Multi–Color Discrepancies

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Contents

Preface

The word ‘discrepancy’ is used to describe the deviation of a situation from the state one would like it to be. In mathematics, discrepancy theory is also called theory of irregularities of distribution. This refers to the theme of ‘classical’ discrepancy theory, namely distributing points in some space such that they are evenly distributed with respect to some (mostly geometrically defined) subsets. The discrepancy (irregularity) measures how far a given distribution deviates from an ideal one.

In combinatorial discrepancy theory the aim is to partition the vertices of a hypergraph into two classes such that this partition induces an even partition on all hyperedges. Describing the partition by a coloring, we try to color the vertices with two colors such that each hyperedge contains the same number of vertices in each color.

The central theme of this thesis is the investigation of the combinatorial discrepancy problem in arbitrary numbers of colors. This is done mostly in Chapter 2, which is entirely devoted to multi-color discrepancies, but as well in the remainder of the thesis, which touches some related problems from 2–color discrepancy theory. A first problem already is the definition of multi-color discrepancies. For two colors, identifying the colors with -1 and $+1$ is convenient. By summing up the colors occurring in a hyperedge E we obtain the imbalance $\chi(E) := \sum_{x \in E} \chi(x)$ of E with respect to the coloring χ.

For the multi-color problem this is more difficult. To express discrepancies in $c \in \mathbb{N}$ colors, we take a special set of c–dimensional vectors as colors. Thus we can describe the discrepancy of a hyperedge E with respect to a coloring χ (taking values in this vector set) by the expression $\| \sum_{x \in E} \chi(x) \|_\infty$. Via tensor products of matrices the hypergraph notion of c–color discrepancies is extended to matrices.

Having fixed this notation, we investigate a recursive approach and an extension of the ‘floating color’ method to construct low discrepancy colorings and prove upper bounds on the c–color discrepancy. The idea of recursive coloring is to use 2–color discrepancy results to iteratively partition the color classes into two until the desired number c of classes is reached. To include the case that c is not a power of 2 we need a weighted version of 2–color discrepancy. A second problem is how to organize the partitioning process. For example, to get a 17–coloring — contrary to what one would expect — it is better to start with a

1 : 16 split than to partition according to the ratio 8 : 9 in the first step. Minimizing over the possible partitioning procedures we show that the c–color discrepancy of a hypergraph is at most 2.0005 times its hereditary discrepancy.

The recursive approach is more effective if we know that induced subhypergraphs on fewer vertices have smaller discrepancy. Roughly speaking we show that if all induced subhypergraphs on n_0 vertices have discrepancy at most Cn_0^α for some $C > 0, \alpha \in]0,1[$, then the c–color discrepancy is at most $Cc_\alpha(\frac{n}{c})^\alpha$, where c_α is a constant depending on α only and n is the number of vertices of the hypergraph.

This result has several consequences. We derive a multi-color analogue of Spencer's 'six standard deviation' result [Spe85], namely that the discrepancy of hypergraphs having n vertices and n hyperedges is at most $K\sqrt{\frac{n}{c}\log c}$ for some absolute constant K. For the famous hypergraph of arithmetic progressions we show that the c–color discrepancy is at most $\mathcal{O}(c^{-0.16}n^{0.25})$ if $c \le \sqrt[4]{n}$. In consequence, the discrepancy is $\Theta(\sqrt[4]{n})$ in every fixed number of colors, extending the result of Matoušek and Spencer [MS96] to arbitrary numbers of colors. We also obtain a general bound of $\mathcal{O}(\sqrt{\frac{n}{c}\log m})$ for arbitrary hypergraph having n vertices and m edges as well as an extension of the discrepancy bounds in terms of the primal and dual shatter functions due to Matoušek [Mat95] and Matoušek, Welzl and Wernisch [MWW84].

In addition to these multi-color results the recursive approach yields 2–color results not known so far. For example, we find that the weighted discrepancy (each hyperedges shall have a given ratio $p \in]0,1[$ of its vertices in the first color class) in the 'six standard deviation' situation has an upper bound of $\mathcal{O}(\sqrt{p\log\frac{1}{p}})$. Similar results hold for the arithmetic progressions and the primal and dual shatter function bounds.

Using the tensor product representation of discrepancies of matrices, we extend a lower bound result of Lovász and Sós to the multi-color case. This gives a lower bound for the arithmetic progressions of $\frac{1}{\sqrt{c}}\sqrt[4]{n}$. The Hadamard matrices construction due to Spencer [Spe87] provides examples of hypergraphs having n vertices and n hyperedges that have c–color discrepancy $\Omega(\sqrt{\frac{n}{c}})$. Thus our upper bound results are nearly tight.

A different approach is necessary to prove a c–color analogue of the linear discrepancy version of the Beck–Fiala theorem [BF81]. In general, a recursive approach cannot give good results for the linear discrepancy. Extending the 'floating color' method of Beck and Fiala to vector colorings we show that the linear discrepancy in c colors is less than twice the degree of the hypergraph. We also derive a result for the on-line version of the discrepancy problem, namely a multi-color version of the theorem of Barany and Grunberg [BG81].

New results on 'classical' problems in 2 colors form the remainder of this thesis. In Chapter 3 we investigate the relation of linear and hereditary discrepancy. Results of Beck and Spencer [BS84a] as well as Lovász, Spencer and Vesztergombi [LSV86] show that lindisc($\mathcal{H}$) ≤ 2 herdisc($\mathcal{H}$) holds for any hypergraph. This bound is known not to be sharp. In [Spe87], Spencer improved the constant to $2(1 - 2^{-2^n})$, where n denotes the number of vertices. Spencer conjectures that lindisc($\mathcal{H}$) $\leq 2(1 - \frac{1}{n+1})$ herdisc($\mathcal{H}$) should be true. Since that time the problem is open without further progress. We use a game theoretic approach and derive lindisc($\mathcal{H}$) $\leq 2(1 - \frac{1}{2m})$ herdisc($\mathcal{H}$), where m denotes the number of hyperedges. Interestingly, this game represents a particular on-line discrepancy problem. We investigate it therefore in detail in Chapter 7.

The linear discrepancy so far was not completely understood even for totally unimodular matrices. This is surprising, as the hereditary discrepancy problem for totally unimodular matrices has been solved a long time ago. Already in 1962, Ghouila-Houri [GH62] showed that totally unimodular matrices have hereditary discrepancy 1 (except, of course, matrices with zero entries only). Only for some special classes of totally unimodular matrices an upper bound of lindisc(A) $\leq 2(1 - \frac{1}{n+1})$ was known. Already for strongly unimodular matrices a recent result [PY00] just shows $2(1 - 3^{\frac{n+1}{2}})$.

We solve the problem in Chapter 4 and show that lindisc(A) $\leq 2(1 - \frac{1}{n+1})$ holds for all totally unimodular matrices. This bound is optimal as shown by an example due to Spencer [Spe87]. The key idea of the proof is to represent the linear discrepancy problem as a specific linear program and then apply the theory of linear programming in the case of totally unimodular constraint matrices. A similar idea allows to solve the lattice approximation problem for totally unimodular matrices optimally in polynomial time. Here the useful observation is that the set of approximations respecting a certain approximation error d forms an integral polyhedron. Therefore is suffices to find the smallest d such that the corresponding polyhedron is non-empty, and then find an extremal point of this polyhedron. All this can be done efficiently. Surprisingly, all these ideas fail for the linear discrepancy problem in higher number of colors. The situations seems to be different there. We show that already for three colors the linear discrepancy of a totally unimodular matrix can exceed one.

Random colorings are treated in Chapter 5. Instead of coloring the vertices independently, we use suitable dependencies. This improves the general discrepancy bound by a factor of $\sqrt{2}$ and allows to prescribe that some sets have to be colored perfectly balanced. More important is the fact that this approach allows to take into account structural information about the hypergraph. For the example of d–dimensional boxes this improves the current-best bound derived from independent randomly coloring by a factor of $2^{\frac{d}{2}}$.

In Chapter 6 we investigate the discrepancy problem in higher dimensions, that is, of direct products of hypergraphs. Petra Wehr [Weh97] showed that the discrepancy of d–dimensional arithmetic progressions in $[n]^d$ is $\Omega(n^{\frac{d}{4}})$. She also proved that this bound is sharp up to a polylogarithmic factor. We are able to remove this factor from the upper bound. This shows a discrepancy of $\Theta(n^{\frac{d}{4}})$, where the implicit constant depends on d only. We also show an analogous multi-color version, where the implicit constants also depend on the number of colors.

In general though the discrepancy of a direct product is not the product of the discrepancies of its factor. There are examples of hypergraphs having non-zero discrepancy such that their product has discrepancy zero. Already for seemingly simple examples like the hypergraph of 2–dimensional boxes in $[n]^2$, which is the two-fold direct product of the complete hypergraph $([n], 2^{[n]})$, the discrepancy is hard to determine. We finally solve this problem by showing that this discrepancy is $\Theta(n^{\frac{3}{2}})$.

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Chapter 1

Introduction to Discrepancy Theory

1.1 Discrepancy Theory and its Applications

Discrepancy theory deals with questions of the kind “How far does an optimal solution of a problem deviate from an ideal solution?” There are several problems of this type:

- Geometric discrepancy: Distribute n points in the d–dimensional unit cube $[0,1]^d$ such that the proportion of the points contained in each rectangle $\prod_{i=1}^{d}[a_i, b_i]$ (roughly) equals the volume of the rectangle. Instead of rectangles also other sets like triangles and circles have been investigated.
- Discrepancy of sequences: Construct sequences $x : \mathbb{N} \to [0,1]^d$ such that all subsequences $x_1, \dots, x_n$ are nicely distributed in the above sense.
- Combinatorial discrepancy: Color the vertices of a hypergraph using two colors in such a way that all hyperedges have roughly the same number of vertices in each color.

All three problems are connected to each other. Some of the geometric problems can be solved by investigating related combinatorial problems. The first chapter of Matoušek's excellent book [Mat99] on geometric discrepancy gives the details. There is also a strong connection between geometric discrepancies in d dimensions and discrepancies of sequences in $d-1$ dimensions. We will explain this after giving the precise definitions of these discrepancy notions shortly.

There are applications of discrepancy theory in several areas of pure and applied mathematics as well as computer science. One of the most striking ones is the connection to numerical integration, especially in higher dimensions.

Let $d \in \mathbb{N}$ denote the dimension we are in, and let $f : [0,1]^d \to \mathbb{R}$ be a function that has to be integrated. One way to do so is to approximate the integral $\int_{[0,1]^d} f(x)dx$ by the arithmetic mean $\frac{1}{|P|}\sum_{x\in P} f(x)$ for some finite $P \subset [0,1]^d$. The approximation error can be bounded in terms of the geometric discrepancy.

Write $D(P, \mathcal{C}_d)$ for the geometric discrepancy of P with respect to corners in $\mathbb{R}^d$. Formally: For $b \in [0,1]^d$ we call the set $C_b := \prod_{i\in[d]}[0, b_i]$ the corresponding corner. Clearly, its volume is $\operatorname{vol}(C_b) = \prod_{i\in[d]} b_i$. Denote by $\mathcal{C}_d$ the set of all corners. Then the discrepancy of P with respect $\mathcal{C}_d$ is

$$D(P, \mathcal{C}_d) := \max_{C\in\mathcal{C}_d} \left||P \cap C| - |P| \operatorname{vol}(C)\right|.$$

For the approximation error $\left|\int_{[0,1]^d} f(x)dx - \frac{1}{|P|}\sum_{x\in P} f(x)\right|$ the Koksma–Hlawka inequality [Kok43, Hla61] states

$$\left|\int_{[0,1]^d} f(x)dx - \frac{1}{|P|}\sum_{x\in P} f(x)\right| \leq \frac{1}{|P|} D(P, \mathcal{C}_d) V(f),$$

where $V(f)$ is the so-called variation in the sense of Hardy and Krause, which we will not define here. For our purpose it is enough to see that the integration error is proportional to a geometric discrepancy of the point set used and a constant depending on the function f only. Hence sets of small geometric discrepancy are useful for numeric integration.

It remains to state that there actually are sets such that $D(P, \mathcal{C}_d)$ is small. An early construction of n–point sets P_n in $[0,1]^2$ having discrepancy $D(P_n, \mathcal{C}_2) = \mathcal{O}(\log(n))$ can be found in van der Corput [vdC35a, vdC35b]. Van Aardenne-Ehrenfest [vAE45, vAE49] was the first to prove that a constant bound does not exist. Schmidt [Sch72] proved that Van der Corput's construction is optimal apart from multiplicative constants. For arbitrary dimension d Halton [Hal60] and Hammersley [Ham60] constructed n–point sets P_n in $[0,1]^d$ having discrepancy $D(P_n, \mathcal{C}_d) = \mathcal{O}(\log(n)^{d-1})$. In this general setting however a tight lower bound is missing so far. Apart from tiny improvements, Roth's [Rot54] lower bound for the L_2 discrepancy of $\Omega(log(n)^{\frac{d-1}{2}}$ (which is also a lower bound for $D(P, \mathcal{C}_d)$) is still the best result available.

Sometimes low discrepancy sequences are the preferred tool for numerical integration. For a sequence $x : \mathbb{N} \to [0,1]^d$ we define its discrepancy function by

$$D(x, \mathcal{C}_d, n) := D(\{x_1, \ldots, x_n\}, \mathcal{C}_d).$$

We call x uniformly distributed, if $\lim_{n\to\infty} \frac{1}{n} D(x, \mathcal{C}_d, n) = 0$. From the above inequality we see that in this case the sequence $\frac{1}{n}\sum_{i\in[n]} f(x_i)$ converges to $\int_{[0,1]^d} f(x)dx$, where the rate of convergence depends on the discrepancy function of x.

There is strong connection between low discrepancy sequences and sets. There are constants c_1, c_2 such that the following holds:

- Every finite set $P \in [0,1]^d$ yields a sequence x in $[0,1]^{d-1}$ such that $D(x, \mathcal{C}_{d-1}, n) \leq c_1 D(P, \mathcal{C}_d)$ for all $n \leq |P|$.

- For every sequence x in $[0,1]^d$ and $n \in \mathbb{N}$ there is an n–point set $P \in [0,1]^{d+1}$ such that $D(P, C_{d+1}) \leq c_2 \max_{k \in [n]} D(x, \mathcal{C}_d, k)$.

Another field where discrepancy results have been applied successfully is computer science. The book [Cha00] is a good reference for this area. Chazelle [Cha94] showed a connection between lower bounds for geometric discrepancies and lower bounds for the computational complexity of a database problem (*range searching*). ε–approximations (which also have a strong connection with combinatorial discrepancies) have been used by Matoušek and several others to derandomize computational geometry algorithms. See the survey [Mat96b] for more information on this.

