j, S)$, it follows that $\xi^*(j, L, U) = \xi(j, S \setminus \{i\})$, (since $S \setminus \{i\}$ and S are the only sets, within the restriction of the mechanism we consider, that contain j). From condition (b) of Fence Monotonicity, there must be a set S_i with $L \subseteq S_i \subseteq U$, such that $i \in S_i$ and for all $k \in S_i \setminus L$, $\xi(k, S_i) = \xi^*(k, L, U)$. Since $\xi^*(j, L, U) = \xi(j, S \setminus \{i\})$, it is impossible that $S_i = S$ and consequently $S_i = S \setminus \{j\}$. As player i achieves her minimum payment at $S \setminus \{j\}$ we get that $\xi(i, S \setminus \{j\}) \leq \xi(i, S)$.

Now we will use condition (c) of Fence Monotonicity to show that $\xi(i, S \setminus \{j\}) \geq \xi(i, S)$. Let $L = S \setminus \{j\}$, $U = S$, $C \cup \{j\} = S \setminus \{i\}$. Now $\xi(j, S \setminus \{i\}) < \xi(j, S) = \xi^*(j, L, U)$. Since $L \setminus (S \setminus \{i\}) = \{i\}$, $T = \{i\}$ and condition (c) of Fence Monotonicity implies that $\xi(i, S \setminus \{i\} \cup T) = \xi^*(i, L, U)$, i.e. $\xi(i, S) = \xi^*(i, S \setminus \{j\}, S)$. Consequently $\xi(i, S) \leq \xi(i, S \setminus \{j\})$. Putting both inequalities together we get $\xi(i, S) = \xi(i, S \setminus \{j\})$. \square

Missing proofs from Subsection 3.2, Necessity of Fence Monotonicity

Proof of Lemma 1 It is obvious that the minimum payment of player i can only decrease as the set of outcomes, over which the minimum in the definition of ξ^* is taken, becomes larger. \square

Here we prove the necessity of Fence Monotonicity. The proof is by induction on the the cardinality of $|U \setminus L|$. **Base:** For $|U \setminus L| = 0$, i.e., $L = U$ every part of Fence Monotonicity is trivially satisfied as follows.

Condition (a): It holds that $\xi^*(i, U, U) = \xi(i, U)$, since the minimum in the definition of ξ is taken over the single possible outcome U .

Condition (b): This condition holds trivially as $U \setminus L = \emptyset$.

Condition (c): Regardless of whether the if condition of this part is true, it holds that for all $C \subset U$ we can set $T = U \setminus C$ since for all $j \in T$, $\xi(j, C \cup T) = \xi(j, U) = \xi^*(j, L, U)$.

Induction Step: Suppose that Fence Monotonicity is satisfied for all sets U, L such that $|U \setminus L| = m - 1$ we will show that it is also satisfied when $|U \setminus L| = m$ and more specifically when L .

Proof of Proposition 2 (i) We show this using the induction hypothesis at every $L \cup \{i\}, U$ for every $i \in U \setminus L$ and more specifically condition (c) of Fence Monotonicity.

To show anti-symmetry we need to consider only elements of $U \setminus L$, as it is impossible that some $i \in L$ harms any other elements.

Consider two distinct $i, j \in U \setminus L$ and assume that i harms j , that is $\xi^*(j, L, U) < \xi^*(j, L \cup \{i\}, U)$. We will show that j does not harm i . From the definition of ξ^* there is a set S_j , where $j \in S_j$, $L \subseteq S_j \subseteq U$ and $\xi(j, S_j) = \xi^*(j, L, U)$. Notice that that $i \notin S_j$, since $\xi^*(j, L \cup \{i\}, U) > \xi(j, S_j)$. It follows that $S_j \subset U$ and now we apply the induction hypothesis, and specifically condition (c) of Fence Monotonicity at $L \cup \{i\}, U$ (the only non-empty subset of $(L \cup \{i\}) \setminus S_j$ is $\{i\}$, so $T = \{i\}$). We get that $\xi(i, S_j \cup \{i\}) = \xi^*(i, L \cup \{i\}, U)$ or equivalently $\xi(i, S_j \cup \{i\}) = \xi^*(i, L, U)$. Since $L \cup \{j\} \subseteq S_j \cup \{i\} \subseteq U$ we also get that $\xi^*(i, L \cup \{j\}, U) \leq \xi(i, S_j \cup \{i\})$. Putting the last two relations together and applying Lemma 1 we get $\xi^*(i, L \cup \{j\}, U) = \xi^*(i, L, U)$, which means that j does not harm i .

We show transitivity of the harm relation in a similar manner. Consider three distinct players i, j and k , where $i, j \in U \setminus L$ and $k \in U$ (may also belong to L) such that i harms j and j harms k . Assume towards a contradiction that i does not harm k . Since $\xi^*(k, L, U) = \xi^*(k, L \cup \{i\}, U)$, there is some set S_k , where $L \cup \{i\} \subseteq S_k \subseteq U$ such that $\xi(k, S_k) = \xi^*(k, L, U)$. Using similar arguments like in the previous part we show that $\xi(j, S_k \cup \{j\}) = \xi^*(j, L, U)$ and as $L \cup \{i\} \subseteq S_k \cup \{j\} \subseteq U$ we reach a contradiction from our assumption that $\xi^*(j, L, U) < \xi^*(j, L \cup \{i\}, U)$.

(ii) If j is not a sink of $G[U \setminus L]$, there must be an path starting from j , which ends at a sink k of this graph. Transitivity implies that j harms k .

(iii) Suppose that there is some $i \in U \setminus L$ that does not harm j . If i is a sink of $G[U \setminus L]$, setting $k = i$ completes our proof. Otherwise, there must be path that goes through i but does not stop there. Let k be the sink of this path (dag). Notice that transitivity implies that i harms k . Thus, it is impossible that k harms j , since using transitivity again we would deduce that i harms j contradicting our assumption. \square

Proposition 2 allows the effective use of the induction hypothesis in order to identify a successful coalition if some part is violated.

The core idea that we will use for the proof of conditions (a) and (b) is to construct bid vectors, where VP and CS restrict the possible outcomes to be subsets of U that contain every player in L , while for condition (c) we will restrict the possible outcomes to be subsets of U . Therefore, we assume that every player in L has bidden a very high value and every player in $\mathcal{A} \setminus U$ has bidden a negative value.

Induction step for proving Fence Monotonicity (a) We first use the induction hypothesis and the properties of the harm relation to prove that the following Lemma, which is a condition somewhat milder (notice that here not all players in S_j but just the players in $S_j \setminus L$ are guaranteed to pay their minimum payment) than condition (a) of Fence Monotonicity.

Lemma 8. *For every $j \in L$, there is some set S_j , where $L \subseteq S_j \subseteq U$ such that for all $i \in S_j \setminus L$, $\xi(i, S_j) = \xi^*(i, L, U)$ and $\xi(j, S_j) = \xi^*(j, L, U)$.*

Proof. (a) Consider some $j \in L$. We take two cases as Proposition 2(ii) indicates:

Case 1: Suppose that every $i \in U \setminus L$ harms j . This implies that $\xi^*(j, L, U) = \xi(j, L)$ and hence we can set $S_j = L$, and Lemma 8 is trivially satisfied, since $S_j \setminus L = \emptyset$.

