Vector Balancing Games with Aging
— Extended Abstract —

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Abstract

In this article we study an extension of the vector balancing game investigated by Spencer and Olson (which corresponds to the on-line version of the discrepancy problem for matrices). We assume that decisions in earlier rounds become less and less important as the game continues.

Key words: Vector balancing games, on-line algorithms, discrepancy.

1 Introduction

A *vector balancing game* is a two-player perfect information game: Each round the first player selects a vector x from some given set $X \subseteq \mathbb{R}^d$. The second player then chooses a sign $\varepsilon \in \{-1, +1\}$ and the position vector p, initially set to zero, is changed to $p + \varepsilon x$. The first player's aim is to maximize $\|p\|$ for some given norm $\|\cdot\|$, while the second player tries to minimize this quantity. We call the value of $\|p\|$ at the end of the game the *pay-off* for the first player.

Vector balancing games can be seen as the on-line version of the discrepancy problem. Strategies for the second player correspond to on-line algorithms for the vector balancing problem, whereas strategies for the first player give lower bounds on how good an on-line algorithm can possibly be. For a deeper insight into discrepancy theory we recommend the survey of Beck and Sós [BS95].

2 Previous Results

Several forms of vector balancing games have been studied. They differ in the set of vectors available to the first player and the norm that is used to

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determine the pay-off. We mention a type that comes closest to ours:

Unit Ball Games: For a fixed norm $\|\cdot\|$ the first player may choose any vector with norm at most one, i. e. $X = \{x \in \mathbb{R}^d \mid \|x\| \leq 1\}$, and the pay-off is measured using the same norm. For the maximum norm $\|\cdot\|_\infty$, Spencer [Spe77] gave an upper bound of $\sqrt{2n\ln(2d)}$. For the d round game he proved a lower bound of $\sqrt{d\log d}(1-o(1))$ in [Spe87].

3 Our Contribution

All games investigated so far relate to on-line problems without temporal aspects. By this we mean that a decision is the same important throughout the game. In this article we assume that a decision made in the past (i. e. in an earlier round of the game) is less important than a newer one. This is represented by the different update rule $p := \frac{1}{q}p + \varepsilon x$ for some aging parameter $q > 1$. Hence the current decision is q times more important than the preceding one. We restrict ourselves to the maximum norm unit ball game, that is, the first player selects vectors from $\{x \in \mathbb{R}^d \mid \|x\|_\infty \leq 1\}$, and the pay-off is measured using $\|\cdot\|_\infty$ on p as well. We also assume $q \geq 2$.

Immediately we see that the pay-off is bounded by $\frac{q}{q-1}$. This is due to the update rule which rescales the importance of decisions in the past relative to the actual one. A different approach working with absolute values is the following: In round i the first player chooses a vector $x^{(i)}$ with norm at most q^{i-1} and the the second player updates the position vector either to $p := p + x^{(i)}$ or $p := p - x^{(i)}$. The values of an n round game then differs from our approach by a factor of q^{n-1}. Hence we lose nothing by investigating the first approach which we find more natural.

Contrary to the games described in the previous section, in our setting there are reasonable strategies such that the maximum value for $\|p\|_\infty$ does not necessarily occur after the last round. This motivates the distinction of two versions of the game: First, the value of $\|p\|_\infty$ after the last round is the pay-off for the first player, and second, the maximum value of $\|p\|_\infty$ occurred during the game is the pay-off for the first player. We call the two versions the *fixed end version* and *continuous version* respectively. The second version also refers to the case that the game is played for a fixed number of rounds which is not known to the second player.

Fixed end version: Assume that the game lasts at least $\log_2 d + 1$ rounds. Then the value of the fixed end version game is

$$\frac{q - q^{-\lfloor \log_2 d \rfloor}}{q-1}.$$

Note that the number $\frac{q-q^{-\lfloor\log_2 d\rfloor}}{q-1}$ is the maximum imbalance that can occur in a $\lfloor\log_2 d\rfloor+1$ round game by putting all vectors into the same partition class. We may thus interpret our result like this: The optimal strategy for the second player in the fixed end vector balancing game leads to a partition which is perfect apart from the last $\lfloor\log_2 d\rfloor+1$ vectors. This seems to be a more intuitive way of stating the result. Let us therefore define

$$v_q(r) := \tfrac{q-q^{-r+1}}{q-1},$$

the maximum imbalance that can occur in an r round game by putting all vectors into the same partition class. Hence the fixed end version game has value $v_q(\lfloor\log_2 d\rfloor+1)$.

The fixed end version with $q=2$ has a nice application. In [Doe00] it is used to prove that $\operatorname{lindisc}(A) \le 2\,(1-\frac{1}{2m})\operatorname{herdisc}(A)$ holds for any matrix $A \in \mathbb{R}^{m\times n}$. This improves an earlier result in this direction by Lovász et al. [LSV86] and Beck and Spencer [BS84], and is a step towards Spencer's conjecture $\operatorname{lindisc}(A) \le 2(1-\frac{1}{n+1})\operatorname{herdisc}(A)$.

Continuous version: For the continuous version we show that the first player can get a pay-off of at least $\frac{q-2q^{-\lfloor\log_2 d\rfloor-\lfloor\log_2\log_2 d\rfloor+1}}{q-1}$ (again assuming sufficiently many rounds played), while the second player has a strategy keeping $\|p\|_\infty$ below $\frac{q}{q-1} - q^{-\log_2 d-\log_2\log_2 d-4}$ throughout the game. In particular, the value v of this game satisfies

$$v_q(\lfloor\log_2 d\rfloor + \lfloor\log_2\log_2 d\rfloor - 1) \;\le\; v \;\le\; v_q(\log_2 d + \log_2\log_2 d + 5).$$

References

[BS84] J. Beck and J. Spencer. Integral approximation sequences. *Math. Programming*, 30:88–98, 1984.

[BS95] J. Beck and V. T. Sós. Discrepancy theory. In R. Graham, M. Grötschel, and L. Lovász, editors, *Handbook of Combinatorics*. 1995.

[Doe00] B. Doerr. Linear and hereditary discrepancy. *Combinatorics, Probability and Computing*, 9:349–354, 2000.

[LSV86] L. Lovász, J. Spencer, and K. Vesztergombi. Discrepancies of set-systems and matrices. *Europ. J. Combin.*, 7:151–160, 1986.

[Spe77] J. Spencer. Balancing games. *J. Combin. Theory Ser. B*, 23:68–74, 1977.

[Spe87] J. Spencer. *Ten Lectures on the Probabilistic Method.* SIAM, 1987.