Discrepancy is also connected to communication complexity. Recall that the task in communication complexity is to design a protocol which allows a given function $f : [m] \times [n] \to \{-1, 1\}$ to be evaluated on a pair (x, y) by two players each holding just one of the two parts x, y of the input.[1] The quality of such a protocol is determined by the number of bits the two players have to exchange in the worst-case. The quality of an optimal protocol is called *communication complexity* of the function f. Viewing f as a matrix, Srinivasan [Sri97] detected that lower bounds on the combinatorial discrepancy of f are implied by lower bounds on its communication complexity: If the communication complexity of f is d, then

$$\operatorname{disc}(f) = \mathcal{O}(\min\{2^d, \sqrt{2^d \log(\max\{2, m2^{-d}\})}\}).$$

On the other hand, if we define a hypergraph $\mathcal{H}$ on $[m] \times [n]$ by taking all rectangles as hyperedges, then upper bounds on $\operatorname{disc}(\mathcal{H}, f)$ imply lower bounds on the communication complexity of f. We have

$$d \geq \log_2 \left(\frac{nm}{\operatorname{disc}(\mathcal{H}, f)} \right).$$

A similar result holds for some randomized communication complexity notion. See the comprehensive book [KN97].

Yet another connection exists between the notion of linear discrepancy and *approximate solutions of integer linear programs.*

[1] The sets $[m], [n], \{-1, 1\}$ are of course completely irrelevant, it is only their sizes that are important. Usually, as we are in computer science, one therefore takes $\{0, 1\}$ as the range of f.

1.2 Combinatorial Discrepancy Theory

1.2.1 Discrepancy of Hypergraphs and Matrices

In this thesis we restrict ourselves to combinatorial discrepancies. The objective is to partition the vertices of a hypergraph into two classes in such a way that all hyperedges are split into roughly equal parts. An ideal solution would be one where every hyperedge has the same number of points in one class as in the other. To be more precise:

Let $\mathcal{H} = (X, \mathcal{E})$ denote a finite[2] *hypergraph*, i. e. X is a finite set (of *vertices*) and $\mathcal{E}$ is a family of subsets of X (called *hyperedges* or *edges* for short). Let us agree for the rest of this work that without further notice the number of vertices shall be denoted by n and the number of edges by m.

A partition into two classes can be represented by a *coloring* $\chi : X \to \{-1, +1\}$. We call -1 and $+1$ *colors*. The color-classes $\chi^{-1}(-1)$ and $\chi^{-1}(+1)$ form the corresponding partition. For a hyperedge $E \in \mathcal{E}$ set $\chi(E) := \sum_{x \in E} \chi(x)$. The *discrepancy* of $\mathcal{H}$ with respect to χ is defined by

$$\operatorname{disc}(\mathcal{H}, \chi) = \max_{E \in \mathcal{E}} |\chi(E)|$$

and the discrepancy of $\mathcal{H}$ by

$$\operatorname{disc}(\mathcal{H}) = \min_{\chi : X \to \{-1,+1\}} \operatorname{disc}(\mathcal{H}, \chi).$$

We note that this discrepancy does not measure exactly the deviation of the optimal solution from the ideal one but gives twice the value. The reason is simple: This way all numbers occurring are integers.

To get some intuition for this concept let us have a look at two extreme cases: If all edges of $\mathcal{H}$ intersect trivially (i. e. $E_1 \cap E_2 = \emptyset$ for any two distinct edges E_1, E_2), the discrepancy is zero, if all edges are even, and one, if there is an odd cardinality edge. We may simply partition the edges one by one. The other extreme is marked by the complete hypergraph $(X, 2^X)$. In this case the discrepancy is $\lceil \frac{1}{2}|X| \rceil$. Any partition will contain a class of at least this size, and this set is also an edge. We note that discrepancy somehow measures how chaotic the hyperedges of $\mathcal{H}$ intersect.

It seems to be very difficult to connect the discrepancy to a single parameter of the hypergraph. Here are two examples. Set $n = 4k, k \in \mathbb{N}$ and $\mathcal{H}_n = ([n], \{E \subseteq [n] \mid |E \cap [2k]| = |E \setminus [2k]|\})$. Now $\mathcal{H}_n$ has more than $\binom{\frac{n}{2}}{\frac{n}{4}}^2 = \Theta(\frac{1}{n}2^n)$ edges and discrepancy zero. On

[2]All hypergraphs considered in this work will be finite, hence this assumption. For the definition of discrepancy only the finiteness of the hyperedges is required. There are some results on discrepancies of infinite hypergraphs, e. g. [BS84b]

the other hand there are hypergraphs having n edges only and discrepancy $\Theta(\sqrt{n})$, cf. Theorem 2.31.

Above we saw that if each two edges intersect trivially, then the discrepancy is zero. If we loosen this constraint just minimally to allow each to edges to intersect in at most one single vertex, then the situation is completely different: Finite projective plains have discrepancy $\Theta(n^{\frac{1}{4}})$ (lower bound: [BS95], upper bound: [Mat95]).

These were two results showing that discrepancy behaves different from some hypergraph parameters. This seems to be a general phenomenon. Therefore one usually has to determine lower and upper bounds using different aspects of the hypergraph under consideration.

Having seen some counterexamples, let us mention a few positive results. Investigating a random coloring, Alon and Spencer [AS00] show

Theorem 1.1. *For any hypergraph $\mathcal{H} = (X, \mathcal{E})$ we have $\operatorname{disc}(\mathcal{H}) \leq \sqrt{2s\ln(2m)}$, where $s := \max\{|E| : E \in \mathcal{E}\}$. A random coloring obtained by independently choosing a color with equal probability for each vertex has discrepancy at most $\sqrt{2s\ln(4m)}$ with probability at least $\frac{1}{2}$.*

Using a different random experiment we improve the constants in the general case and show an upper bound of $\sqrt{n\ln(2m)}$ in Chapter 5. A much more sophisticated approach using the entropy function was necessary to prove

Theorem 1.2 (Spencer [Spe85]). *For any hypergraph $\mathcal{H}$ such that $m \geq n$ we have* $\operatorname{disc}(\mathcal{H}) = \mathcal{O}\left(\sqrt{n\log\left(\frac{m}{n}\right)}\right)$.

Of course this particularly interesting for $m = \mathcal{O}(n)$. In the case $m = n$, $\operatorname{disc}(\mathcal{H}) \leq 6\sqrt{n}$ can be shown for n large enough. Therefore, this result is usually known to as 'Six Standard Deviations Suffice'. It is considered to be one of the milestones of discrepancy theory. The entropy method has seen numerous other applications, e. g. in the proof of the tight upper bound for the arithmetic progressions of Matoušek and Spencer [MS96] or the upper bound in terms of the primal shatter function due to Matoušek [Mat95]. Further references are Srinivasan [Sri97], Matoušek [Mat96a] and Matoušek [Mat98].

Another beautiful result is due to Beck and Fiala [BF81]. They bound the discrepancy by the maximum degree of $\mathcal{H}$:

Theorem 1.3. *Assume that each vertex in contained in at most t edges. Then*

$$\operatorname{disc}(\mathcal{H}) < 2t.$$

It is a famous open problem whether this bound can be improved or not. Beck and Fiala conjectured that $\operatorname{disc}(\mathcal{H}) = \mathcal{O}(\sqrt{t})$, but little progress has been made so far in this direction. Bednarchak and Helm [BH97] and Helm [Hel99] improved the Beck–Fiala bound in tiny steps to $\mathrm{disc}(\mathcal{H}) < 2t - 3$ (for a slightly restricted situation). A corollary of Beck's paper [Bec81] — the first time the notion of discrepancy explicitly appeared — shows $\mathrm{disc}(\mathcal{H}) \leq C\sqrt{t \log m} \log n$ for some constant C. The latest improvement in this direction is due to Banaszczyk [Ban98]: $\mathrm{disc}(\mathcal{H}) = \mathcal{O}(\sqrt{t} \log n)$.

The Beck–Fiala conjecture is by far not the only famous open problem in discrepancy theory. There are many more. Most of them are easy to state, but surprisingly hard. Let us mention a few:

Three Permutation Conjecture: Let $X = [n]$. For a permutation $\sigma \in S_n$ set $\mathcal{E}_\sigma := \{\sigma(I) \mid \exists a, b \in [n] : I = [a, b] \cap \mathbb{N}\}$. Is it true that for all $n \in \mathbb{N}$ and any three permutations $\sigma_1, \sigma_2, \sigma_3 \in S_n$ the discrepancy of $([n], \mathcal{E}_{\sigma_1} \cup \mathcal{E}_{\sigma_2} \cup \mathcal{E}_{\sigma_3})$ is bounded by an absolute constant?

This is clear for two permutations, where an upper bound of two can be shown easily. For any number l of permutations, Bohus [Boh90] gave an upper bound of $\mathcal{O}(l \log n)$. This was improved by A. Srinivasan [Sri97] to $\mathcal{O}(\sqrt{l} \log n)$.

Arithmetic Progressions with first term 0: The hypergraph of arithmetic progressions with first term 0 is $([n-1] \cup \{0\}, \{d\mathbb{N}_0 \cap [0, k] \mid d, k \in [n-1] \cup \{0\}\}$. Is it true that there is no common upper bound for all these hypergraphs? This conjecture is due to Erdős and worth over \$ 500!

Update Problem: Given a hypergraph such that all induced subhypergraphs have discrepancy at most 1. Assume we add a single further hyperedge. Can the discrepancy be bounded by a constant (independent of n)? See the next subsection for more on the update problem.

The concept of discrepancy may be generalized to matrices in a natural way. Let $A = (a_{ij})$ be any $m \times n$–matrix and set

$$\mathrm{disc}(A) := \min_{\chi \in \{-1,1\}^n} \|A\chi\|_\infty .$$

Let $X = \{x_1, \ldots, x_n\}$, $\mathcal{E} = \{E_1, \ldots, E_m\}$ and define a matrix $A = (a_{ij})$ by $a_{ij} = 1$ if $x_j \in E_i$ and $a_{ij} = 0$ else. A is called the incidence matrix of $\mathcal{H}$.[3] We have $\mathrm{disc}(A) = \mathrm{disc}(\mathcal{H})$, so discrepancy of matrices is a more general concept. If of course we restrict ourselves to $0, 1$ matrices both concepts are equivalent. Sometimes even for hypergraphs the matrix notion is more convenient, e. g. to prove the Beck–Fiala theorem.

[3]Of course A depends on the enumeration of the vertices and edges. A purist might prefer to say 'an' incidence matrix and remark that all incidence matices of a given hypergraph hav ethe same discrepancy.

1.2.2 Hereditary Discrepancy

For any set of vertices $X_0 \subseteq X$ we call $\mathcal{H}_{|X_0} := (X_0, \{E \cap X_0 | E \in \mathcal{E}\})$ an induced subhypergraph of $\mathcal{H}$. We will write $\mathcal{H}_0 \leq \mathcal{H}$ if $\mathcal{H}_0$ is an induced subgraph of $\mathcal{H}$. It is easy to see that there is little correlation between the discrepancy of a hypergraph and its induced subhypergraphs. In particular the discrepancy of an induced subhypergraph can be large even though $\text{disc}(\mathcal{H}) = 0$.[4] Looking at this from the opposite direction we see that any hypergraph can be transformed to a zero discrepancy hypergraph by adding at most n more vertices (namely by 'doubling' each vertex). For these reasons it makes sense to define the *hereditary discrepancy* by

$$\text{herdisc}(\mathcal{H}) := \max_{\mathcal{H}_0 \leq \mathcal{H}} \text{disc}(\mathcal{H}_0).$$

Two more reasons justify this notion. Firstly, many results are hereditary. All theorems from the preceeding subsection also hold with $\text{disc}(\mathcal{H})$ replaced by $\text{herdisc}(\mathcal{H})$. Secondly, hte hereditary discrepancy is a very powerful tool. The recursive method presented in section 2.4 heavily relies on hereditary assumptions. Theorem 1.4 is another example.

Let us consider the *update–version* of the discrepancy problem again: Assume we know the discrepancy of a given hypergraph — what can we say about the discrepancy of the hypergraph that results from adding one more edge? Nothing. There are hypergraphs having discrepancy zero, but if you add an appropriate further edge, the discrepancy increases to a constant fraction of n.[5]

For the hereditary discrepancy things are different. It is another well-known open problem whether the hereditary discrepancy of the hypergraph with an additional edge can be bounded by a function of $\text{herdisc}(\mathcal{H})$ (independent of n). Matoušek (private communication 2000) very recently showed an upper bound of $\mathcal{O}(\log n \, \text{herdisc}(\mathcal{H}))$, but beyond this nothing is known. Nevertheless, this example shows that for the hereditary discrepancy a rather strong result holds, whereas nothing can be said for the ordinary discrepancy.

For matrices we write $A_0 \leq A$ to indicate that the matrix A_0 consists of some columns of the matrix A. Then $\text{herdisc}(A) := \max_{A_0 \leq A} \text{disc}(A_0)$ is the natural matrix extension of the notion of hereditary discrepancy for hypergraphs.

1.2.3 Linear Discrepancy

There is a third notion of discrepancy: The *linear discrepancy* refers to the problem where each vertex has a weight assigned to it describing the ratio it should in average belong to

[4] $\mathcal{H}_n$ from the preceeding subsection is such an example: We have $\text{disc}(\mathcal{H}_n) = 0$ and $\text{disc}((\mathcal{H}_n)_{|[\frac{n}{2}]} = \frac{n}{4}$.