Case 2: Consider some sink k that *does not harm* j . Applying the induction hypothesis, part (a) of Fence Monotonicity is satisfied at $L \cup \{k\}, U$, and there is a set S , where $L \cup \{k\} \subseteq S \subseteq U$ such that for all $i \in S$, $\xi(i, S) = \xi^*(i, L \cup \{k\}, U)$. Using the fact that k is a sink and also *does not harm* j we get that for all $i \in S' \setminus L$, $\xi^*(i, L \cup \{k\}, U) = \xi^*(i, L, U)$ and $\xi^*(j, L \cup \{k\}, U) = \xi^*(j, L, U)$. As a result, we can set $S_j = S$. \square

Using this we can prove Lemma 2, which implies that the cost-sharing scheme satisfies condition (a) of Fence Monotonicity.

Proof of Lemma 2 By CS and VP the output of the mechanism satisfies $L \subseteq O(b) \subseteq U$. First note that in any case a player $i \in U \setminus L$ has utility zero: either she is not serviced, or by VP, if she is serviced her payment cannot exceed her bid and cannot be less than her minimum payment $\xi^*(i, L, U)$ so $\xi(i, O(b)) = b_i = \xi^*(i, L, U)$. What remains is to show that every $j \in L$ also pays $\xi^*(j, L, U)$.

Suppose towards a contradiction that for some player $j \in L$, $\xi(j, O(b)) > \xi^*(j, L, U)$ (If the set $U \setminus L$ is empty $U = L$ it is impossible that some $j \in L$ is charged more than $\xi^*(j, L, U) = \xi(j, L)$). Then she could form a coalition with the players in $U \setminus L$, who would enforce the set S_j , where S_j is the set guaranteed to exist by Lemma 8, to be output i.e. the players $i \in U \setminus L$ could change their bids to b_i^* if $i \in S_j$ and -1 if $i \notin S_j$, so that by CS and VP the output is S_j .

Since for every $i \in S_j \setminus L$, $\xi(i, S_j) = \xi^*(i, L, U)$ their utilities remain zero after this manipulation, while the utility of player j strictly increases, and thus the coalition is successful indeed.

Consequently for all $j \in L$, $j \in O(b)$ and $\xi(j, O(b)) = \xi^*(j, L, U)$ and the players in $O(b) \setminus L$ are charged at their minimum payment as well. \square

Induction step for proving Fence Monotonicity (b) Using Proposition 2(ii), we can split the proof in two cases: First we consider the sinks, as the fulfillment of the second condition of Fence Monotonicity for a sink is an immediate consequence of the induction hypothesis.

Lemma 9. *For every sink k of $G[U \setminus L]$ condition (b) of Fence Monotonicity is satisfied at L, U .*

Proof. Using the induction hypothesis at $L \cup \{k\}, U$ we get from part (a) of Fence Monotonicity that there is a set S , where $L \cup \{k\} \subseteq S \subseteq U$, such that for all $i \in S$, $\xi(i, S) = \xi^*(i, L \cup \{k\}, U)$. Since k is a sink, she does not harm any other player from $U \setminus L$, and so we have that for all $i \in S \setminus L$, $\xi^*(i, L, U) = \xi^*(i, L \cup \{k\}, U)$. Thus, setting $S_k = S$ condition (b) of Fence Monotonicity is satisfied for k at L, U . \square

We already know that Fence Monotonicity is satisfied for sinks and the induction hypothesis also guarantees that it is satisfied for $L \cup \{i\}, U$, i.e. if we add to L some player $i \notin L$. To use this fact in order to prove Fence Monotonicity for all other cases as well we prove that if Fence Monotonicity (b) is satisfied just for one player j and L', U' , then if the input is b (as defined in the next lemma), this player receives service and is charged $\xi^*(j, L', U')$.

Lemma 10. *Consider some $L' \subseteq U' \subseteq \mathcal{A}$ suppose that the set S_j as in condition (b) of Fence Monotonicity exists just for some player $j \in U' \setminus L'$. Then at any bid vector b with $b_i = b_i^*$ for $i \in L'$, $b_j > \xi^*(j, L', U')$, $b_i = \xi^*(i, L', U')$ for all $i \in U' \setminus (L' \cup \{j\})$, and $b_i = -1$ for all $i \notin U'$, it holds that $j \in O(b^j)$ and $\xi(j, O(b)) = \xi^*(j, L', U')$.*

Proof. Like in the proof of Lemma 2 we show that By CS and VP the output of the mechanism satisfies $L \subseteq O(b) \subseteq U$. Moreover all players in $U \setminus (L \cup \{j\})$ have utility zero. We need to show that j is serviced and charged $\xi^*(j, L, U)$.

Suppose towards a contradiction that player j , is either not serviced or she is charged $\xi(j, O(b)) > \xi^*(j, L, U)$. Obviously if she is serviced at a higher payment than she prefers the set S to the current outcome. If she is not serviced she again prefers S , as we assumed that $b_j > \xi^*(j, L, U) = \xi(j, S)$ and thus her utility would become positive if the outcome was S . In a similar manner like in the proof of Lemma 2 she could form a coalition with the players in $U \setminus L$ by enforcing the output S . Again our assumption about the payments of the rest of the agents in $S \setminus L$ implies that their utility remains unchanged thus this coalition is successful. Now consider an player j in $U \setminus L$, that is *not a sink* of $G[U \setminus L]$ and let k be one of its *sinks* such that j *harms* k . In order to prove that the second condition is satisfied for j , we will involve group-strategyproofness at certain bid vectors (trying to generalize Example 2 where j takes the role of player 3 and k the role of player 4). Prior to defining these inputs, we prove another allocation property. \square

We continue with the rest players in $U \setminus L$. We will gradually reason about the allocation of the bid vectors b^k, b^{jk} and finally b^j .

| bids | $i \in L$ | j | k | $i \in U \setminus L$ | $i \in \mathcal{A} \setminus U$ |
|----------|-----------|-----------------------------|-----------------------------|-----------------------|---------------------------------|
| b^k | b_i^* | $\xi^*(j, L, U)$ | $\xi^*(k, L, U) + \epsilon$ | $\xi^*(i, L, U)$ | -1 |
| b^{jk} | b_i^* | $\xi^*(j, L, U) + \epsilon$ | b_k^* | $\xi^*(i, L, U)$ | -1 |
| b^j | b_i^* | $\xi^*(j, L, U) + \epsilon$ | $\xi^*(k, L, U) + \epsilon$ | $\xi^*(i, L, U)$ | -1 |

| output | payment of j | payment of the sink k |
|----------|--------------------------------------|-----------------------------------|
| b^k | $j \notin O(b^k)$ | $\xi(k, O(b^k)) = \xi^*(k, L, U)$ |
| b^{jk} | $\xi(j, O(b^{jk})) = \xi^*(j, L, U)$ | $\xi(k, O(b^k)) > \xi^*(k, L, U)$ |
| b^j | $\xi(j, O(b^j)) = \xi^*(j, L, U)$ | $k \notin O(b^j)$ |

Claim. At the bid vector b^k the following hold: (a) player k is serviced and charged $\xi^*(k, L, U)$,

(b) player j is not serviced.

Proof. (a) We get that player k is serviced and charged $\xi^*(k, L, U)$ by applying Lemma 10 for k with $L' = L$ and $U' = U$.