[5] Consider e. g. the hypergraph $\mathcal{H}_{4k}$ from the last subsection again. Add the edge $[2k]$. Let χ be any coloring. If $\chi^{-1}(1) \cap [2k]$ or $\chi^{-1}(2) \cap [2k]$ contain less than $\frac{k}{2}$ points, then $|\chi([2k])| > k$. Otherwise there is an edge having k points all in the same color. Hence $\text{disc}(\mathcal{H} \cup \{[2k]\}) \geq k$.

the partition classes induced on the edges. The natural notion for this would be

$$\operatorname{lindisc}_{01}(\mathcal{H}) := \max_{p:X\to[0,1]} \min_{\chi:X\to\{0,1\}} \max_{E\in\mathcal{E}} |(p-\chi)(E)|\,.$$

To keep consistent with the above notion we usually prefer

$$\operatorname{lindisc}(\mathcal{H}) := \max_{p:X\to[-1,1]} \min_{\chi:X\to\{-1,1\}} \max_{E\in\mathcal{E}} |(p-\chi)(E)|\,.$$

Both notions are used by different authors. Obviously they just differ by a constant factor of 2. For all hypergraphs $\mathcal{H}$ we have

$$\operatorname{lindisc}(\mathcal{H}) = 2\operatorname{lindisc}_{01}(\mathcal{H}).$$

For matrices we set

$$\operatorname{lindisc}(A) := \max_{p\in[-1,1]^n} \min_{\chi\in\{-1,1\}^n} \|A(p-\chi)\|_\infty.$$

The $0,1$ version $\operatorname{lindisc}_{01}(A) := \max_{p\in[0,1]^n} \min_{\chi\in\{0,1\}^n} \|A(p-\chi)\|_\infty$ of this notion has a close connection to integer linear programming problems. One approach to solve these problems is to solve their linear relaxation and then try to round the solution to an integer one. This might of course lead to a violation of the constraints, therefore these solutions are called approximate solutions. An approach like this is feasable if slight violations in the constraints are tolerable due to the nature of the problem or due to the fact that the constraints already have some error inflicted. See Chapter 14 of [Chv83] for more on this topic. The linear discrepancy now measures to what extent the constraints have to be violated even by the optimal rounding. Formally, we have

Remark. *Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ such that the linear system $Ax = b$ has a solution $x \in \mathbb{R}^n$. Let $p \in [0,1]^n$ such that $p - x \in \mathbb{Z}^n$. Then there is a $z \in \mathbb{Z}^n$ such that $\|x - z\|_\infty \le 1$ and $\|Az - b\|_\infty \le \operatorname{lindisc}_{01}(A,p) := \min_{\chi\in\{0,1\}^n} \|A(p-\chi)\|_\infty$.*

This problem of finding an integral vector z such that $\|Az - b\|_\infty$ is small is also called *lattice approximation problem.* The reason is obvious: The set $A\mathbb{Z}^n = \{Az | z \in \mathbb{Z}^n\}$ forms lattice, and we are looking for a lattice point that is closest (with respect to $\|\cdot\|_\infty$) to b. See Raghavan [Rag88] for more on this aspect, and [SS96, SS93] for natural generalizations to weighted and quadratic versions.

Looking at the interrelation between the different discrepancy notions we immediately note $\operatorname{disc}(\mathcal{H}) \le \operatorname{herdisc}(\mathcal{H})$ and $\operatorname{disc}(\mathcal{H}) \le \operatorname{lindisc}(\mathcal{H})$. The relation between $\operatorname{lindisc}(\mathcal{H})$ and $\operatorname{herdisc}(\mathcal{H})$ is less obvious. An extremely useful result due to Beck and Spencer [BS84a] and Lovász, Spencer and Vesztergombi [LSV86] shows

Theorem 1.4. *For any $A \in \mathbb{R}^{m\times n}$ we have* $\operatorname{lindisc}(A) \le 2\operatorname{herdisc}(A)$.

Moreover this result is constructive in the following sense: A 2–coloring χ such that $\|A(p-\chi)\|_\infty \leq 2h + \varepsilon\|A\|_\infty$ holds can be computed by $\mathcal{O}(\log \varepsilon^{-1})$ times computing a coloring having discrepancy at most h for some induced subhypergraph. This result will be crucial for the recursive method for multi-colorings in Chapter 2. In Chapter 3 we will investigate the relation between linear and hereditary discrepancy in more detail and improve the bound of Theorem 1.4. This is constructive again. In Chapter 4, we determine the linear discrepancy of totally unimodular matrices, and thus prove Spencer's conjecture $\operatorname{lindisc}(A) \leq 2(1 - \frac{1}{n+1}) \operatorname{herdisc}(A)$ in this case.

There are some further discrepancy notions, which we discuss the in the following chapters. As a general reference we would like to recommend the chapter of Beck and Sós [BS95] and the fourth chapter of Matoušek's book [Mat99].

1.2.4 Algorithmic Aspects

The discrepancy problem is known to be notoriously hard. It is even NP–hard to decide whether an given hypergraph has discrepancy 0 or not. Notions like approximation factors do not make sense for the discrepancy problem. The situation resembles to the one in graph coloring. The graph coloring problem is to find a vertex coloring such that no edge is monochromatic, or in other words, no two adjacent vertices receive the same color. Even for 3–colorable graphs, the best known efficient algorithm ([BK97], following work of [Wig83], [KMS98] and [Blu94]) uses up to $\widetilde{\mathcal{O}}(n^{3/14})$ colors. On the other hand, the best known inapproximability result [KLS00] only shows that it is NP–complete to approximate the chromatic number within a factor of $\frac{4}{3}$.

For the general case the probabilistic method of Theorem 1.1 or Chapter 5 is the only result available. An approach via semi-definite programming was investigated in [Leh99], but the experimental results were not significantly different from the probabilistic method. For a derandomization of these probabilistic results we refer to Srivastav [Sri95, Sri01].

Fortunately in many special situations, e. g. the one of the Beck–Fiala theorem 1.3, the proofs of the best known discrepancy bound can be transformed into a polynomial-time algorithm.

1.2.5 Ramsey Theory and Property B

There is a second problem concerned with coloring hypergraphs: A hypergraph $\mathcal{H} = (X, \mathcal{E})$ is said to have *Property B* (or to be 2–*colorable*) if there is a 2–coloring $\chi : X \to \{-1, 1\}$ such that no hyperedge is monochromatic.

V. Sós (at the workshop "Discrepancy Theory and its Applications" in Kiel 1998) called this the Ramsey-type aspect of the hypergraph balancing problem. The connection to

Ramsey-theory is straight-forward: Recall (e. g. from [GRS90]) that the diagonal Ramsey number R_k is defined to be the minimum n such that any 2–coloring of the edges of K_n (the complete graph on n vertices) produces a monochromatic copy of K_k. Write $\binom{X}{n}$ to denote the set of n–element subsets of a set X. Then $\left(\binom{[n]}{2}, \{\binom{T}{2} | T \in \binom{[n]}{k}\}\right)$ has property B if and only if $n < R_k$.

The connections to discrepancy theory, 'the other side of the coin' (Sós), are vague. Trivially, an r–uniform hypergraph (i. e. all hyperedges have size r) has discrepancy less than r if and only if it has property B. Hence in this situation discrepancy is a stronger concept. For hypergraphs having edges of different sizes, Ramsey-type results and discrepancy results cannot be translated into another that well.

To show that Ramsey theory and discrepancy are different aspects of the same problem, let us consider the hypergraph of arithmetic progressions in $[n]$. A probabilistic argument (also to be found in [GRS90]) shows that if $n \leq \frac{2^k}{2ek}(1 + o(1))$, then there is a 2–coloring such that no arithmetic progressions of length k is monochromatic. On the other hand, already for $n \geq k^2$ any 2–coloring of $[n]$ has discrepancy at least $\Omega(\sqrt{k})$ with respect to the arithmetic progressions of length at most k. This was shown by Roth [Rot64].

One last remark which nicely leads to the next chapter: Ramsey theory does not only deal with 2–colorability, but as well with the problem of coloring with any number of colors such that no hyperedge becomes monochromatic. Already Ramsey's theorem [Ram30], van der Waerden's theorem [vdW27] and Schur's theorem [Sch16] — all three proved at least seventy years ago — deal with the partitioning problem into arbitrary numbers of classes. In this context it is surprising that discrepancy theory so far restricted itself to 2 colors only. In the next chapter we try to change this.

Chapter 2

Multi–Color Discrepancies

In this chapter we introduce the notion of multi-color discrepancies. We present two methods that will extend several classical 2–color results to arbitrary numbers of colors. It turns out that there is no general rule concerning the discrepancies of a given hypergraph in different numbers of colors. In fact, there are extreme hypergraphs having zero discrepancy in one number of colors and high discrepancy in almost all other numbers of colors. For many classical examples though the situation is much nicer. For hypergraph having n vertices and n hyperedges, the c–color discrepancy is $\mathcal{O}(\sqrt{\frac{n}{c}\log c})$. This bound is almost sharp. There are hypergraphs of this type showing a lower bound of $\Omega(\sqrt{\frac{n}{c}})$.

Up to now the c–partitioning problem was not investigated very much. A special case occurs in a paper concerning communication complexity [BHK98]. Apart from that we found two results.

Theorem 2.1 ([BS95]). *Let $\mathcal{H} = (X, \mathcal{E})$ be any hypergraph such that the incidence matrix of $\mathcal{H}$ is totally unimodular. Then for any number $c \in \mathbb{N}$ there is a c–partition $X = X_1 \dot\cup \ldots \dot\cup X_c$ of the vertex set such that for any edge $E \in \mathcal{E}$ and $i \in [c]$*

$$\left| |E \cap X_i| - \frac{|E|}{c} \right| < 1.$$

In the notation introduced shortly, this means that these hypergraphs have c–color discrepancy less than one. A second result on multi-color discrepancies is hidden in the paper by Beck and Fiala [BF81] as will be pointed out in Section 2.3, where a similar result will be proven.

2.1 Definition of Multi-Color Discrepancies

Throughout this chapter let $c \in \mathbb{N}$ denote the number of classes we want to partition the vertices of $\mathcal{H}$ into. It is natural to realize the partition by a coloring: A c–coloring of $\mathcal{H} = (X, \mathcal{E})$ is simply a mapping $\chi : X \to M$, where M is any set of cardinality c. For convenience, normally one takes $M = [c] := \{1, \dots, c\}$, though sometimes a different set M will be of advantage. In an application to communication complexity [BHK98] choosing M as a finite abelian group proved to be effective. The basic idea of measuring the deviation from the ideal motivates the definitions of the *discrepancy of an edge $E \in \mathcal{E}$ in color $i \in M$ with respect to χ* by

$$\operatorname{disc}_{\chi,i}(E) := \left| |\chi^{-1}(i) \cap E| - \frac{|E|}{c} \right|,$$

the *discrepancy of $\mathcal{H}$ with respect to χ* by

$$\operatorname{disc}(\mathcal{H}, \chi) := \max_{i \in M, E \in \mathcal{E}} \operatorname{disc}_{\chi,i}(E)$$

and the *discrepancy of $\mathcal{H}$ in c colors* by

$$\operatorname{disc}(\mathcal{H}, c) := \min_{\chi : X \to [c]} \operatorname{disc}(\mathcal{H}, \chi).$$

Immediately we see

Remark 2.2.

$$\operatorname{disc}(\mathcal{H}, 2) = \tfrac{1}{2} \operatorname{disc}(\mathcal{H}).$$

This seems a little unsatisfactory at first, but we have good reason for this. Of course we could multiply all occurring numbers with c. Both notions would coincide then, and all numbers would be integers. The disadvantage is that we cannot compare discrepancies in different numbers of colors anymore. To do this properly we need to stick more closely to the "deviation of the optimum from the ideal" concept.

In this notion Theorem 2.1 states that totally unimodular hypergraphs have discrepancy less than one in any number of colors. To get some more intuition and comparison let us examine two examples: We determine the multi-color discrepancies of the complete hypergraph, which is of course the worst case. In a second lemma we show that the 4–color discrepancy of a perfectly 2–balanced hypergraph can well be a constant fraction of the number of vertices, i. e. relatively large. In particular, this shows that a hypergraph may have very different discrepancies in different numbers of colors. This phenomenon is investigated in more detail in Section 2.7.

Lemma 2.3. *Let $n \in \mathbb{N}$. The complete hypergraph $\mathcal{H} = ([n], 2^{[n]})$ on n vertices has c–color discrepancy*

$$\operatorname{disc}(\mathcal{H}) = \left(1 - \tfrac{1}{c}\right) \left\lceil \tfrac{n}{c} \right\rceil .$$

Proof. Let $\chi : [n] \to [c]$ be any coloring. Then there is a color $j \in [c]$ such that $E := \chi^{-1}(j)$ has at least $\lceil \frac{n}{c} \rceil$ points. As E is an edge of $\mathcal{H}$, we have

$$\operatorname{disc}(\mathcal{H}, c) \geq \left| |E \cap \chi^{-1}(j)| - \tfrac{1}{c}|E| \right| = \left(1 - \tfrac{1}{c}\right) |E| \geq \left(1 - \tfrac{1}{c}\right) \left\lceil \tfrac{n}{c} \right\rceil .$$

For the upper bound let χ be a c–coloring such that all color classes deviate in size by at most 1, i. e. for all $j \in [c]$ we have $\lfloor \frac{n}{c} \rfloor \leq |\chi^{-1}(j)| \leq \lceil \frac{n}{c} \rceil$. From the definition of multi-color discrepancy it is clear that we just have to consider the two extreme cases of an edge E^+ having exactly $\lceil \frac{n}{c} \rceil$ points in one color j^+ and an edge E^- having $n - \lfloor \frac{n}{c} \rfloor$ points and avoiding one color class $\chi^{-1}(j^-)$. We compute

$$\begin{aligned} \operatorname{disc}_{\chi, j^+}(E^+) &= \left(1 - \tfrac{1}{c}\right) \left\lceil \tfrac{n}{c} \right\rceil , \\ \operatorname{disc}_{\chi, j^-}(E^-) &= \tfrac{1}{c} \left(n - \left\lfloor \tfrac{n}{c} \right\rfloor\right) . \end{aligned}$$

Hence $\operatorname{disc}(\mathcal{H}) \leq \max\{\operatorname{disc}_{\chi,j^+}(E^+), \operatorname{disc}_{\chi,j^-}(E^-)\} = \left(1 - \frac{1}{c}\right) \lceil \frac{n}{c} \rceil$. □

For the next result we invoke our favorite counterexample from Chapter 1 again.

Lemma 2.4. *Let $k \in \mathbb{N}$ and $n = 4k$. Set $\mathcal{H}_n = ([n], \{E \subseteq [n] \mid |E \cap [\frac{n}{2}]| = |E \setminus [\frac{n}{2}]|\})$. Obviously, $\mathcal{H}_n$ has discrepancy zero, but* $\operatorname{disc}(\mathcal{H}_n, 4) = \frac{1}{8}n$.

Proof. Let $\chi : [n] \to [4]$ be any 4–coloring. Let $i \in [4]$ be a color such that $|\chi^{-1}(i)| \leq \frac{1}{4}n$. Then there are sets $E_1 \subseteq [\frac{n}{2}]$, $E_2 \subseteq [n] \setminus [\frac{n}{2}]$ such that $|E_j| = \frac{1}{4}n$ and $\chi^{-1}(i) \cap E_j = \emptyset$ hold for $j = 1, 2$. Thus $E_1 \cup E_2$ is an edge in $\mathcal{H}_n$ that has discrepancy $\frac{1}{8}n$ in color i. On the other hand $\chi : x \mapsto \lceil \frac{4x}{n} \rceil$ is a coloring having discrepancy $\frac{1}{8}n$. □

Let us return to the definitions of multi-color discrepancies. In the notion introduced above we can not express discrepancies simply by sums of colors as we could in the 2–color case. As this is very practical sometimes and a step towards the notion of multi-color discrepancies of matrices, here is a solution. We describe the color $i \in [c]$ by a vector $m^{(i)} \in \mathbb{R}^c$ defined by

$$m_j^{(i)} := \begin{cases} \frac{c-1}{c} & \text{if } i = j \\ -\frac{1}{c} & \text{otherwise.} \end{cases}$$

Then for a coloring $\chi : X \to [c]$,

$$\operatorname{disc}(\mathcal{H}, \chi) = \max_{E \in \mathcal{E}} \left\| \sum_{x \in E} m^{(\chi(x))} \right\|_\infty .$$

Set $M_c := \{m^{(i)} | i \in [c]\}$. Apparently, we have

$$\operatorname{disc}(\mathcal{H}, c) := \min_{\chi : X \to M_c} \max_{E \in \mathcal{E}} \left\| \sum_{x \in E} \chi(x) \right\|_\infty .$$

As in 2 colors, the notion of multi-color discrepancy has a natural extension to matrices. Let $A \in \mathbb{R}^{m\times n}$ be any matrix and denote its columns by $a^{(1)}, \dots, a^{(n)}$. Then the c–color discrepancy of A measures how well the columns of A can be partitioned into c classes such that the entries of each row of A are evenly distributed among these classes. Formally, for a coloring $\chi : [n] \to [c]$ we define

$$\operatorname{disc}(A,\chi) = \max_{i\in[c]} \left\| \sum_{k\in\chi^{-1}(i)} a^{(k)} - \frac{1}{c}\sum_{k\in[n]} a^{(k)} \right\|_\infty$$

and its discrepancy in c colors by

$$\operatorname{disc}(A,c) = \min_{\chi:[n]\to[c]} \operatorname{disc}(A,\chi).$$

The vector colors $m^{(i)}$ enable us to express these discrepancies in a smoother way. Let $\overline{A}$ be the matrix which results from replacing every a_{ij} in A by $a_{ij}I_c$, where I_c shall denote the identity matrix of dimension c. Identifying a $\chi : [n] \to M_c$ by a cn–dimensional vector in the natural way, we get

$$\begin{aligned} \operatorname{disc}(A,\chi) &= \left\|\overline{A}\chi\right\|_\infty, \\ \operatorname{disc}(A,c) &= \min_{\chi:[n]\to M_c} \operatorname{disc}(A,\chi). \end{aligned}$$

It is easily seen that both notions coincide, and furthermore that the c–color discrepancy of a hypergraph equals the one of its incidence matrix.

We continue with the definitions of weighted, linear and hereditary discrepancy. Both as a technical tool as well as out of independent interest the notion of weighted discrepancy[1] is introduced. It refers to the partitioning problem where the vertices of each hyperedge shall not be spread evenly among the partition classes but respecting a given ratio.

A vector $p \in [0,1]^c$ such that $\|p\|_1 = \sum_{i\in[c]} p_i = 1$ is called a *weight* for c colors. Let $\chi : X \to [c]$ be a c–coloring. Then the *weighted discrepancy* of an edge $E \in \mathcal{E}$ with respect to χ, p and the color $i \in [c]$ is

$$\operatorname{wd}(E,\chi,p,i) := \left|\left|E\cap\chi^{-1}(i)\right| - p_i|E|\right|.$$

Naturally, we define

$$\begin{aligned} \operatorname{wd}(\mathcal{H},\chi,p) &:= \max_{E\in\mathcal{E},i\in[c]} \operatorname{wd}(E,\chi,p,i) \\ \operatorname{wd}(\mathcal{H},c,p) &:= \min_{\chi:X\to[c]} \operatorname{wd}(\mathcal{H},\chi,p) \\ \operatorname{wd}(\mathcal{H},c) &:= \max_{p} \operatorname{wd}(\mathcal{H},c,p). \end{aligned}$$

[1] This notion of discrepancy is called ‘diagonal discrepancy’ in [BS95]

This is easily extended to matrices. Let $A \in \mathbb{R}^{m\times n}$. For any c–dimensional vector p set $\overline{p} : [n] \to \mathbb{R}; i \mapsto p$. Denote by E_c the standard basis of $\mathbb{R}^c$ and by $\overline{E}_c$ its convex hull, which are just the c–color weights. It is now easy to see that the following definitions extend the corresponding hypergraph ones. We may use colorings $\chi : [n] \to E_c$ — recall that we are free to choose any c–element set as range of a c–coloring — and view χ as a cn–dimensional vector again. We define

$$\begin{aligned} \mathrm{wd}(A,\chi,p) &:= \|\overline{A}(\overline{p}-\chi)\|_\infty \\ \mathrm{wd}(A,c,p) &:= \min_{\chi:[n]\to E_c} \mathrm{wd}(\mathcal{H},\chi,p) \\ \mathrm{wd}(A,c) &:= \max_{p\in\overline{E}_c} \mathrm{wd}(A,c,p). \end{aligned}$$

Of course nothing changes if we replace E_c and $\overline{E}_c$ by M_c and $\overline{M}_c := \{\sum_{i\in[c]} \lambda_i m^{(i)} | \lambda \in [0,1]^c, \sum_{i\in[c]} \lambda_i = 1\}$. We then get $\mathrm{wd}(A,c) := \max_{p\in\overline{M}_c} \min_{\chi:[n]\to M_c} \left\|\overline{A}(\overline{p}-\chi)\right\|_\infty$, which is closer to the usual matrix discrepancy notion introduced above.

Following this, the *linear discrepancy* in c colors with regard to $p : [n] \to \overline{M}_c$ and in general can be defined by

$$\begin{aligned} \mathrm{lindisc}(A,c,p) &:= \min_{\chi:[n]\to M_c} \left\|\overline{A}(p-\chi)\right\|_\infty \\ \mathrm{lindisc}(A,c) &:= \max_{p:[n]\to\overline{M}_c} \mathrm{lindisc}(A,c,p). \end{aligned}$$

Finally the hereditary discrepancy in c colors is

$$\mathrm{herdisc}(A,c) := \max_{A_0\leq A} \mathrm{disc}(A_0,c). \tag{2.1}$$

The notions of linear and hereditary discrepancy shall be defined for hypergraphs as well by taking the incidence matrix of the hypergraph, e. g. for a hypergraph $\mathcal{H}$ with incidence matrix A we set $\mathrm{lindisc}(\mathcal{H},c) := \mathrm{lindisc}(A,c)$. For the hereditary discrepancy we simply have $\mathrm{herdisc}(\mathcal{H},c) = \max_{\mathcal{H}_0\leq\mathcal{H}} \mathrm{disc}(\mathcal{H},c)$. Like in Remark 2.2, these other discrepancy notions are identical with the ones of Chapter 1 up to the constant factor of 2. When citing 2–color results we will use the conventional notation which has no parameter c in it, e. g. $\mathrm{herdisc}(\mathcal{H})$, so there is no danger of confusion.

2.2 Tensor Products

As we will use the $\|\overline{A}\chi\|_\infty$ expression of discrepancy several times, let us analyze this substituting matrices into another briefly: For any two matrices $A_k \in \mathbb{C}^{m_k\times n_k}$, $k = 1, 2$ the tensor (or Kronecker) product $A_1 \otimes A_2$ is the matrix $B = (b_{ij}) \in \mathbb{C}^{m_1 m_2\times n_1 n_2}$ defined by

$$b_{(i_1-1)m_1+i_2,(j_1-1)n_1+j_2} = a_{i_1 j_1} a_{i_2 j_2}$$

for all $i_k \in [m_k]$, $j_k \in [n_k]$, $k = 1, 2$. Simply speaking, B is produced by replacing every entry a_{ij} of A_1 by $a_{ij}A_2$.

Lemma 2.5. *The following laws hold for the tensor product:*

(i) Associativity: All matrices A, B, C fulfill $(A \otimes B) \otimes C = A \otimes (B \otimes C)$.

(ii) Distributivity with +: For all matrices A, B, C such that $A + B$ is defined we have $(A + B) \otimes C = A \otimes C + B \otimes C$ and $C \otimes (A + B) = C \otimes A + C \otimes B$.

(iii) 'Mixed Product Rule': $(AB) \otimes (CD) = (A \otimes C)(B \otimes D)$ for all matrices A, B, C, D such that AB and CD are defined.

(iv) $\otimes$ is compatible with inversion: $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$ for all non-singular matrices A and B.

(v) The (complex) eigenvalues of $A \otimes B$ are exactly the products of an eigenvalue of A and one of B.

(vi) $\mathrm{rank}(A \otimes B) = \mathrm{rank}(A)\,\mathrm{rank}(B)$.

(vii) $\det(A \otimes B) = (\det A)^{n_B}(\det B)^{n_A}$ *for all matrices* $A \in \mathbb{C}^{n_A \times n_A}$ *and* $B \in \mathbb{C}^{n_B \times n_B}$.

Some patience and knowledge of linear algebra is enough to prove Lemma 2.5. Most books on multilinear algebra contain these results in one form or another in chapters concerning tensor products of linear mappings and matrices. An elementary approach can be found in [Gra81].

In the tensor product notation we get

$$\begin{aligned}
\mathrm{disc}(A, c) &= \min_{\chi:[n]\to M_c} \|(A \otimes I_c)\chi\|_\infty \\
\mathrm{lindisc}(A, c) &= \max_{p:[n]\to \overline{M}_c} \min_{\chi:[n]\to M_c} \|(A \otimes I_c)(p - \chi)\|_\infty \\
\mathrm{herdisc}(A, c) &= \max_{A_0 \leq A} \min_{\chi:[\mathrm{cols}(A_0)]\to M_c} \|(A_0 \otimes I_c)\chi\|_\infty .
\end{aligned}$$

2.3 Elementary Probabilistic Method

An elementary probabilistic approach is to consider a random coloring. We color each vertex independently with a random color. Using the so–called Chernoff–bound, we prove that with positive probability our random coloring is balanced to a certain extent. For the 2–color case, this is Theorem 1.1. We show

Theorem 2.6. *Let $\mathcal{H} = (X, \mathcal{E})$ be any hypergraph. Set $m := |\mathcal{E}|$ and $s = \max_{E \in \mathcal{E}} |E|$. Then*

$$\operatorname{disc}(\mathcal{H}, c) \leq \sqrt{\tfrac{1}{2} s \ln(2mc)}.$$

Proof. Define a random c–coloring χ by independently picking a random color uniformly distributed from $[c]$ for every vertex $x \in X$. Define random variables $X_{i,x}$ by

$$X_{i,x} := \begin{cases} \frac{c-1}{c} & \text{if } \chi(x) = i \\ -\frac{1}{c} & \text{else} \end{cases}$$

for all $x \in X$, $i \in [c]$. Set $X_{i,E} := \sum_{x \in E} X_{i,x}$ for all $E \in \mathcal{E}$, $i \in [c]$. From [AS00, Theorem A.4] we know

$$P(|X_{i,E}| > \alpha) < 2e^{-2\alpha^2/|E|}$$

for all real α.

Set $\alpha = \sqrt{\frac{1}{2} s \ln(2mc)}$. Thus we have

$$\begin{aligned} P(\operatorname{disc}(\mathcal{H}, \chi) \leq \alpha) &= P(\forall i \in [c], E \in \mathcal{E} : |X_{i,E}| \leq \alpha) \\ &= 1 - P(\exists i \in [c], E \in \mathcal{E} : |X_{i,E}| > \alpha) \\ &\geq 1 - \sum_{i \in [c], E \in \mathcal{E}} P(|X_{i,E}| > \alpha) \\ &> 1 - \sum_{i \in [c], E \in \mathcal{E}} 2e^{-2\alpha^2/|E|} \\ &\geq 1 - cm\, 2e^{-2\alpha^2/s} = 0 \end{aligned}$$

by choice of α. Hence with positive probability our random χ has discrepancy not greater than $\sqrt{\frac{1}{2} s \ln(2mc)}$, thus such a coloring exists. □

Note that with $\alpha = \sqrt{\frac{1}{2} s \ln(4mc)}$ our random coloring χ has discrepancy at most α with probability greater than $\frac{1}{2}$. This observation leads to a randomized algorithm.

We cannot derive Theorem 1.1 from our theorem. Specializing our result to 2–color discrepancy all we get is $\operatorname{disc}(\mathcal{H}) \leq \sqrt{2s \ln(4m)}$ (instead of $\operatorname{disc}(\mathcal{H}) \leq \sqrt{2s \ln(2m)}$). The reason is that in the 2–color case one needs to keep the discrepancy small with respect to just one color as both colors generate the same discrepancy.

There are several ways to improve this result. If α can be chosen significantly smaller than $\frac{n}{c}$, then a sharper bound on the tail probabilities (also to be found in [AS00]) yields an upper bound of $\mathcal{O}(\sqrt{\frac{n}{c} \ln(2mc)})$. Large hyperedges, which have a big contribution to the failure probability, can be handled by an equi-coloring approach. This is most effective for

2 colors, and therefore done in a separate chapter of this work (Chapter 5). This 2–color result plus a recursive approach are combined in Section 2.4.

The following result can be found in [BF81, Proof of Theorem 3], the famous paper by Beck and Fiala.

Theorem 2.7. *In the notation from above,* $\operatorname{disc}(\mathcal{H}, c) < (12 - \frac{4}{c-1})\sqrt{2m\log(2m)}$ *holds.*

We mention this result here mainly for the reason that it is the second of the two results we found on multi-color discrepancies. It is a little difficult to compare it with Theorem 2.6 due to the fact that the result is independent of the number of vertices. There are reductions by linear algebra, cf. [Spe87]. Since we find the situation that there are more hyperedges than vertices much more common — and then Theorem 2.6 is clearly superior — we do not want to exhibit the details here.

2.4 Recursive Coloring

For some 2–color discrepancy results the proofs seem to rely heavily on the fact that only two colors are used. One example is Spencer's $\mathcal{O}(\sqrt{n})$ bound for hypergraphs having n vertices and edges. A key step in the proof is to construct a low discrepancy partial coloring $\chi := \frac{1}{2}(\chi_1 - \chi_2)$ from two colorings χ_1, χ_2 with $\chi_1(E) \approx \chi_2(E)$ for all $E \in \mathcal{E}$. It is not clear to us how this idea can be generalized to c colors.

As the partial coloring method has been a major break-through in 2–color discrepancy theory, it is desirable to have a similar method for c colors as well. What we do in this section is not partial coloring, i. e. enlarging the partition classes by successively coloring points, but recursive 2–coloring, i. e. successively enlarging the number of partition classes. The basic idea is to find a suitable 2–coloring of X with color classes X_1, X_2 and then to iterate this process on the subhypergraphs induced by X_1 and X_2. If the weighted 2–color discrepancies of suitable induced subhypergraphs are bounded, such a recursive method can be analyzed, even if c is not a power of 2. This will lead to a generalization of the 'six standard deviation' theorem of Spencer [Spe85], the discrepancy bound of Beck–Fiala [BF81] and the bounds using the primal and dual shatter function of Matoušek, Welzl and Wernisch [MWW84] and Matoušek [Mat95].