(b) Suppose towards a contradiction that $j \in O(b^k)$. The payment of player k would be lower bounded by $\xi^*(k, L \cup \{j\}, U)$, since $L \cup \{j\} \subseteq O(b^k) \subseteq U$, which contradicts with the fact that k is charged $\xi^*(k, L, U)$, which is strictly lower by our assumption that j harms k . \square

Claim. At the bid vector $b^{j,k}$ the following hold

- (a) For all $i \in U \setminus (L \cup \{j, k\})$, $b_i^{j,k} = \xi^*(i, L \cup \{k\}, U)$ and $b_j^{j,k} = \xi^*(j, L \cup \{k\}, U) + \epsilon$.
- (b) Player j is serviced and charged $\xi^*(j, L, U)$.
- (c) Player k is serviced and charged more than $\xi^*(k, L, U)$.

Proof. (a) Since k is a sink of $G[U \setminus L]$, we have that for all $i \in U \setminus (L \cup \{j, k\})$, $\xi^*(i, L \cup \{k\}, U) = \xi^*(i, L, U) = b_i^{j,k}$. Similarly, we get that $\xi^*(j, L \cup \{k\}, U) + \epsilon = \xi^*(j, L, U) + \epsilon = b_j^{j,k}$.

(b) We apply Lemma 10 for j with $L' = L \cup \{k\}$ and $U' = U$, since condition (b) is satisfied for j at this pair (induction hypothesis) and the bid vector satisfies the requirements of this Lemma. ($b_i^{j,k} = b_i^*$ for all $i \in L \cup \{k\}$, $b_i^{j,k} = -1$ for all $i \notin U$ and using Claim B (a).) As a result, $j \in O(b^{j,k})$ and $\xi(j, O(b^{j,k})) = \xi^*(j, L \cup \{k\}, U) = \xi^*(j, L, U)$.

(c) From part (b) we get that the set $O(b^{j,k})$ satisfies that $L \cup \{j\} \subseteq O(b^{j,k}) \subseteq U$, and thus the payment of player k is lower bounded by $\xi^*(k, L \cup \{j\}, U)$. Since j harms k we get that $\xi^*(k, L \cup \{j\}, U) > \xi^*(k, L, U)$ completing our proof. \square

Finally, we construct the bid vector b^j , where player j bids according to $b^{j,k}$ ($\xi^*(j, L, U) + \epsilon$) player k bids according to b^k ($\xi^*(k, L, U) + \epsilon$), and every other player bids the same value as in both bid vectors.

Claim. At the bid vector b^j the following hold

- (a) Player k is not serviced at b^j .
 - (b) Player j is serviced at b^j .
 - (c) $L \subseteq O(b^j) \subseteq U$ and every player $i \in O(b^j) \setminus L$, is charged $\xi^*(i, L, U)$.
- Setting $S_j = O(b^j)$ we get that Fence Monotonicity is satisfied for every $j \in U \setminus L$ that is not a sink.

Proof. (a) Assume that player k is serviced at b^j . Notice that by VP and the definition of ϵ , if k is serviced at b^j then her payment cannot exceed $\xi^*(k, L, U)$. Additionally, notice that the only coordinate b^j differs from $b^{j,k}$ is the bid of player k . Thus, strategyproofness is violated by $b^{j,k}$, since from Claim B the payment of k decreases.

(b) Suppose that j is not serviced at b^j . Then $\{j, k\}$ can form a successful when true values are b^k bidding b^j , since from Claim B the utility of k increases from zero to ϵ and the utility of j is kept to zero (she is not serviced at either input).

(c) By VP and CS we get that $L \subseteq O(b) \subseteq U$, thus the payment of every serviced player i is lower bounded by $\xi^*(i, L, U)$. By definition of b^j and VP of

the mechanism we get the equality for every player in $O(b) \setminus (L \cup \{j\})$. Moreover, from the definition of ϵ , we conclude the same for j . \square

One can gain more intuition about these allocation properties considering the case of two players j, k . The bid vectors used to prove condition (b) of Fence Monotonicity are shown in Figure 1.

Induction step for proving Fence Monotonicity (c)

Lemma 11. *For every bid vector b , where for all $i \in L$, $b_i > \xi(i, S)$, for all $i \in U \setminus L$, $b_i = \xi^*(i, L, U)$ and for all $i \notin U$, $b_i = -1$, it holds that $L \subseteq O(b) \subseteq U$ and for all $i \in O(b)$, $\xi(i, O(b)) = \xi^*(i, L, U)$.*

Proof. We will prove our statement by induction on the cardinality of the set $T = \{i \in L \mid b_i \neq b_i^*\}$.

Base: Since $T = \emptyset$, we simply apply Lemma 2.

Induction step: For the induction step we will show that $L \subseteq O(b) \subseteq U$ and for all $i \in L$, $\xi(i, O(b)) = \xi^*(i, L, U)$. Then it is easy to see that every $i \in O(b) \setminus L$ must be serviced at $\xi^*(i, L, U)$ by VP and the definition of ξ^* .

By the construction of the bid vector we have that $O(b) \subseteq U$. Consider now some $j \in T$. The induction hypothesis implies that j is serviced and charged $\xi^*(j, L, U)$ at the bid vector (b_j^*, b_{-j}) . Suppose that j is not serviced at b resulting in zero utility. Since $b_j - \xi^*(j, L, U) > 0$ she can misreport b_j^* so as to increase her utility to a strictly positive quantity and thus violate strategyproofness of the mechanism. As a result, j must be serviced at b and hence by strategyproofness j must be serviced at the same payment. (since j is an arbitrary element of T , the same holds for all $j \in T$).

Now since j is indifferent between the two outcomes group-strategyproofness requires the same about the rest of the players. This is only possible if every $i \in L \setminus T$ ($i \in O(b)$ by CS since $b_i = b_i^*$) is charged the same payment in either input i.e. $\xi(i, O(b)) = \xi(i, O(b_j^*, b_{-j})) = \xi^*(i, L, U)$ (induction hypothesis). \square

Lemma 12. *For every bid vector b , where for all $i \in L$, $b_i > \xi(i, S)$, for all $i \in U \setminus L$, $b_i \in \mathbb{R}$ and for all $i \notin U$, $b_i = -1$, it holds that: for all $i \in O(b)$, $\xi(i, O(b)) \geq \xi^*(i, L, U)$.*

Proof. Let b^0 be any bid vector satisfying the conditions of Lemma 11, which means that for all $i \in L$, $b_i^0 > \xi^*(i, L, U)$, for all $i \in U \setminus L$, $b_i^0 = \xi^*(i, L, U)$ and for all $i \notin U$, $b_i^0 = -1$. Then we relax the constrains we put on the bids of the players in $U \setminus L$ (the rest of the players bid always according to b^0) and prove that no player becomes serviced at a price strictly less than her minimum payment at L, U , by using induction on $|T|$, where $T := \{i \in U \setminus L \mid b_i \neq b_i^0\}$.

Base: For $T = \emptyset$ we have that $b = b^0$ and the allocation property follows from Lemma 11.

Induction step: Assume that there is some $j \in O(b)$ that is charged less than $\xi^*(j, L, U)$. Since by VP $O(b) \subseteq U$, either $j \in L$ which implies that $b_j^0 > \xi^*(j, L, U)$ or $j \in U \setminus L$ and thus $b_j^0 = \xi^*(j, L, U)$. In both cases we have that

$$j \in O(b) \text{ and } \xi(j, O(b)) < \xi^*(j, L, U) \leq b_j^0. \quad (1)$$

We will prove that there exists *at least one* successful coalition, contradicting the assumed group-strategyproofness of the mechanism. The key of our proof is the definition of the following special subset of T

$$R := \{i \in T \mid i \in O(b) \text{ and } \xi(i, O(b)) > b_i^0\} \quad (2)$$

(notice that $j \notin R$).