At the end of this long section we will show the limits of the recursive approach. For example, for the linear discrepancy in c colors recursive methods fail, and we need other methods, which will be introduced in the next section.

2.4.1 The Recursive Method

The following lemma analyses a single step in the recursion. It shows that an imbalance inflicted in the first step of the recursion is evenly split up in the remainder of the partitioning process.

Lemma 2.8. *Let C be a set of colors with $c = |C|$ and let $\{C_1, C_2\}$ be a partition of C. Let p be a weight of C, i. e. $p \in [0,1]^c$ such that $\|p\|_1 = 1$. Denote by $p_{|C_j}$ the vector taking the components of p with indices corresponding to the colors in the set C_j, and set $q_j = \|p_{|C_j}\|_1$, $j \in [2]$. Let χ_0 be a 2–coloring of X, set $X_1 := \chi_0^{-1}(1)$, $X_2 := \chi_0^{-1}(-1)$. Let $\chi_j : X_j \to C_j$ be any colorings. Set $\chi := \chi_1 \cup \chi_2$. For all $E \in \mathcal{E}$, $j \in [2]$ and $i \in C_j$ the discrepancy of E with respect to the color i, the coloring χ and the weight p is*

$$\big||E \cap \chi^{-1}(i)| - p_i|E|\big| \le \tfrac{p_i}{q_j}\,||E \cap X_j| - q_j|E|| + ||E \cap X_j \cap \chi_j^{-1}(i)| - \tfrac{p_i}{q_j}|E \cap X_j||.$$

In particular

$$\mathrm{wd}(\mathcal{H}, c, p) \le \max_{j \in [2], i \in C_j} \left(\tfrac{p_i}{q_j}\,\mathrm{wd}(\mathcal{H}, 2, (q_1, q_2)) + \max_{\mathcal{H}_0 \le \mathcal{H}} \mathrm{wd}(\mathcal{H}_0, |C_j|, \tfrac{1}{q_j} p_{|C_j}) \right).$$

Proof. Let $j \in [2]$, $i \in C_j$, $E \in \mathcal{E}$. Then

$$\begin{aligned} &\big||E \cap \chi^{-1}(i)| - p_i|E|\big| \\ &= \big||E \cap X_j \cap \chi_j^{-1}(i)| - p_i|E|\big| \\ &\le \left||E \cap X_j \cap \chi_j^{-1}(i)| - \tfrac{p_i}{q_j}|E \cap X_j|\right| + \left|\tfrac{p_i}{q_j}|E \cap X_j| - p_i|E|\right|. \end{aligned}$$

If the χ_j, $j = 0,1,2$ are chosen such that $||E \cap X_j| - q_j|E|| \le \mathrm{wd}(\mathcal{H}, 2, (q_1, q_2))$ and $\left||E \cap X_j \cap \chi_j^{-1}(i)| - \frac{p_i}{q_j}|E \cap X_j|\right| \le \mathrm{wd}(\mathcal{H}_{X_j}, |C_j|, \frac{1}{q_j} p_{|C_j})$ for all $E \in \mathcal{E}$, $j \in [2]$, $i \in C_j$, then the second claim follows from the first. □

As this section is quite lengthy, here is a short overview of what is going to come. We first analyze recursive coloring assuming that we have a uniform bound on the weighted 2–color discrepancies of the induced subhypergraphs. We derive a first result for the weighted c–color discrepancy and then improve it in the case of equi-weighted discrepancy. Finally we replace the uniformity assumption with the assumption that subhypergraphs on n_0 vertices have weighted discrepancy $\mathcal{O}(n_0^\alpha)$. With this stronger precondition we get a bunch of beautiful results, among them a near tight c–color analogue of Spencer's 'six standard deviation' theorem.

2.4.2 Weighted Discrepancy

In this subsection we analyze the case that all induced subgraphs have a common bound on all weighted discrepancies in two colors. This is an important case for two reasons: Firstly, the proof of some results on two-color discrepancy provides some information about the weighted discrepancy of the induces subgraphs (e. g. in the Beck–Fiala setting). Secondly, the linear discrepancy and thus also the weighted discrepancies of all subgraphs, note $\mathrm{wd}(\mathcal{H}, 2) \leq \frac{1}{2}\mathrm{lindisc}(\mathcal{H})$, are bounded by the hereditary discrepancy: From Theorem 1.4 we get

Remark 2.9. *For all induced subhypergraphs $\mathcal{H}_0$ of $\mathcal{H}$ we have* $\mathrm{wd}(\mathcal{H}_0, 2) \leq \mathrm{herdisc}(\mathcal{H})$.

Hence a bound on the hereditary discrepancy is also sufficient to apply the main theorems of this section. Bounds on the hereditary discrepancy are often encountered in situations where the partial coloring method is used in the 2–color case — for the simple reason that the uncolored points induce a subhypergraph which has to be colored in the next iteration.

It will be convenient to represent the iterated partitioning of the set of colors C by a binary tree. We call a binary rooted tree $T = (V_T, E_T)$ a *partition tree* for C, if the following conditions are satisfied: The root of T is C, all nodes are subsets of C, all leaves are singletons of C and each two son nodes form a partition of their common father node. For every color $i \in C$ there is a unique path $C = C_0^{(i)} \supset C_1^{(i)} \supset \ldots \supset C_{k(i)}^{(i)} = \{i\}$ in the partition tree. We write $h(T)$ for the height of T, that is the length of a longest path connecting a leaf and the root.

Recall that a function $p : C \to [0,1]$ is called a weight of C if $\sum_{i \in C} p_i = 1$. For $D \subseteq C$ set $p(D) := \|p_{|D}\|_1 = \sum_{i \in D} p_i$. For a color $i \in C$ set $v(T,p,i) := \sum_{l=1}^{k(i)} \frac{p_i}{p(C_l^{(i)})}$ and $v(T,p) = \max_{i \in C} v(T,p,i)$. As the next theorem shows, these constants reflect the influence of the partition tree chosen for the recursive coloring process. In Lemma 2.11 and 2.13 we will give partition trees for which these values (and hence the resulting discrepancy) is small.

Theorem 2.10. *Let $\mathrm{wd}(\mathcal{H}_0, 2) \leq K$ for all induced subgraphs $\mathcal{H}_0$ of $\mathcal{H}$. Let C be a set of colors with $c = |C|$ and let p be a weight of C. Let $T = (V_T, E_T)$ be a partition tree of C. Then there is a coloring $\chi : X \to C$ such that for all colors $i \in C$ and all $E \in \mathcal{E}$ we have*

$$\big||E \cap \chi^{-1}(i)| - p_i|E|\big| \leq K v(T,p,i).$$

In particular, $\mathrm{wd}(\mathcal{H}, p, c) \leq K v(T,p)$.

Proof. We use induction on the height $h(T)$ of T. For $h(T) = 0$ we have just one color and both sides of the inequality become zero. Let T be of height $h(T) > 0$ and assume that the theorem is true for all partition tree of height strictly less than $h(T)$. Let C_1 and

C_2 be the sons of C in T. Set $q_j := p(C_j) = \sum_{k \in C_j} p_k$, $j = 1, 2$. By assumption of this theorem there is a 2–coloring $\chi_0 : X \to [2]$ such that

$$\left| |E \cap \chi_0^{-1}(j)| - q_j |E| \right| \leq \mathrm{wd}(\mathcal{H}, 2, (q_1, q_2)) \leq K \tag{2.2}$$

for all colors $j \in [2]$ and edges $E \in \mathcal{E}$. Put $X_j := \chi_0^{-1}(j)$, $j = 1, 2$. Denote by T_j the subtree having C_j as its root. Then the hypergraph $\mathcal{H}_{|X_j}$ together with the set of colors C_j, the weight $\frac{1}{q_j} p_{|C_j}$ and the partition tree T_j fulfills the assumption of this theorem. By induction hypothesis there are colorings $\chi_j : X_j \to C_j$, $j \in 1, 2$ such that

$$\left| |E \cap X_j \cap \chi_j^{-1}(i)| - \tfrac{1}{q_j} p_i |E \cap X_j| \right| \leq K v(T_j, \tfrac{1}{q_j} p_{|C_j}, i) \leq K \sum_{l=2}^{k(i)} \frac{\frac{p_i}{q_j}}{\frac{1}{q_j} p(C_l^{(i)})} \tag{2.3}$$

for all $i \in C_j$. Set

$$\chi = \chi_1 \cup \chi_2 : x \mapsto \begin{cases} \chi_1(x) & \text{if } x \in X_1 \\ \chi_2(x) & \text{else} \end{cases} .$$

Let $j \in [2]$ and $i \in C_j$. Then $C_1^{(i)} = C_j$ and $q_j = p(C_1^{(i)})$. Let $E \in \mathcal{E}$. From (2.2), (2.3) and Lemma 2.8 we get

$$\begin{aligned} \left| |E \cap \chi^{-1}(i)| - p_i |E| \right| &\leq \left| |E \cap X_j \cap \chi_j^{-1}(i)| - \tfrac{p_i}{q_j} |E \cap X_j| \right| + \tfrac{p_i}{q_j} \left| |E \cap X_j| - q_j |E| \right| \\ &\overset{(2.2),(2.3)}{\leq} \sum_{l=2}^{k(i)} \frac{\frac{p_i}{q_j}}{\frac{1}{q_j} p(C_l^{(i)})} K + \tfrac{p_i}{q_j} K \\ &= K \sum_{l=1}^{k(i)} \frac{p_i}{p(C_l^{(i)})} = K v(T, p, i). \end{aligned}$$

Hence χ satisfies the claim. □

In the following corollary we give an upper bound on the constant $v(T, p, i) = p_i \sum_{l=1}^{k(i)} \frac{1}{p(C_l^{(i)})}$. An improvement in the case of equi-weighted discrepancy will be discussed in more detail in Section 2.4.3.

Lemma 2.11. *In the situation of Theorem 2.10 there is a partition tree T such that $v(T, p, i) < 4$ for all $i \in [c]$. Thus* $\mathrm{wd}(\mathcal{H}, p, c) < 4K$.

Proof. Recursively we construct a partition tree T for C with $v(T, p) \leq 4$. We start with the tree consisting of the unique node C. For a leaf C_0 of cardinality greater than 1 let us define sons by the following rule: If there is a color $i \in C_0$ with weight $p_i \geq \frac{1}{2} p(C_0)$, then the sons of C_0 shall be $\{i\}$ and $C_0 \setminus \{i\}$. Otherwise partition C_0 in any way (C_1, C_2) such that $p(C_j) \in [\frac{1}{3} p(C_0), \frac{2}{3} p(C_0)]$. Repeat this process until all leaves are singletons. The resulting tree T is a partition tree for C. All father-son pairs (C_0, C_1) in the resulting tree fulfill $\frac{2}{3}p(C_0) \geq p(C_1)$ or $|C_1| = 1$ and $p(C_0) > p(C_1)$. In the notation of Theorem 2.10 we have $p(C^{(i)}_{k(i)}) = p_i$, $p(C^{(i)}_{k(i)-1}) \geq p_i$ and $p(C^{(i)}_{k(i)-1-l}) \geq \left(\frac{3}{2}\right)^l p_i$ for all $l \in [k(i)-1]$. Now

$$v(T,p) \leq \max_{i \in C} \sum_{l=1}^{k(i)} \frac{p_i}{p(C^{(i)}_l)} \leq \max_{i \in C} p_i \left(\frac{1}{p_i} + \sum_{l=0}^{k(i)-2} \frac{1}{\left(\frac{3}{2}\right)^l p_i} \right) \leq 1 + \sum_{l=0}^{h(T)-2} \left(\frac{2}{3}\right)^l < 4,$$

and Theorem 2.10 gives the bound $\mathrm{wd}(\mathcal{H}, p, c) \leq 4K$. □

2.4.3 Equi-Weighted Discrepancy

In this subsection we consider the case of equi-weighted discrepancy in c colors. Hence our assumptions are identical with the ones from the preceding subsection except that we always have $p = \frac{1}{c}\mathbf{1}_c$. In this case only the size of the color sets is important, as all colors are equivalent. Therefore the following simpler structure can be investigated:

A *partition tree* for a positive integer n is a binary tree $T = (V_T, E_T)$ together with a labeling $l : V_T \to [n]$ such that the following conditions are satisfied:

- The root r is labeled $l(r) = n$.
- For every non-leaf v with sons s_1 and s_2 we have $l(v) = l(s_1) + l(s_2)$.
- The leaves are labeled 1.

Note that we can not assume l to be injective anymore. For a path $P : r = v^{(i)}_0, v^{(i)}_1, \ldots, v^{(i)}_{k(i)}$ connecting the root r and a leaf $v^{(i)}_k$ labeled i we call $v(T,P) = \sum_{l=1}^{k(i)} \frac{1}{l(v^{(i)}_l)}$ the value of P and $v(T)$ the maximum $v(T,P)$ over all these paths P. Finally $v(n)$ is the minimum $v(T)$ over all partition trees T of n.

There is a natural correspondence between partition trees for sets of colors and for positive integers. Let $T = (V_T, E_T)$ denote a partition tree for the set of colors C. Define a labeling $l_T : V_T \to |C|; v \mapsto |v|$. Then T together with l_T is a partition tree for $|C|$.

Now let T together with l denote a partition tree for a positive integer c. Let C be any set of colors such that $|C| = c$. We construct a partition tree T^* for C such that $l_{T^*} = l$. Define $f : V_T \to 2^{[c]}$ recursively: Set $f(r) = C$ for the root r of T. For every node v with sons s_1 and s_2 such that $f(v)$ is already defined choose $f(s_1)$ to be any subset of $f(v)$ of size $l(s_1)$ and $f(s_2) = f(v) \setminus f(s_1)$. Note that f is injective, and by replacing every $v \in V_T$ by $f(v)$ we get a partition tree T^* for C. Clearly, $l_{T^*} = l$.

Furthermore, we have

$$v(T^*, \tfrac{1}{c}\mathbf{1}_c) = \max_{i\in C} \frac{1}{c} \sum_{l=1}^{k(i)} \frac{1}{\frac{1}{c}|C_l^{(i)}|} = \max_{i\in C} \sum_{l=1}^{k(i)} \frac{1}{l(v_l^{(i)})} = v(T).$$

Corollary 2.12. *Let* $\mathrm{wd}(\mathcal{H}_0, 2) \le K$ *for all induced subgraphs* $\mathcal{H}_0$ *of* $\mathcal{H}$.

Then $\mathrm{disc}(\mathcal{H}, c) \le v(c)K$.

Proof. Let $T = (V_T, E_T)$ together with l be a partition tree for c such that $v(T) = v(c)$. We build T^* as above and apply Theorem 2.10 on T^* and $p = \frac{1}{c}\mathbf{1}_c$:

$$\mathrm{disc}(\mathcal{H}, c) = \mathrm{wd}(\mathcal{H}, p, c) \le Kv(T^*, p) = Kv(T) = Kv(c).$$

□

The exact calculation of $v(c)$ seems to be a difficult task. In particular, the optimal partition trees are in general not of minimal height. Put $\lfloor c\rfloor_2 := 2^{\lfloor \log_2 c\rfloor}$ and $\lceil c\rceil_2 := 2^{\lceil \log_2 c\rceil}$. Denote by $n_1(c)$ the number of 1's in the binary expansion of c (e. g. $n_1(9) = 2$). We give a lower bound and an upper bound on $v(c)$. If c is a power of 2, both bounds coincide.

Lemma 2.13. *For all* $c \in \mathbb{N}$, $c \ge 2$ *we have*

$$2 - \tfrac{2}{\lceil c\rceil_2} \le v(c) \le 2 + (n_1(c) - 3)\tfrac{1}{\lfloor c\rfloor_2}.$$

In particular, $v(c) \le 2.0005$.

Proof. Let $T = (V_T, E_T)$ together with l be any partition tree for c. Then there is a path $v_0, \dots, v_k$ of length $k \ge \log_2\lceil c\rceil_2$ such that v_k is a leaf and $l(v_{i-1}) \le 2l(v_i)$ for all $i \in [k]$. Thus $\sum_{i=1}^{k} \frac{1}{l(v_i)} \ge \sum_{i=0}^{k-1} 2^{-i} = 2 - \frac{1}{2^{k-1}} \ge 2 - \frac{1}{\lceil c\rceil_2}$.

For the upper bound we recursively construct a partition tree T for c. For a vertex v labeled $\sum_{i\in[k]} a_i 2^k \ne 1$, $a_i \in \{0,1\}$, we add sons $s_1(v)$ and $s_2(v)$ labeled $l(s_1(v)) = 2^{\min\{i\in[k] | a_i = 1\}}$ and $l(s_2(v)) = l(v) - l(s_1(v))$, if $l(v)$ is not a power of two, and labeled $l(s_1(v)) = l(s_2(v)) = \frac{1}{2}l(v)$ otherwise. Immediately we see that we only need to investigate the path $P : r, s_2(r), s_2(s_2(r)), \dots$ — if r denotes the root of T —, because the labels of all other paths occur also on this path. Thus $v(P)$ is maximal. The labels of the first $n_1(c)$ vertices of P are greater than or equal to $\lfloor c\rfloor_2$, so their contribution to $v(P)$ is not greater than $(n_1(c) - 1)\frac{1}{\lfloor c\rfloor_2}$. The rest of the vertices are labeled by $\frac{2}{\lfloor c\rfloor_2}, \frac{4}{\lfloor c\rfloor_2}, \dots$ up to 1. This sums up to $2 - \frac{2}{\lfloor c\rfloor_2}$ and the inequality is proven.

The last assertion is clear for $c \ge c_0 := 2^{15} - 1$, as $(n_1(c) - 3)\frac{1}{\lfloor c\rfloor_2} \le \frac{\log_2(\lfloor c\rfloor_2) - 2}{\lfloor c\rfloor_2} \le \frac{\log_2(\lfloor c_0\rfloor_2) - 2}{\lfloor c_0\rfloor_2}$. For the remaining small numbers, $v(c)$ can be computed in $\mathcal{O}(c^2)$–time and attains its maximum value for $c = 909$, namely $v(909) = 2.000450$. □

Now Corollary 2.12 and Lemma 2.13 yield

Theorem 2.14. *Let* $\mathrm{wd}(\mathcal{H}_0, 2) \le K$ *for all induced subgraphs* $\mathcal{H}_0$ *of* $\mathcal{H}$. *Then* $\mathrm{disc}(\mathcal{H}, c) \le 2.0005K$ *in any number* c *of colors.*

We apply Theorem 2.14 on the Beck–Fiala setting and get

Theorem 2.15. *For any hypergraph* $\mathcal{H}$ *we have*

$$\mathrm{disc}(\mathcal{H}, c) < v(c)\,\Delta(\mathcal{H}) \le 2.0005\,\Delta(\mathcal{H}).$$

Proof. The Beck–Fiala theorem states that $\mathrm{lindisc}(\mathcal{H}) < 2\Delta(\mathcal{H})$ holds for any hypergraph $\mathcal{H}$. In particular, we have $\mathrm{wd}(\mathcal{H}_0, 2) \le \frac{1}{2}\mathrm{lindisc}(\mathcal{H}_0) < \Delta(\mathcal{H}_0) \le \Delta(\mathcal{H})$ for all induced subhypergraphs $\mathcal{H}_0$ of $\mathcal{H}$. From Corollary 2.12 and Lemma 2.13 we conclude $\mathrm{disc}(\mathcal{H}, c) \le v(c)\,\mathrm{wd}(\mathcal{H}, 2) < 2.0005\Delta$. ☐

The following example shows that this recursive approach is nearly optimal in the general case. Let $n = kc$ for some $k \in \mathbb{N}$. Set $\mathcal{H} = \left([n], \binom{[n]}{k}\right) = ([n], \{E \subseteq [n] \,|\, |E| = k\})$. Any c–coloring for $\mathcal{H}$ produces a monochromatic hyperedge which has discrepancy $k(1 - \frac{1}{c})$. Hence $\mathrm{disc}(\mathcal{H}, c) \ge k(1 - \frac{1}{c})$ (equality holds as well, but we do not need this). Not let $(p, 1-p)$ be any 2–color weight. Assume without loss of generality that $p \le \frac{1}{2}$. Put $\chi : [n] \to [2]; i \mapsto 2$. Now each hyperedge has weighted discrepancy $k(1-p)$ with respect to χ and $(p, 1-p)$. Thus $\mathrm{wd}(\mathcal{H}, c) \le \frac{1}{2}k$, and of course this holds as well for any induced subhypergraph $\mathcal{H}_0$ of $\mathcal{H}$. This shows

$$\mathrm{disc}(\mathcal{H}, c) \ge 2(1 - \tfrac{1}{c}) \max_{\mathcal{H}_0 \le \mathcal{H}} \mathrm{wd}(\mathcal{H}_0, 2).$$

In particular, the recursive method yields optimal results of this hypergraph if c is a power of 2, and it is asymptotically optimal for $c \to \infty$.

Note that Theorem 2.14 also yields the following bounds:

- For any hypergraph $\mathcal{H} = (V, \mathcal{E})$ with $n := |V| = |\mathcal{E}|$ sufficiently large we have
$$\mathrm{disc}(\mathcal{H}, c) < 12\sqrt{n}.$$
- Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph on n points. Let $d > 1$. If $\pi_{\mathcal{H}} = \mathcal{O}(m^d)$, then $\mathrm{disc}(\mathcal{H}, c) = \mathcal{O}(n^{\frac{1}{2} - \frac{1}{2d}})$. If $\pi^*_{\mathcal{H}} = \mathcal{O}(m^d)$, then $\mathrm{disc}(\mathcal{H}, c) = \mathcal{O}(n^{\frac{1}{2} - \frac{1}{2d}} \log n)$. In both cases the implicit constants are independent of c.
- The hypergraph $\mathcal{A}_n$ of arithmetic progressions in $[n]$ fulfills
$$\mathrm{disc}(\mathcal{A}_n, c) \le v(c) C \sqrt[4]{n} \le 2.0005 C \sqrt[4]{n},$$
where C is the constant of Matoušek and Spencer such that $\mathrm{disc}(\mathcal{A}_n) \le C\sqrt[4]{n}$.

Using the fact that in these cases the discrepancies of smaller induced subhypergraphs are decreasing, we improve these bounds in the next subsection.

2.4.4 Recursive Coloring with Decreasing Discrepancies in Induced Subhypergraphs

In this subsection we extend the recursive approach to make use of the additional assumption that subhypergraphs on fewer vertices have smaller discrepancy. This is a natural assumption as many results are of this type (see below where we prove their multi-color analogies). Roughly speaking we show that if the 2–color discrepancy of the subhypergraphs on n_0 vertices is bounded by $\mathcal{O}(n_0^\alpha)$, then the c–color discrepancy is bounded by $\mathcal{O}((\frac{n}{c})^\alpha)$. It seems a little surprising that this bound is achievable by a recursive approach, as the first step in the recursion will find a 2–coloring for the whole hypergraph with discrepancy guarantee $\mathcal{O}(n^\alpha)$ only. We still get the $\mathcal{O}((\frac{n}{c})^\alpha)$–discrepancy for the final coloring due to the fact that imbalances inflicted in earlier rounds of the recursion are split up in a balanced manner by later steps. It turns out that this effect even exceeds the effect of decreasing discrepancy of smaller subhypergraphs. Crucial therefore is the last step of the recursion where colorings for hypergraphs on roughly $\frac{2n}{c}$ vertices are looked for.

There are two points though that need further attention: Firstly, like in the case where we only assumed a uniform bound on the discrepancies of the induced subhypergraphs, this simple approach only works if the number of colors is a power of 2. This is the reason why we have to use weighted discrepancies again.

A second point is that to use the assumption of decreasing discrepancies we need to make sure that the vertex sets considered actually become smaller. Unfortunately, in general we do not know the size of the color classes generated by a low discrepancy coloring. If the whole vertex set is a hyperedge, we know at least that the sizes of the color classes deviate from the aimed at value by at most the discrepancy guarantee. This is not too bad if the discrepancy is relatively small, but even then keeping track of these deviations during the recursion is tedious. Better bounds seem achievable by the cleaner approach of only investigating *fair* colorings, that is, those which have discrepancy less than one on the set of all vertices. Hence, apart from fractional parts, for each color i the corresponding color class contains $p_i|X|$ vertices.

One remark that eases work with the fractional parts: Let us call a weight $p \in [0,1]^c$ *integral* (with respect to $\mathcal{H} = (X, \mathcal{E})$) if all $p_i, i \in [c]$ are multiples of $\frac{1}{|X|}$. From the definition it is clear that a fair coloring χ with respect to an integral weight p fulfills $|\chi^{-1}(i)| = p_i|X|$ for all colors $i \in [c]$. On the other hand, suppose that we know that for a given hypergraph and for all integral weights p there is a fair coloring that has discrepancy at most k. Then there are fair colorings having discrepancy at most $k+1$ for any weight: For an arbitrary weight p there is an integral weight p' such that $|p_i - p'_i| < \frac{1}{|X|}$ holds for all $i \in [c]$. Therefore, a fair coloring with respect to p' is also fair with respect to p, and its discrepancy with respect to p is larger (if at all) than the one with respect to p' by less than one. For these reasons we may restrict ourselves to the more convenient case that all weights are integral.

Using the following recoloring argument we will construct fair colorings having a similar discrepancy as given by the discrepancy bound.

Lemma 2.16. *Let $\mathcal{H} = (X, \mathcal{E})$ be a hypergraph such that $X \in \mathcal{E}$. Let $p = (q, 1-q)$ be a 2–color weight. Then there is a fair coloring χ such that* $\operatorname{wd}(\mathcal{H}, \chi, p) \le 2 \operatorname{wd}(\mathcal{H}, c, p)$.

Proof. Let χ be a coloring such that $\operatorname{wd}(\mathcal{H}, \chi, p) = \operatorname{wd}(\mathcal{H}, c, p)$. Set $x := q|X| - |\chi^{-1}(1)|$. Since X is an edge in $\mathcal{H}$, $|x| \le \operatorname{wd}(\mathcal{H}, c, p)$. Let $\overline{\chi}$ denote a coloring arising from χ by changing the color of $\lfloor |x| \rfloor$ points in such a way that $|q|X| - |\overline{\chi}^{-1}(1)|| < 1$. Now $\overline{\chi}$ is a fair coloring with respect to the weight $(q, 1-q)$. For an edge $E \in \mathcal{E}$ we compute

$$\begin{aligned}
&|\, q|E| - |\overline{\chi}^{-1}(1) \cap E||\\
&\le\ |\, q|E| - |\chi^{-1}(1) \cap E|| + ||\chi^{-1}(1) \cap E| - |\overline{\chi}^{-1}(1) \cap E||\\
&\le\ |\, q|E| - |\chi^{-1}(1) \cap E|| + \lfloor |x| \rfloor\\
&\le\ 2 \operatorname{wd}(\mathcal{H}, c, p).
\end{aligned}$$

□

Lemma 2.16 requires the whole vertex set to be a hyperedge. Fortunately, most discrepancy results are relatively robust concerning the addition of a single hyperedge. In these cases we may just replace the hypergraph under consideration by the one obtained from adding X as additional edge and still get a useful bound.

To state the main theorem in its strongest form we need the following constants. Let $\alpha \in]0, 1[$. For each $p \in]0, 1[$ define $v_\alpha(p)$ to be

$$\max \left\{ \sum_{i=1}^{k} \prod_{j=1}^{i} q_j^\alpha \prod_{j=i+1}^{k} q_j \,\middle|\, k \in \mathbb{N}, q_1, \ldots, q_{k-1} \in [0, \tfrac{2}{3}], q_k \in [0, 1], \prod_{j=1}^{k} q_j = p \right\}.$$

Set $c_\alpha := 1 + \sum_{i=0}^{\infty} \left(\frac{2}{3}\right)^{(1-\alpha)i}$. Then we have

Lemma 2.17. *Let $\alpha \in]0, 1[$.*

(i) Let $0 < p < q \le \frac{2}{3}$. Then $q^\alpha v_\alpha(\frac{p}{q}) + q^\alpha \frac{p}{q} \le v_\alpha(p)$.

(ii) For all $p \in [0, 1]$, $v_\alpha(p) \le c_\alpha p^\alpha$.

Proof. Let $k \in \mathbb{N}, q_1, \ldots, q_{k-1} \in [0, \frac{2}{3}], q_k \in [0, 1]$ such that $\prod_{j=1}^{k} q_j = \frac{p}{q}$ and $v_\alpha(\frac{p}{q}) = \sum_{i=1}^{k} \prod_{j=1}^{i} q_j^\alpha \prod_{j=i+1}^{k} q_j$. With $q_0 := q$ we have

$$\begin{aligned}
q^\alpha v_\alpha(\tfrac{p}{q}) + q^\alpha \tfrac{p}{q} &= q_0^\alpha \sum_{i=1}^{k} \prod_{j=1}^{i} q_j^\alpha \prod_{j=i+1}^{k} q_j + q_0^\alpha \prod_{j=1}^{k} q_j\\
&= \sum_{i=0}^{k} \prod_{j=0}^{i} q_j^\alpha \prod_{j=i+1}^{k} q_j \le v_\alpha(p),
\end{aligned}$$

since $\prod_{j=0}^{k} q_j = q \prod_{j=1}^{k} q_j = p$. This is (i).