This set is a special subset of T that can be interpreted as follows: If the true valuations are given by b^0 and the players in T bid according to b , then the set R represents the players whose utility becomes negative, and since every $i \in R$ had zero utility at b^0 (Lemma 11), this gives us the set of *liars that sacrificed their utilities*. Notice that the set R not being empty renders the manipulation we considered *unsuccessful*. For proving the induction step we consider two cases regarding R .

Case 1: If $R \subset T$, we construct the bid vector b' , where for every $i \in R$, $b'_i = b_i$ and for every $i \notin R$, $b'_i = b_i^0$. We complete the proof of this case by showing that $(T \setminus R) \cup \{j\}$ form a coalition when their true values are b' bidding b .

First, we prove that in the truthful scenario every player $i \in T \setminus R$ has zero utility and non-negative after the misreporting (profile b). By using the induction hypothesis at b' (differs with b^0 in $|R| < |T|$ coordinates) we get that for every $i \in T \setminus R$ such that $i \in O(b')$ (the rest of the players have obviously zero utility), it holds that $\xi^*(i, L, U) \leq \xi(i, O(b')) \leq b'_i$, where the last inequality follows from VP. Since $i \in T \setminus R$, we get that $b'_i = b_i^0 = \xi^*(i, L, U)$ and thus $b'_i = \xi(i, O(b'))$. Now consider the outcome after the misreporting. Either $i \notin O(b)$ and hence she has zero utility after the misreporting or $i \in O(b)$ and her utility becomes $b'_i - \xi(i, O(b)) = b_i^0 - \xi(i, O(b)) \geq 0$, where the last inequality follows from Equation 2 as $i \in T \setminus R$.

Second, we prove that player j strictly increases her utility. Considering the truthful scenario there are two cases for j : She is serviced at b' and charged $\xi(j, O(b')) \geq \xi^*(j, L, U)$ from the induction hypothesis. From Equation 1 we get that j is serviced after the misreporting and charged $\xi(j, O(b)) < \xi^*(j, L, U)$, and thus her payment decreases, while she is still serviced. Now if j is not serviced at b' , then she has zero utility. From Equation 1 we get that $b_j^0 > \xi(j, O(b))$ and since $j \notin R$ it follows that $b_j^0 = b'_j$. As a result, her utility increases to $b'_j - \xi(j, O(b)) > 0$.

Case 2: Otherwise if $R = T$, we construct the bid vector b'' , where every $i \in R$, $b''_i = \xi(i, O(b))$ and every $i \notin R$, $b''_i = b_i$. Notice that for all $i \in R$, $b_i \geq b''_i$ by VP at b since $R \subseteq O(b)$. Moreover, for every $i \notin R$, $b''_i = b_i^0$, since $R = T$.

Claim. If $T = R$ and there is some j that satisfies Equation 1, then for the bid vector b'' , it holds that $R \cup \{j\} \subseteq O(b'')$ and for all $i \in R \cup \{j\}$, $\xi(i, O(b'')) = \xi(i, O(b))$.

Proof. For all $S \subseteq R$, we define the bid vector b^S , where for all $i \in S$, $b_i^S = b''_i$ and for all $i \notin S$, $b_i^S = b_i$. Notice that since we assumed that $T = R$, it holds that $b_i^S = b_i^0$ for all $i \notin R$ and all $S \subseteq R$. We show that for all $S \subseteq R$, $R \cup \{j\} \subseteq O(b^S)$

and for all $i \in R \cup \{j\}$, $\xi(i, O(b^S)) = \xi(i, O(b))$, with induction on $|S|$. We will refer to this induction as *second* induction to discriminate it from the *first*. After proving this statement, we can set $S = R$ ($b^R = b''$) and complete the proof.

Base (second): If $S = \emptyset$, i.e. $b^\emptyset = b$, then every condition holds trivially.

Induction step (second): If for some $i \in S$, it holds that $b_i'' = b_i$, then the induction step follows trivially from induction hypothesis as $b^S = (b_i, b_{-i}^S) = b^{S \setminus \{i\}}$.

Now assume that for all $i \in S$, $b_i'' < b_i$ and consider some $i \in S$. Player i is serviced at (b_i, b_{-i}^S) and charged $\xi(i, O(b))$ (induction hypothesis). Strategyproofness implies that by lowering her bid up to her payment, which gives us the bid vector b^S , if she is serviced, then her payment must be the same, i.e.,

$$\text{for all } i \in S \cap O(b^S), \xi(i, O(b^S)) = \xi(i, O(b)). \quad (3)$$

This Equation implies that if the true values are given by b^S , then every player in S has zero utility. Moreover, by misreporting b they are charged an amount equal to their true value and thus their utilities is kept at zero. Group-strategyproofness implies that no player in $\mathcal{A} \setminus S$ has incentives for this misreporting. Notice that $b_j^S = b_j^0$ since $j \notin R = T$. Moreover, from Equation 1 we have that $b_j^0 \geq \xi^*(j, L, U) > \xi(j, O(b))$ and thus her utility after the manipulation we described is given by $b_j^S - \xi(j, O(b)) > 0$. In order that this amount is not bigger than her utility in the truthful scenario (b^S), it holds that

$$j \in O(b^S) \text{ and } \xi(j, O(b^S)) \leq \xi(j, O(b)), \quad (4)$$

which together with Equation 1 and the fact that $b_j^S = b_j^0$ implies that

$$\xi(j, O(b^S)) < \xi^*(j, L, U) \leq b_j^S. \quad (5)$$

Using the latter equation we show that $R \subseteq O(b^S)$. Assume that some $i \in R$ is not serviced at b^S . First, we show that i is not serviced at (b_i^0, b_{-i}^S) . There are two cases for i : Either $b_i^S = b_i''$ if $i \in S$ or $b_i^S = b_i \geq b_i''$ if $i \in R \setminus S$. In any case $b_i^S \geq b_i'' > b_i^0$, where the second inequality follows from the definition of R (Equation 2) since $b_i'' = \xi(i, O(b))$. Strategyproofness implies that if the true values are b^S and i reports b_i^0 (the bid vector becomes (b_i^0, b_{-i}^S)), then she is not serviced as otherwise her payment cannot exceed b_i^0 by VP and thus her utility increases to $b_i^S - b_i^0 > 0$.

Now we show that if i is not serviced at (b_i^0, b_{-i}^S) , then $\{i, j\}$ form a successful coalition when true values are (b_i^0, b_{-i}^S) bidding b^S . The utility of player i is kept to zero, as she is excluded in both inputs from the outcome. Notice that by the *first* induction hypothesis at the bid vector (b_i^0, b_{-i}^S) (differs with b^0 in $|R| - 1 = |T| - 1$ coordinates) there are two cases for j : Either j is serviced and charged $\xi(j, O((b_i^0, b_{-i}^S))) \geq \xi^*(j, L, U)$ and thus her payment decreases to $\xi(j, O(b^S)) < \xi^*(j, L, U)$ (from Equation 5) or she is not serviced and her utility becomes $b_j^S - \xi(j, O(b^S)) > 0$ (from Equation 5).