Let $k \in \mathbb{N}$, $q_1, \ldots, q_{k-1} \in [0, \frac{2}{3}]$ and $q_k \in [0,1]$ such that $\prod_{j=1}^{k} q_j = p$ and $v_\alpha(p) = \sum_{i=1}^{k} \prod_{j=1}^{i} q_j^\alpha \prod_{j=i+1}^{k} q_j$. For $i \in [k]$ set $x_i := \prod_{j=1}^{i} q_j^\alpha \prod_{j=i+1}^{k} q_j$. Then $x_k = p^\alpha$ and $x_{k-1} \leq x_k$. For $i \in [k-2]$ we have

$$\frac{x_{k-1-i}}{x_{k-1-i+1}} = \frac{q_{k-1-i}}{q_{k-1-i}^\alpha} = q_{k-1-i}^{1-\alpha} \leq \left(\tfrac{2}{3}\right)^{1-\alpha},$$

and hence $x_{k-1-i} \leq \left(\frac{2}{3}\right)^{(1-\alpha)i} x_k$. Thus

$$v_\alpha(p) = \sum_{i=1}^{k} x_i \leq p^\alpha \left(1 + \sum_{i=0}^{\infty} \left(\tfrac{2}{3}\right)^{(1-\alpha)i}\right) = c_\alpha p^\alpha.$$

□

We are now in the situation to state the main result, which is a little technical. In addition to what we already explained about this recursive approach, there is one further detail. As we do recursive partitioning, we will never need a discrepancy result concerning induced subhypergraphs on fewer than $\frac{n}{c}$ vertices (in the equi-weighted case). This observation will be useful in some applications, e. g. in the case $|\mathcal{E}| = |X|$, where our bound for the discrepancies of induced subhypergraphs on n_0 vertices is $C\sqrt{n_0 \log(\frac{n}{n_0})}$.

Theorem 2.18. *Let $\mathcal{H} = (X, \mathcal{E})$ be a hypergraph. Let $p_0, \alpha \in]0,1[$ and $C > 0$. Assume that for all $X_0 \subseteq X$ such that $|X_0| \geq p_0|X|$ and all $q \in [0,1]$ such that $(q, 1-q)$ is integral with respect to $\mathcal{H}_{|X_0}$ there is a fair 2–coloring χ having discrepancy at most $C|X_0|^\alpha$, i. e. we have $|X_0 \cap \chi^{-1}(1)| = q|X_0|$ and $||E \cap X_0 \cap \chi^{-1}(1)| - q|E \cap X_0|| \leq C|X_0|^\alpha$ holds for all $E \in \mathcal{E}$.*

Then for each integral weight $p \in [0,1]^c$ there is a fair c–coloring of $\mathcal{H}$ with respect to p such that the discrepancy with respect to p is at most $Cv_\alpha(p_i)n^\alpha \leq Cc_\alpha(p_i n)^\alpha$ in all colors $i \in [c]$ such that $p_i \geq p_0$. In particular, the c–color discrepancy of $\mathcal{H}$ is bounded by

$$\operatorname{disc}(\mathcal{H}, c) \leq Cc_\alpha \left(\frac{n}{c}\right)^\alpha + 1$$

for all $c \leq \frac{1}{p_0}$.

Proof. We start with

Claim 1: For each integral (with respect to $\mathcal{H}$) weight $(2^{-k}, 1-2^{-k})$, $2^{-k} \geq p_0$, $k \in \mathbb{N}$ there is a fair 2–coloring with respect to this weight such that $\operatorname{wd}(\mathcal{H}, \chi, (2^{-k}, 1-2^{-k})) \leq \sum_{i=0}^{k-1} 2^{-k+1+i} 2^{-\alpha i} C n^\alpha$.

We proceed by induction. For $k = 1$, there is nothing to show. Let $k > 1$. Let $\chi_0 : X \to [2]$ be a fair $(0.5, 0.5)$–coloring having discrepancy at most Cn^α.[2] Set $X_1 := \chi_0^{-1}(1)$. Let $\chi_1 : X_1 \to [2]$ be a fair $(2^{-k+1}, 1 - 2^{-k+1})$–coloring (note that $(2^{-k+1}, 1 - 2^{-k+1})$ is integral for $\mathcal{H}_{|X_1}$). By induction we may assume that χ_1 has discrepancy at most $\sum_{i=0}^{k-2} 2^{-k+2+i} 2^{-\alpha i} C(\frac{n}{2})^\alpha$. Define a coloring $\chi : X \to [2]$ by $\chi(x) = 1$ if and only if $\chi_0(x) = 1$ and $\chi_1(x) = 1$. Then χ is a fair $(2^{-k}, 1 - 2^{-k})$–coloring. Using a similar argument as in Lemma 2.8, we compute the discrepancy of an edge $E \in \mathcal{E}$ with respect to $(2^{-k}, 1 - 2^{-k})$ in color 1:

$$
\begin{aligned}
&\big||E \cap \chi^{-1}(1)| - 2^{-k}|E|\big| \\
&= \big||E \cap \chi_0^{-1}(1) \cap \chi_1^{-1}(1)| - 2^{-k}|E|\big| \\
&\le \big||E \cap \chi_0^{-1}(1) \cap \chi_1^{-1}(1)| - 2^{-k+1}|E \cap \chi_0^{-1}(1)|\big| \\
&\quad + \big|2^{-k+1}|E \cap \chi_0^{-1}(1)| - 2^{-k}|E|\big| \\
&\le \big||(E \cap X_1) \cap \chi_1^{-1}(1)| - 2^{-k+1}|E \cap X_1|\big| + 2^{-k+1}\big||E \cap \chi_0^{-1}(1)| - 0.5|E|\big| \\
&\le \sum_{i=0}^{k-2} 2^{-k+2+i} 2^{-\alpha i} C(\tfrac{n}{2})^\alpha + 2^{-k+1} C n^\alpha \\
&= \sum_{i=0}^{k-1} 2^{-k+1+i} 2^{-\alpha i} C n^\alpha.
\end{aligned}
$$

As 2–colorings have the same discrepancy in both colors, this proves Claim 1. From our assumptions on $\mathcal{H}$ it is clear that the assertion of Claim 1 also holds for any induced subgraph $\mathcal{H}_{|X_0}$ of $\mathcal{H}$ as long as $2^{-k}|X_0| \le p_0|X|$. We will use this fact to prove

Claim 2: For each integral (with respect to $\mathcal{H}$) weight $(q, 1-q), q \ge p_0$ there is a fair 2–coloring with respect to this weight that has discrepancy at most $\mathrm{wd}(\mathcal{H}, \chi, (q, 1-q)) \le \frac{2}{2^{1-\alpha}-1} C\,(qn)^\alpha$.

Let $q' \le 1$ be maximal subject to the condition that $\frac{q'}{q}$ is a power of 2. Let $q' = 2^k q$. Let $\chi_0 : X \to [2]$ be a fair $(q', 1-q')$–coloring having discrepancy at most Cn^α. Let $\chi_1 : \chi_0^{-1}(1) \to [2]$ be a fair $(\frac{q}{q'}, 1 - \frac{q}{q'})$–coloring. From Claim 1 we may assume that χ_1 has discrepancy at most $\sum_{i=0}^{k-1} 2^{-k+1+i} 2^{-\alpha i} C(q'n)^\alpha$. Define a coloring $\chi : X \to [2]$ by $\chi(x) = 1$ if and only if $\chi_0(x) = 1$ and $\chi_1(x) = 1$. Then χ is a fair $(q, 1-q)$–coloring. For an edge $E \in \mathcal{E}$ we compute its discrepancy in color 1 as above:

---

[2] To keep this proof a little more readable, we will use this expression to say that χ_0 is fair with respect to $(0.5, 0.5)$ and furthermore has weighted discrepancy at most Cn^α with respect to $(0.5, 0.5)$.

$$
\begin{aligned}
&\big||E \cap \chi^{-1}(1)| - q|E|\big| \\
&= \big||E \cap \chi_0^{-1}(1) \cap \chi_1^{-1}(1)| - q|E|\big| \\
&\le \left||E \cap \chi_0^{-1}(1) \cap \chi_1^{-1}(1)| - \tfrac{q}{q'}|E \cap \chi_0^{-1}(1)|\right| + \left|\tfrac{q}{q'}|E \cap \chi_0^{-1}(1)| - q|E|\right| \\
&\le \big||E \cap \chi_0^{-1}(1) \cap \chi_1^{-1}(1)| - 2^{-k}|E \cap \chi_0^{-1}(1)|\big| + 2^{-k}\big||E \cap \chi_0^{-1}(1)| - q'|E|\big| \\
&\le \sum_{i=0}^{k-1} 2^{-k+1+i}2^{-\alpha i}C(q'n)^\alpha + 2^{-k}Cn^\alpha \\
&= \left(2^{-k+1}q'^\alpha \frac{2^{(1-\alpha)k}-1}{2^{1-\alpha}-1} + 2^{-k}\right) Cn^\alpha \\
&< 2q'^\alpha \frac{2^{-\alpha k}}{2^{1-\alpha}-1} Cn^\alpha = \frac{2}{2^{1-\alpha}-1} C\,(qn)^\alpha.
\end{aligned}
$$

This proves Claim 2.

For the main part of the proof let $p \in [0,1]^c$ be an integral weight. Again we proceed by induction. Choose a partition $\{C_1, C_2\}$ of the set of colors $[c]$ such that $\|p_{|C_1}\|_1, \|p_{|C_2}\|_1 \le \frac{2}{3}c$ or C_1 contains a single color with weight at least $\frac{1}{3}$. In particular, $\|p_{|C_2}\|_1 \le \frac{2}{3}c$ holds in both cases. Set $(q_1, q_2) := (\|p_{|C_1}\|_1, \|p_{|C_2}\|_1)$. Choose a fair coloring $\chi_0 : X \to [2]$ with respect to the weight (q_1, q_2). From Claim 2 we may assume that χ_0 has discrepancy at most $C(q_i n)^\alpha$ in color $i = 1, 2$ if $q_i \ge p_0$ (of course the discrepancy is the same in both colors, but we do not need this). Set $X_i := \chi^{-1}(i)$ for $i = 1, 2$. If $|C_i| > 1$, then by induction, there is a fair coloring $\chi_i : X_i \to C_i$ with respect to the weight $\frac{1}{q_i}p_{|C_i}$ having discrepancy at most $Cv_\alpha(\frac{p_j}{q_i})(q_i n)^\alpha$ in each color $j \in C_i$, $p_j \ge p_0$. If $C_i = \{j\}$ for some $j \in [c]$, set $\chi_i : X_i \mapsto \{j\}$. Set $\chi = \chi_1 \cup \chi_2$. We compute the discrepancy of an edge $E \in \mathcal{E}$ with respect to χ and p in color $j \in C_i$ in the case $|C_i| > 1$ and $p_j \ge p_0$:

$$
\begin{aligned}
&\big||E \cap \chi^{-1}(j)| - p_j|E|\big| \\
&= \big||E \cap \chi_0^{-1}(i) \cap \chi_i^{-1}(j)| - p_j|E|\big| \\
&\le \left||E \cap \chi_0^{-1}(i) \cap \chi_i^{-1}(j)| - \tfrac{p_j}{q_i}|E \cap \chi_0^{-1}(i)|\right| + \left|\tfrac{p_j}{q_i}|E \cap \chi_0^{-1}(i)| - p_j|E|\right| \\
&\le \left||(E \cap X_i) \cap \chi_i^{-1}(j)| - \tfrac{p_j}{q_i}|E \cap X_i|\right| + \tfrac{p_j}{q_i}\big||E \cap \chi_0^{-1}(i)| - q_i|E|\big| \\
&\le Cv_\alpha(\tfrac{p_j}{q_i})(q_i n)^\alpha + \tfrac{p_j}{q_i}C(q_i n)^\alpha \\
&= Cv_\alpha(p_j)n^\alpha
\end{aligned}
$$

by Lemma 2.17 (i). On the other hand, if C_i contains a single color j, then $p_j = q_i$ and $||E \cap \chi^{-1}(j)| - p_j|E|| = \big||E \cap \chi_0^{-1}(i)| - p_j|E|\big| \le C(p_j n)^\alpha \le Cv_\alpha(p_j)n^\alpha$.

The last assertion follows from $p = \frac{1}{c}\mathbf{1}_c$ and the remark about integral and arbitrary weights at the beginning of this subsection. □

General Hypergraphs

Let $\mathcal{H} = (X, \mathcal{E})$ denote an arbitrary hypergraph. Set $n := |V|$ and $m := |\mathcal{E}|$ for convenience. Let us denote by $\overline{\mathcal{H}}$ the hypergraph obtained from $\mathcal{H}$ by adding the whole vertex set as an additional hyperedge. Note that we have $\overline{(\mathcal{H}_{|X_0})} = (\overline{\mathcal{H}})_{|X_0}$ for all $X_0 \subseteq X$.

In Section 2.3 it was shown that a random coloring generated by coloring each vertex independently with each color with probability $\frac{1}{c}$ has discrepancy $\sqrt{0.5n \ln(4mc)}$ with probability $\frac{1}{2}$. In this section we show that via the recursive approach of Theorem 2.18 a much better bound can be achieved. In particular, the discrepancy decreases for larger numbers of colors.

Theorem 2.19. *For an arbitrary hypergraph $\mathcal{H}$ the c–color discrepancy is bounded by*

$$\mathrm{disc}(\mathcal{H}, c) \leq 13\sqrt{\frac{n}{c}\ln(2m)} + 1.$$

Proof. Let $X_0 \subseteq X$. For any induced subhypergraph $\mathcal{H}_1 = (X_1, \mathcal{E}_1)$ of $\mathcal{H}_{|X_0}$ Theorem 5.2 shows the existence of a fair coloring χ such that $\mathrm{disc}(\mathcal{H}_1, \chi) \leq \sqrt{|X_1| \ln(2m)}$. From $|X_1| \leq |X_0|$ and the fairness of these χ we get $\mathrm{herdisc}(\overline{\mathcal{H}_{|X_0}}) \leq \sqrt{|X_0| \ln(2m)}$.

From Remark 2.9 we have $\mathrm{wd}(\overline{\mathcal{H}_{|X_0}}, 2, p) \leq \sqrt{|X_0| \ln(2m)}$ for any 2–color weight p. Now Lemma 2.16 gives a fair coloring χ such that $\mathrm{disc}(\mathcal{H}_{|X_0}, \chi, p) \leq 2\sqrt{|X_0| \ln(2m)}$. Therefore we may apply Theorem 2.18 with $\alpha = \frac{1}{2}$ and $C = 2\sqrt{\ln(2m)}$. Finally $c_\alpha \leq 6.5$ proves the discrepancy bound. □

Note that all ingredients of this proof are constructive: The randomized colorings of Theorem 5.2 respect a slightly weaker bound with probability a half, Theorem 1.4 allows an arbitrary close approximation and the fair coloring lemma and the recursive method yield no problems at all.