As a result, it holds that $S \subseteq R \subseteq O(b^S)$ and together with 4 we get that for every $i \in S$, $\xi(i, O(b^S)) = \xi(i, O(b))$. Therefore, the players in S are indifferent

between the two inputs, i.e regardless of which of the two we consider as true values, they can misreport the other without losing utility. Group-strategyproofness implies that the other players are indifferent as well, which is only possible if for all $i \in R \setminus S$, $\xi(i, O(b^S)) = \xi(i, O(b))$. Moreover, from Equation 4 we get that $j \in O(b^S)$ and for similar reasons it holds that $\xi(j, O(b^S)) = \xi(j, O(b))$. \square

By VP we have that $O(b'') \subseteq U$ (for every $i \in \mathcal{A} \setminus U$, it holds that $i \notin R$ and thus $b_i'' = b_i^0 = -1$). Furthermore, from the previous Claim and Equation 1 we get that $\xi(j, O(b'')) < \xi^*(j, L, U)$. Therefore, the definition of ξ^* implies that $L \not\subseteq O(b'')$.

Assume that the true values are given by b'' . Every player $k \in (L \setminus O(b''))$ has obviously zero utility. From the previous Claim we deduce the same for every $i \in R$. We complete our proof by showing that $R \cup (L \setminus O(b''))$ form a successful coalition bidding b^0 .

First, we prove that the utility of every $i \in R$ remains non-negative after the misreporting. Consider some $i \in R$ such that $i \in O(b^0)$ (obviously the rest players have zero utility). From Lemma 11 it holds that $\xi(i, O(b^0)) = \xi^*(i, L, U)$ and since $\xi^*(i, L, U) = b_i^0$ ($i \in U \setminus L$) and $b_i^0 < \xi(i, O(b)) = b_i''$ (Equation 2), we conclude that her utility increases to $b_i'' - \xi(i, O(b^0)) > 0$.

Second, we show that the utility of every $k \in L \setminus O(b'')$ increases, as it becomes strictly positive. Since $k \notin R$ we get that $b_k'' = b_k^0$. Moreover, since $k \in L$ we have $b_k^0 > \xi^*(k, L, U)$ and from Lemma 11 we get that $k \in O(b^0)$ and $\xi(k, O(b^0)) = \xi^*(k, L, U)$. As a result, her utility becomes $b_k'' - \xi(k, O(b^0)) > 0$. \square

Proof of Lemma 3 (a) Immediate by applying Lemma 12.

(b) VP and CS imply that $C_j \subseteq O(b^c) \subseteq L \cup C_j$. The strict inclusion $C_j \subset O(b^c)$ follows from (a) and the initial assumption of the Lemma that $\xi(j, C_j) < \xi^*(j, L, U)$.

(c) From (a) we have that for all $i \in O(b^c) \setminus C_j$, $\xi(i, O(b^c)) \geq \xi^*(i, L, U)$. However all of these players have bidden $\xi^*(i, L, U) + \epsilon$, where ϵ is an arbitrarily small positive number, smaller than any payment difference, and by VP would not receive service if they had to pay higher than their bid. So $\xi(i, O(b^c)) = \xi^*(i, L, U)$. Setting $S_j = O(b^j)$, we conclude that condition (b) of Fence Monotonicity is satisfied for any $j \in U \setminus L$ that is not a sink of $G[U \setminus L]$ as well. \square

Proving the Necessity of Stability and Validity of the tie breaking rule

Proof of Lemma 4 Notice that Lemma 11 implies the same allocation property in the special case, where every $i \in \mathcal{A} \setminus U$ has bidden -1 . In order to show this Lemma we will use the same induction, i.e. we show that $L \subseteq O(b)$ and for all $i \in O(b)$, $\xi(i, O(b)) = \xi^*(i, L, U)$ using induction on $m = |\{i \in L \mid b_i \neq b_i^*\}|$.

Base: First, by CS we have that $L \subseteq U$. Next, we show that not player in $\mathcal{A} \setminus U$ is serviced. Suppose that the set $R = O(b) \setminus U \neq \emptyset$. We will show that VP is violated. By the definition of the bid vectors we consider it holds that there is

some $i \in R$, such that $b_i < \xi^*(i, L, U \cup R) \leq \xi(i, O(b))$ (since $L \subseteq O(b) \subseteq U \cup R$) and thus we reach a contradiction.

As a result, it holds that $L \subseteq O(b) \subseteq U$. Thus, for every $i \in O(b)$, we have that $\xi(i, O(b)) \geq \xi^*(i, L, U)$. By VP we get the equality for every $i \in O(b) \setminus L$. Now assume that the previous inequality is strict for some $j \in L$. We show that the players in $(\mathcal{A} \setminus U) \cup \{j\}$ can form a successful coalition, where every $i \notin U$ announces -1 .

Trivially the utilities of these players are kept to zero after the misreporting and applying Lemma 11 we get that j is serviced and charged $\xi^*(j, L, U) < \xi(j, O(b))$.

Induction Step: We show that $L \subseteq O(b)$ and for all $i \in L$, $\xi(i, O(b)) = \xi^*(i, L, U)$ exactly like in the proof of Lemma 11. Now since $L \subseteq O(b)$ in similar way we show that $U \subseteq O(b)$. Thus, this restriction together with the definition of the bid vector, implies that for every $i \in O(b) \setminus L$, $\xi(i, O(b)) = \xi^*(i, L, U)$. \square

B.1 Missing Proofs of Subsection 3.3 – Sufficiency

Proof of Lemma 5 Assume towards a contradiction that there exist two distinct sets U_1, U_2 none of which is a subset of the other that both satisfy this property. Then we could construct the set $U_1 \cup U_2$ which also satisfies the same property and is a proper superset of both U_1 and U_2 reaching a contradiction. Indeed for all $i \in U_1 \setminus L$ we have $b_i \geq \xi^*(i, L, U_1) \geq \xi^*(i, L, U_1 \cup U_2)$ as the minimum payment of player i can only decrease as the set of outcomes, over which the minimum in the definition of ξ^* is taken, becomes larger. A similar inequality holds for the players in $i \in U_2 \setminus L$ completing the proof.

Now we show that if U is the maximal set with this property, then L, U satisfy property 3. of stability. Consider some non-empty $R \subseteq \mathcal{A} \setminus U$. Assume that for all $i \in R$, $b_i \geq \xi^*(i, L, U \cup R)$. From Lemma 1 we have that for all $i \in U \setminus L$, $b_i \geq \xi^*(i, L, U \cup R)$ and thus we reach contradiction by the maximality of U . \square

Prior to proving Lemma 6 we provide some technical Lemmas

Lemma 13. *Suppose that L, U is a stable pair at the bid vector b and that S is set with the property that for all $i \in S$ we have that $b_i - \xi(i, S) \geq 0$. (If the mechanism would output S then all players are would have been served with non-negative utility.) If (a) $S \not\subseteq U$, or*

(b) if $S \subseteq U$ and for some $i \in S$ we have $\xi^(i, L, U) > \xi(i, S)$, there exists some non-empty set $T \subseteq L \setminus S$, such that for all $j \in T$ we have $\xi(j, S \cup T) < b_j$.*

Proof. We will first show that if $S \not\subseteq U$ then there exists some $i \in S$ we have $\xi^*(i, L, U \cup S) > \xi(i, S)$. Since L, U is stable (property 3.) there exists some $i \in S \setminus U$ such that $b_i < \xi^*(i, L, U \cup S)$ and since from the initial assumption $\xi(i, S) \leq b_i$ we get that $\xi(i, S) < \xi^*(i, L, U \cup S)$

The rest of the proof is the same for both cases (note that in what follows, if $S \subseteq U$ then $S \cup U = U$). Note that $S \subset U \cup S$, since $L \not\subseteq S$. Applying part

(c) of Fence Monotonicity we get that there exists a non-empty $T \subseteq L \setminus S$ such that for all $j \in T$ we have $\xi(j, S \cup T) = \xi^*(j, L, U \cup S) \leq \xi^*(j, L, U)$, where the last inequality is by the definition of ξ^* , since the minimum cannot decrease as the set of outcomes over which it is taken becomes larger. From property 1. of stability and since $T \subseteq L$, for all $j \in T$ we have $\xi^*(j, L, U) < b_j$. Consequently for all $j \in T$ we have $\xi(j, S \cup T) < b_j$. \square

Lemma 14. *Suppose that $L \subseteq S \subseteq U$ and that there exists a non-empty $T \subseteq \mathcal{A} \setminus S$ such that for all $i \in T$ we have $b_i \geq \xi(i, S \cup T)$ and that for at least one player from T the inequality is strict. Then L, U is not a stable set at the bid vector b .*

Proof. Suppose towards a contradiction that L, U is stable at b and that there exists some $T \subseteq \mathcal{A} \setminus S$, such that for all $i \in T$, $b_i \geq \xi(i, S \cup T)$ and that for at least one player the inequality is strict.

Since $L \subseteq S \cup T \subseteq U \cup T$ from the definition of ξ^* we get that for all $i \in T$ we have $\xi^*(i, L, U \cup T) \leq \xi^*(i, L, U) \leq \xi(i, S \cup T)$.

If $T \subseteq U$, then there exists some $i \in T$ such that $b_i > \xi(i, S \cup T) \geq \xi^*(i, L, U)$. Since $T \subseteq \mathcal{A} \setminus L$ this contradicts with stability (condition 2.) of b at L, U .

If $T \not\subseteq U$, then $T \setminus U$ is not empty and for all $i \in T \setminus U$ we have $b_i \geq \xi(i, S \cup T) \geq \xi^*(i, L, U \cup T)$, which contradicts with stability (condition 3.) of b at L, U . \square

Proof of Lemma 6 Uniqueness of Stable Pairs

Assume first that $L_1 \neq L_2$ and without loss of generality that $L_2 \not\subseteq L_1$. Consequently there exists some $j \in L_2 \setminus L_1$.

By Fence Monotonicity (part (a)) there exists a set S_2 such that $L_2 \subseteq S_2 \subseteq U_2$ and $\xi(i, S_2) = \xi^*(i, L_2, U_2)$ for all $i \in S_2$. Notice that for all $j \in S_2$, $b_j \geq \xi(j, S_2)$, where strict inequality holds only if $j \in L_2$. The idea is to apply Lemma 13 and get that there exists a non-empty $T \subseteq L \setminus S_2$, such that for all $i \in T$ we have $b_i > \xi(i, S_2 \cup T)$. We will then apply Lemma 14 to show that L_2, U_2 is not stable at b which contradicts our initial assumption.

It only remains to show that we can apply Lemma 13. If $S_2 \not\subseteq U_2$ this is immediate. Suppose that $S_2 \subseteq U_1$. If $j \in L_2 \setminus L_1$ then also $j \in U_1 \setminus L_1$ thus from stability (condition 2.) of L_1, U_1 , we get that $b_j = \xi^*(j, L_1, U_1)$ and from stability (condition 1.) of L_2, U_2 and since $j \in L_2$, we get that $b_j > \xi(j, S_2)$. Therefore, we get that $\xi(j, S_2) < \xi^*(j, L_1, U_1)$ and consequently S_2 satisfies the requirements of Lemma 13.

We showed that $L_1 = L_2 = L$. Suppose towards a contradiction that $U_1 \neq U_2$. From Lemma 5 there exists a unique maximal set U such that $b_i \geq \xi^*(i, L, U)$ for all $i \in U$. Consequently U_1, U_2 are subsets of U and at least one of them, say U_1 a proper subset of U . Then the players in $U \setminus U_1$ contradict stability (condition 3.). \square

Group-strategyproofness between inputs where a stable pair exists

Let b and b' be two bid vectors, and let L, U and L', U' be their corresponding unique (from Claim 6 (a)) stable pairs and $O(b)$ and $O(b')$ the corresponding

outputs. Assume towards a contradiction that some of the players can form a successful coalition when the true values are b reporting b' .

We will first show that any player i served in the new outcome, $i \in O(b')$, has non-negative utility i.e. $b_i \geq \xi(i, O(b'))$. Take some i that is output in $O(b')$. If $b_i = b'_i$ then it holds trivially since the mechanism satisfies VP at the outcome $O(b')$. If $b_i \neq b'_i$ then i changes his bid to be part of the coalition and consequently his utility after this coalition is non-negative, which gives $b_i - \xi(i, O(b')) \geq 0$.

The next step is to apply Lemma 13 to show that there exists some non-empty $T \subseteq L \setminus O(b')$ such that for all $i \in T$ we have $b_i > \xi(i, O(b') \cup T)$. If $O(b') \not\subseteq U$ then the premises of the Lemma hold trivially.

Suppose that $O(b) \subseteq U$. For the coalition to be successful the utility of at least one player j increases strictly when the players bid b' consequently $j \in O(b')$ and $b_j - \xi(j, O(b')) > 0$. We will show that $\xi^*(j, L, U) > \xi(j, O(b'))$. If j is not served at b and since $O(b') \subseteq U$ from stability we get that $j \in U \setminus L$ and $b_j = \xi^*(j, L, U) > \xi(j, O(b'))$. If j is served at b then her payment equals $\xi^*(j, L, U)$ by the definition of the mechanism and in order that she profits strictly it must be $\xi(j, O(b')) < \xi^*(j, L, U)$, so we can again apply Lemma 13.

Finally we will show that for all $i \in T$ we have $b_i = b'_i$. After the manipulation the players in T are not serviced, while as $T \subseteq L$ from stability we have that in the truthful scenario the players in T are serviced with positive utility. Consequently the players in T wouldn't have an incentive to be part of the coalition and change their bids.

Putting everything together we get that there exists a $T \subseteq \mathcal{A} \setminus O(b')$ such that for all $i \in T$ we have $b'_i > \xi(i, O(b') \cup T)$, which by Lemma 14 contradicts our initial assumption that L', U' is stable at b' . \square

Proof of Lemma 7 Let b_i^* be some value that is big enough so that player i always gets serviced, you can let $b_i^* > \max_{S \subseteq \mathcal{A}} \xi(i, S)$, since the allocation of the mechanism satisfies Stability. We will show that there exists a stable pair at any input b by induction on the number m of coordinates that are less than b_i^* , i.e. on the number $m = |\{i \mid b_i < b_i^*\}|$.

Base: For $m = 0$, we only have to show that there exists a stable pair for the bid vector (b_1^*, \dots, b_n^*) and \mathcal{A}, \mathcal{A} is a stable pair.

Induction Step: Suppose that if a bid vector has $m - 1$ coordinates that are less than b_i^* , then it has a stable pair. We will show that if a bid vector b has m coordinates that are less than b_i^* then it also has a stable pair. We will suppose towards a contradiction that there exists no stable pair at b .

We first need some definitions. Let $L^* := \{i \mid b_i \geq b_i^*\}$ and U^* the corresponding (from Lemma 5) maximal set. Notice that for all $i \in L^*$, $b_i = b_i^* > \xi^*(i, L^*, U^*)$ by the definition of b_i^* , which implies that the pair L^*, U^* satisfies the first condition of stability. Moreover, from Lemma 6 we get that the third condition of stability is satisfied for L^*, U^* as well. Thus, as L^*, U^* cannot be stable at b (from our assumption), the set $W = \{i \mid i \in U^* \setminus L^* \text{ and } b_i > \xi^*(i, L^*, U^*)\}$ should be non-empty.

For each $i \in W$ we define a corresponding pair L_i, U_i as follows: The pair L_i, U_i is the unique (by Lemma 6) stable pair of (b_i^*, b_{-i}) , which exists by the induction hypothesis.

Claim. If L_i, U_i is the stable pair of (b_i^*, b_{-i}) , then

- (a) $L^* \cup \{i\} \subseteq L_i$ and (b) $U_i \subseteq U^*$.
- (c) If $j \in L_i \setminus (L^* \cup \{i\})$ then $j \in W$ and $b_j > \xi^*(L^*, U^*)$.

Proof. (a) Using the definition of L^* we get that $L^* \cup \{i\}$ contains all the players who have bidden higher than any payment of the mechanism in (b_i^*, b_{-i}) . Since L_i is stable at (b_i^*, b_{-i}) , each one of these players, who have bidden strictly higher than any payment of the mechanism, should be serviced (any value higher than any payment satisfies the definition of CS for Fencing Mechanisms), and if they are serviced they obviously have strictly positive utility, and thus $L^* \cup \{i\} \subseteq L_i$.

(b) The idea is to show that for all $j \in (U^* \cup U_i) \setminus L^*$, $b_j \geq \xi^*(j, L^*, U^* \cup U_i)$ and since we defined U^* be the maximal set with this property, we get that $U_i \subseteq U^*$.

From definition of U^* we have that for all $j \in U^* \setminus L^*$, $b_j \geq \xi^*(j, L^*, U^*) \geq \xi^*(j, L^*, U^* \cup U_i)$, where the last inequality follows from Lemma 1.

Now consider some $j \in U_i \setminus U^*$ ($j \neq i$). Since L_i, U_i is stable at (b_i^*, b_{-i}) we get that $b_j \geq \xi^*(j, L_i, U_i) \geq \xi^*(j, L^*, U_i \cup U^*)$ by applying Lemma 1, since from (a) $L^* \subset L_i$.

(c) From (a),(b) and Lemma 1 we have that $\xi^*(j, L_i, U_i) \geq \xi^*(j, L^*, U^*)$. As we defined L_i, U_i to be the stable pair at (b_i^*, b_{-i}) and $j \in L_i$, for $j \neq i$ we have $b_j > \xi^*(j, L_i, U_i) \geq \xi^*(j, L^*, U^*)$ and as $j \notin L^*$, we have $j \in W$. \square

Claim. If there exists no stable pair at b , then for all $i \in W$ we have

- (a) $b_i \leq \xi^*(i, L_i, U_i)$, and (b) $L^* \cup \{i\} \subset L_i$.

Proof. (a) The pair L_i, U_i is stable at (b_i^*, b_{-i}) (by definition of L_i, U_i) but it is not stable at b (from our assumption that there exists no stable pair at b). Since the two bid vectors differ only on the i -th coordinate we deduce that stability (condition 1. as $i \in L_i$) is not satisfied by the i -th coordinate of b , thus $b_i \leq \xi^*(i, L_i, U_i)$.

(b) From Claim B.1 (a) we already have that $L_i \supseteq L^* \cup \{i\}$. Suppose towards a contradiction that $L_i = L^* \cup \{i\}$. From Fence Monotonicity (condition (b)) we get that there exists some S_i , where $L^* \subseteq S_i \subseteq U^*$ and $i \in S_i$ such that for all $j \in S_i \setminus L^*$ we have $\xi(j, S_i) = \xi^*(j, L_i, U_i)$.

The idea is to show that $L_i \subseteq S_i \subseteq U_i$, which implies that $\xi^*(i, L_i, U_i) \leq \xi(i, S_i) = \xi^*(i, L^*, U^*)$. Then considering also that $\xi^*(i, L^*, U^*) < b_i$, because $i \in W$, we get $\xi^*(i, L_i, U_i) < b_i$, contradicting the inequality we showed in (a).

It only remains to show that $L_i \subseteq S_i \subseteq U_i$. Notice first that $L_i \subseteq S_i$ since $L_i = L^* \cup \{i\}$. Moreover, since for all $j \in S_i \setminus L_i$, it holds that $b_j \geq \xi(j, S_i)$ and the bids of these players are the same at (b_i^*, b_{-i}) , stability of L_i, U_i (condition 3) implies that $S_i \subseteq U_i$, because otherwise $S_i \setminus U_i \neq \emptyset$ and its elements would violate condition 3. of stability. \square

Since $W \neq \emptyset$ there exists some $i \in W$ such that L_i has minimum cardinality, i.e. $i = \arg \min_{i \in W} |L_i|$. By Claim B.1 (b) there exists some $j \in L_i \setminus (L^* \cup \{i\})$ and by Claim B.1 (c) $j \in W$. We will show that $L_j \subset L_i$ contradicting the choice of i .

Claim. If there exists no stable pair at b and $j \in L_i \setminus (L^* \cup \{i\})$ then

- (a) The pair L_i, U_i is stable at the bid vector $(b_i^*, b_j^*, b_{-\{i,j\}})$.
- (b) $\xi^*(j, L_j, U_j) > \xi^*(j, L_i, U_i)$ and $i \notin L_j$.
- (c) If there exists some $k \in L_j \setminus L_i$, then the mechanism is not group-strategyproof at inputs where a stable pair exists.

Proof. (a) The pair L_i, U_i is stable at bid vector $(b_i^*, b_j^*, b_{-\{i,j\}})$, since L_i, U_i is stable at (b_i, b_{-i}) and since raising the bid of player j ($j \in L_i$ from our initial assumption) to b_j^* does not effect its stability.

(b) From stability of L_i, U_i at (b_i^*, b_{-i}) we get that $b_j > \xi^*(j, L_i, U_i)$, while from part (a) of Claim B.1 we get that $\xi^*(j, L_j, U_j) \geq b_j$. Consequently $\xi^*(j, L_j, U_j) > \xi^*(j, L_i, U_i)$.

Supposing towards a contradiction that $i \in L_j$ we also get in a similar way as before that the pair L_j, U_j is stable at $(b_i^*, b_j^*, b_{-\{i,j\}})$. By Lemma ?? these two pairs coincide, which is a contradiction because we just showed that the payment of j is different.

(c) We will show that if there exists some $k \in L_j \setminus L_i$ then the mechanism is not GSP at inputs, where a stable pair exists. Observe that $k \neq i, j$.

Consider first the vector $b^{i,j} := (b_i^*, b_j^*, b_{-\{i,j\}})$, which stable pair is L_i, U_i from part (a). As $k \notin L_i$ either k is not serviced, or if k is serviced, then her utility is zero. As for j she is serviced with payment $\xi^*(j, L_i, U_i) < \xi^*(j, L_j, U_j) = \xi(j, O(b_j^*, b_{-j}))$ (from (a)).

Consider then $b^j := (b_j^*, b_{-j})$, where L_j, U_j is stable. As $k \in L_j$ and k is serviced and $b_k > \xi(k, O(b_j^*, b_{-j}))$.

Resuming player j strictly prefers $O(b^{i,j})$ to $O(b^j)$, while for k the situation is exactly the opposite. The idea is to construct a bid vector b' where i has zero utility and either $\{i, j\}$ or $\{i, k\}$ is a successful coalition. Let $b' = (\xi(i, O(b^{i,j})), b_j^*, b_{-\{i,j\}})$ and notice that induction hypothesis implies that there is a stable pair at b' . Moreover, observe that the three bid vectors differ only on the bid of player i and consequently from strategyproofness, at every input i is served, she is charged $\xi(i, O(b^{i,j}))$. This implies that if i is served at b' she is charged an amount equal to her bid. Moreover, i is serviced at b^j in the degenerate case where $b' = b^j$ as otherwise VP would imply that $\xi(j, O(b^j)) \leq b_j^j < b_j^{i,j} = \xi(j, O(b^{i,j}))$. In every case we have that i has zero utility in at b' and b^j .

Observe first that k must be serviced at b' and charged $\xi(k, O(b')) \leq \xi(k, O(b^j))$ as otherwise $\{i, k\}$ would have been able to form a successful coalition when the true values are b' bidding b^j . Similarly we can show that $\xi(j, O(b')) \leq \xi(j, O(b^{i,j}))$, excluding the degenerate case $b' = b^j$, because otherwise $\{i, j\}$ would have been able to form a successful coalition when the true values are b' bidding $b^{i,j}$ (the utility of i is kept to zero as in the truthful scenario).

As a result players j and k strictly prefer $O(b')$ to $O(b^j)$ and $O(b^{i,j})$ respectively. If $i \in O(b')$ then $\{i, k\}$ when true utilities are given by $b^{i,j}$ can form a successful coalition bidding b' strictly increasing the utility of k , while keeping the utility of i constant, since she is still served with the same payment. If $i \notin O(b')$ then we deduce that $\{i, j\}$ when the true utilities are given by b^j can form successful coalition bidding b' strictly increasing the utility of j , while keeping the utility of i to zero. \square

Now since $j \in W$ and from Claim B.1 $i \notin L_j$ and for every $k \in L_j$ we have that also $k \in L_i$, we get that $L_j \subset L_i$, which completes the proof. \square

C Missing Proofs of Section 4

Proof of Theorem 3 Let $\mathcal{A} = \{1, 2, 3\}$. Consider the cost sharing function defined on \mathcal{A} as follows: $C(\{1, 2\}) = C(\{1, 3\}) = 1$, $C(\{1\}) = C(\{2\}) = C(\{3\}) = x$, $C(\{2, 3\}) = x^2 + x$ and $C(\{1, 2, 3\}) = x^3 + x^2 + x$, where $x \geq 1$. We show that there is no $\frac{1}{x}$ -budget balanced cost sharing scheme, that satisfies Fence Monotonicity.

Assume by contradiction that there is a α -budget balanced cost sharing scheme that satisfies Fence Monotonicity for C where $1 \geq \alpha > \frac{1}{x}$.

First, we consider payments of the players 2 and 3 at the sets $\{1, 2\}$ and $\{1, 3\}$ respectively. From the upper bound of budget balance we have that $\xi(2, \{1, 2\}) \leq 1$ and $\xi(3, \{1, 3\}) \leq 1$. From Proposition 1 (b) it cannot be the case that both of these players payment increases at the set $\{1, 2, 3\}$. That is either $\xi(2, \{1, 2, 3\}) \leq 1$ or $\xi(3, \{1, 2, 3\}) \leq 1$ (or both). W.l.o.g. we assume that $\xi(2, \{1, 2, 3\}) \leq 1$. We consider now two cases for $\xi(2, \{2, 3\})$.

Case 1: Suppose that $\xi(2, \{2, 3\}) \leq 1$. From the lower bound of budget-balance at the set $\{2\}$, we have that $\xi(2, \{2\}) > 1$, and thus $\xi(2, \{2, 3\}) < \xi(2, \{2\})$. We apply Proposition 1 (b) and deduce that $\xi(3, \{2, 3\}) \leq \xi(3, \{3\}) \leq C(3) = x$. Thus, the total sum of the shares at the set $\{2, 3\}$ is $\xi(2, \{2, 3\}) + \xi(3, \{2, 3\}) \leq x + 1$, which contradicts our assumption about the budget balance of the mechanism, since $C(\{2, 3\}) = x^2 + x$.

Case 2: Suppose that $\xi(2, \{2, 3\}) > 1 \geq \xi(2, \{1, 2, 3\})$. Notice that the contra-positive implication of Proposition 1 (a) implies that $\xi(3, \{1, 2, 3\}) \leq \xi(3, \{2, 3\}) \leq C(\{2, 3\}) - \xi(2, \{2, 3\}) < x^2 + x - 1$. Moreover, as the payment of player 2 decreases from $\{2, 3\}$ to $\{1, 2, 3\}$, Proposition 1 (b) implies that the payment of player 1 doesn't increase from $\{1, 3\}$ to $\{1, 2, 3\}$, i.e. $\xi(1, \{1, 2, 3\}) \leq \xi(1, \{1, 3\}) \leq C(\{1, 3\}) = 1$. Now summing the shares at the set $\{1, 2, 3\}$, we get that $\xi(1, \{1, 2, 3\}) + \xi(2, \{1, 2, 3\}) + \xi(3, \{1, 2, 3\}) \leq x^2 + x + 1$, which is again contradicts our budget balance assumption, since $C(\{1, 2, 3\}) = x^3 + x^2 + x$. \square

Proof of Theorem 4 Consider the following process, which takes as input a bid vector b and a set L .

repeat

$U \leftarrow L \cup \{i \in U \setminus L \mid b_i \geq \xi^*(i, L, U)\}$

until For all $i \in U \setminus L$, $b_i \geq \xi^*(i, L, U)$

We will prove that if we feed this process with the lower set L of the stable pair at b , then the outcome is the upper set of the stable pair.

Obviously by the definition of the process, for its final set U it holds that for all $i \in U$, $b_i \geq \xi^*(i, L, U)$. First, we show that U is the maximal set with this property.

Assume that there is some U' with $U' \not\subseteq U$, that satisfies this property, then we have that for all $i \in (U \cup U') \setminus L$, $b_i \geq \xi^*(i, L, U \cup U')$. Thus, it is impossible that the players in $U \cup U'$ are removed at any step of the previous process, contradicting our assumption that $U \neq U' \cup U$ is the outcome of this process.

Now consider now the upper set U'' of the stable pair at b . Since U'' satisfies this property it follows that $U'' \subseteq U$. Notice that if $U'' \subset U \neq \emptyset$, then the elements of $U \setminus U''$ would violate stability of L, U'' . Thus, $U'' = U$ and consequently this process outputs the upper set of the stable pair.

As a result, given an outcome S of a GSP mechanism we can compute $L := \{i \in S \mid b_i > \xi(i, S)\}$ and use this process to find the upper set U . The time-complexity of this algorithm is polynomial in the number of players assuming that we have polynomial-time access to every $\xi^*(i, L, U)$ for all $L \subseteq U \subseteq \mathcal{A}$ and all $i \in U$. \square