Six Standard Deviations

The celebrated 'six standard deviations' result due to Spencer [Spe85] states that there is a constant K such that for all hypergraphs $\mathcal{H} = (X, \mathcal{E})$ having n vertices and $m \geq n$ edges

$$\mathrm{disc}(\mathcal{H}) \leq K\sqrt{n \ln(\tfrac{2m}{n})}$$

holds.

The interesting case is of course the one where $m = \mathcal{O}(n)$ and thus $\mathrm{disc}(\mathcal{H}) = \mathcal{O}(\sqrt{n})$. For m significantly larger than n this result is outnumbered by the simple fair coin flip random coloring of Theorem 1.1 or the matching random coloring of Theorem 5.2. The title “Six Standard Deviations Suffice” of this paper comes from the fact that for $n = m$ large enough, $\operatorname{disc}(\mathcal{H}) \leq 6\sqrt{n}$ holds. Using the relation between discrepancies respecting a particular weight and hereditary discrepancy (Remark 2.9) and the recoloring argument (Lemma 2.16), we derive from Spencer's result

Lemma 2.20. *For any $X_0 \subseteq X$ and integral (with respect to $\mathcal{H}_{|X_0}$) weight $p = (q, 1-q)$ there is a fair $(q, 1-q)$–coloring χ of $\mathcal{H}_{|X_0}$ that has $\operatorname{wd}(\mathcal{H}_{|X_0}, \chi, p) \leq 2K\sqrt{|X_0| \ln(\frac{2m+2}{|X_0|})}$.*

Proof. Let $X_0 \subseteq X$. Then any induced subgraph of $\mathcal{H}_{|X_0}$ has discrepancy at most $K\sqrt{|X_0| \ln(\frac{2m}{|X_0|})}$, simply because Spencer's bound is monotone in the number of vertices. From Remark 2.9, we have $\operatorname{wd}(\mathcal{H}_{|X_0}, 2, (q, 1-q)) \leq \operatorname{herdisc}(\mathcal{H}_{|X_0}) \leq K\sqrt{|X_0| \ln(\frac{2m}{|X_0|})}$.

It remains to show the existence of a fair coloring. Let $\overline{\mathcal{H}}$ denote the hypergraph arising from $\mathcal{H}$ by adding the set X as an additional edge (unless of course $X \in \mathcal{E}$ already holds). Then $\overline{\mathcal{H}}_{|X_0}$ has at most $m+1$ edges, and from the previous paragraph we know $\operatorname{disc}(\overline{\mathcal{H}}_{|X_0}, 2, (q, 1-q)) \leq K\sqrt{|X_0| \ln(\frac{2m+2}{|X_0|})}$. Lemma 2.16 now yields the claim. □

Lemma 2.20 and Theorem 2.18 yield

Theorem 2.21. *Let $\mathcal{H} = (X, \mathcal{E})$ denote a hypergraph having n vertices and $m \geq n$ edges and $p \in [0,1]^c$ an integral weight. Set $p_0 := \min_{i \in [c]} p_i$. Then there is a fair coloring with respect to p having discrepancy at most $13K\sqrt{p_i n \ln(\frac{2m+2}{p_0 n})}$ in color i.*

In particular, in the case $|X| = |\mathcal{E}| = n$ we have

$$\operatorname{disc}(\mathcal{H}, c) \leq \mathcal{O}\left(\sqrt{\tfrac{n}{c} \ln c}\right).$$

Proof. By Lemma 2.20 we may apply Theorem 2.18 with $\alpha = \frac{1}{2}$, $C = 2K\sqrt{\ln(\frac{2m+2}{p_0 n})}$ and p_0. This yields a fair coloring with respect to p having discrepancy at most $CC(\frac{1}{2})\sqrt{p_i n}$ in color $i \in [c]$. The claim follows from $C(\frac{1}{2}) \leq 6.44949$. □

This is quite close to the optimum. Theorem 2.31 shows a lower bound of $\Omega(\sqrt{\frac{n}{c}})$ in the case $|X| = |\mathcal{E}| = n$.

The following corollary on 2 color discrepancies seems worth mentioning. Already from combining Claim 2 of the proof of Theorem 2.18 and Lemma 2.20 we derive:

Corollary 2.22. *Let $\mathcal{H} = (X, \mathcal{E})$ denote a hypergraph such that $|X| = |\mathcal{E}| =: n$ and $(q, 1-q)$ an integral 2–color weight. Assume $q \leq \frac{1}{2}$. Then the weighted discrepancy $\operatorname{wd}(\mathcal{H}, 2, (q, 1-q))$ is at most $10K\sqrt{qn \ln(\frac{2}{q})}$.*

Arithmetic Progressions

A third classical example is the hypergraph of arithmetic progressions on the first n numbers. This is probably the most famous of the few non–trivial examples where discrepancy is well–understood. For $a, d, l \in \mathbb{N}$ denote by $A_{adl} := \{a + id | 0 \leq i \leq l - 1\}$ the arithmetic progression with starting point a, difference d and length l. Denote by $\mathcal{E}_n$ the set of all arithmetic progressions in $[n]$, that is $\mathcal{E}_n = \{A_{adl} \cap [n] | a, d, l \in [n]\}$. Set $\mathcal{A}_n = ([n], \mathcal{E}_n)$.

Roth [Rot64] proved the celebrated lower bound $\operatorname{disc}(\mathcal{A}_n) = \Omega(n^{\frac{1}{4}})$. Roth himself believed that this bound was too small and that the discrepancy was close to $n^{\frac{1}{2}}$. This was disproved by Sárközy [Sár74], who showed an upper bound of $\mathcal{O}(n^{\frac{1}{3}+\varepsilon})$. Inventing the partial coloring method, Beck [Bec81] showed a nearly tight bound of $\mathcal{O}(n^{\frac{1}{4}}(\log n)^{\frac{5}{2}})$. Finally Matoušek and Spencer [MS96] solved the discrepancy problem for $\mathcal{A}_n$ by proving the asymptotically tight upper bound $\mathcal{O}(n^{\frac{1}{4}})$.

This bound holds in any fixed number of colors. Moreover, we prove that the discrepancy decreases for larger numbers of colors.

Theorem 2.23. *For an absolute constant C' the following holds: Let $p \in [0,1]^c$ be a weight. Then there is a fair coloring of $\mathcal{A}_n$ with respect to p having discrepancy at most $C' p_i^{0.16} n^{0.25}$ in each color i such that $p_i \geq n^{0.25}$. In particular,*

$$\operatorname{disc}(\mathcal{A}_n, c) = \mathcal{O}(c^{-0.16} n^{0.25})$$

holds for $c \leq n^{0.25}$ colors.

Proof. From Lemma 5.3 of [MS96] we learn than an induced subgraph $\mathcal{H}_0 = (\mathcal{A}_n)_{|X_0}$ of $\mathcal{A}_n$ on $|X_0| = \rho n \geq n^{0.25}$ vertices has discrepancy at most $C_1 \rho^{0.16} n^{0.25}$. We first show that $\operatorname{herdisc}(\mathcal{H}_0) \leq 2C_1 \rho^{0.16} n^{0.25}$:

Let $\mathcal{H}_1 = (X_1, \mathcal{E}_1)$ be an induced subhypergraph of $\mathcal{H}_0$. If $|X_1| \geq n^{0.25}$ we are done by the Lemma of Matoušek and Spencer. Let us therefore assume $|X_1| < n^{0.25}$. We show that $(\mathcal{H}_1)_{|[\frac{n}{2}]}$ and $(\mathcal{H}_1)_{|[n] \setminus [\frac{n}{2}]}$ have discrepancy at most $C_1 \rho^{0.16} n^{0.25}$ and conclude $\operatorname{disc}(\mathcal{H}_1) \leq 2C_1 \rho^{0.16} n^{0.25}$. Consider the hypergraph $\mathcal{H}_2 := \mathcal{H}_{(X_1 \cap [\frac{n}{2}]) \cup \{n - n^{0.25} + |X_1 \cap [\frac{n}{2}]| + 1, \ldots, n\}}$. This hypergraph has exactly $n^{0.25} \leq \rho n$ vertices and thus discrepancy at most $C_1 \rho^{0.16} n^{0.25}$. As every edge of $(\mathcal{H}_1)_{|[\frac{n}{2}]}$ is also an edge of $\mathcal{H}_2$, we conclude $\operatorname{disc}((\mathcal{H}_1)_{|[\frac{n}{2}]}) \leq C_1 \rho^{0.16} n^{0.25}$. A similar argument shows $\operatorname{disc}((\mathcal{H}_1)_{|[n] \setminus [\frac{n}{2}]}) \leq C_1 \rho^{0.16} n^{0.25}$.

Thus $\operatorname{herdisc}(\mathcal{H}_0) \leq 2C_1 \rho^{0.16} n^{0.25}$. The relation between the linear and hereditary discrepancy yields that all weighted discrepancies of $\mathcal{H}_0$ are bounded by $2C_1 \rho^{0.16} n^{0.25}$. As $[n]$ is an arithmetic progression, we may apply Lemma 2.16 and conclude that twice this discrepancy may be achieved by a fair coloring respecting the underlying weight.

Thus we may apply Theorem 2.18 with $C = 4C_1 n^{0.09}$, $\alpha = 0.16$ and $p_0 = n^{0.25}$, which proves our claim. □

Bounded Shatter Functions

The recursive approach also generalizes results of Matoušek, Welzl and Wernisch [MWW84] and Matoušek [Mat95] connecting discrepancy with the primal shatter function $\pi_{\mathcal{H}}$ and dual shatter function $\pi^*_{\mathcal{H}}$ of a hypergraph. Note that this also yields a bound in terms of the VC–dimension $\dim(\mathcal{H})$ of $\mathcal{H}$: Already Vapnik and Chervonenkis [VC71] showed $\pi_{\mathcal{H}} \in \mathcal{O}(n^{\dim(\mathcal{H})})$.

Theorem 2.24. *Let $\mathcal{H} = (X, \mathcal{E})$ be a hypergraph on n points. Let $d > 1$. If $\pi_{\mathcal{H}} = \mathcal{O}(m^d)$, then $\operatorname{disc}(\mathcal{H}, c) = \mathcal{O}((\frac{n}{c})^{\frac{1}{2}-\frac{1}{2d}})$. If $\pi^*_{\mathcal{H}} = \mathcal{O}(m^d)$, then $\operatorname{disc}(\mathcal{H}, c) = \mathcal{O}((\frac{n}{c})^{\frac{1}{2}-\frac{1}{2d}} \log n)$. In both cases the implicit constants are independent of c.*

Proof. Clearly the assumptions on the shatter functions are hereditary in the sense that a shatter function of an induced subhypergraph is less or equal the one of the whole hypergraph. They are also very robust: Adding the whole vertex set as additional edge changes the primal shatter function by at most 1, and does not change the dual shatter function. Without loss of generality we may therefore assume $X \in \mathcal{E}$. The remainder of the proof is standard — bound the weighted discrepancies of the induced subhypergraphs using Remark 2.9, buy fairness at the price of a factor of 2 (Lemma 2.16) and apply Theorem 2.18. □

2.4.5 Recursive Approaches versus Direct Approaches

We see that the recursive method is very effective in situations where we can bound the weighted discrepancy of the induced subgraphs. We always get a uniform bound from the hereditary discrepancy of $\mathcal{H}$ (Remark 2.9). There are situations where the recursive approach is the only result we have. We do not have a direct proof for a result like Theorem 2.21 or Theorem 2.23. We feel that the original proof relies heavily on the fact that only two colors are considered.

Surprisingly, the recursive approach and direct methods are sometimes nearly equally effective. An example is the (equi-weighted) multi–color discrepancy in the case of bounded degree. The direct approach (cf. Section 2.5) yields $\operatorname{disc}(\mathcal{H}, c) \le 2\Delta(\mathcal{H})$, the recursive one gives $\operatorname{disc}(\mathcal{H}, c) \le v(c)\Delta(\mathcal{H})$. Both methods are constructive. For c tending to infinity both methods give the same bound. For $c \ge 100$ the two bounds deviate by less than one percent. As a rough estimate, the recursive approach is worse for large c: For $1 \le c \le 1000$, in 2.5 % of the cases $v(c)$ is greater than 2, for $1 \le c \le 10000$, it is 15.5 %, and for $1 \le c \le 100000$, in about 37.8 % of the cases the direct approach is better.

Still — as the powers of 2 show — there are infinitely many numbers where the recursive approach is better.

On the other hand of course the recursive approch is limited: We can get results on weighted discrepancy, but we do not get any on linear discrepancy, e. g. in the Beck–Fiala setting. This is surprising at first, but there is a good reason. One key argument used several times in this section is that during the recursion the discrepancy already accrued is split up in the same ratio we try to split up the hyperedges. By organizing the recursive coloring process in a clever way we mananged to have some close to fifty-fifty splitting in later stages of the process keeping the final discrepancy low. In the linear discrepancy problem each hyperedges has its own individual ratio it shall be divided according to. Thus we have no chance to be clever — the only thing to do is to split up c colors into $\lfloor \frac{c}{2} \rfloor$ and $\lceil \frac{c}{2} \rceil$ colors to keep the number of steps small. Having no information about splitting up 'old' discrepancies, we end up with a bound of

$$\mathrm{lindisc}(\mathcal{H}, c) \leq \mathcal{O}(\log(c) \max_{\mathcal{H}_0 \leq \mathcal{H}} \mathrm{lindisc}(\mathcal{H}_0)).$$

In the Beck–Fiala situation this yields $\mathrm{lindisc}(\mathcal{H}, c) = \mathcal{O}(\log(c)\Delta)$. Compare this with Theorem 2.25 obtained via the vector color approach! Already the recursive result on the weighted discrepancy is inferior to this.

A second point to keep in mind is that to apply recursion, we need a two–color result on the hereditary discrepancy, even in the case that c is a power of 2. See the examples in Section 2.1 or Section 2.7.

2.5 Vector-Coloring

In this section we present a direct method to construct low discrepancy c–colorings. We extend the Beck–Fiala theorem and the Barany–Grunberg theorem to any number of colors. In the 2–color case both are proved using 'floating colors'. Colors initially floating in $[-1, 1]$ are successively changed to colors in $\{-1, 1\}$. Linear algebra is the key tool there.

To prove c–color versions we need the vector colors and the matrix calculus introduced in Section 2.1. In the Beck–Fiala situation we derive a bound independent on the number of colors (and twice the bound of the original result), whereas in the Barany–Grunberg case our bound is $(c-1)$ times the original bound (and thus coincides with the original result in the case $c = 2$).

2.5.1 The Theorem of Beck and Fiala

Denote by $\Delta(\mathcal{H}) := \max_{x \in X} |\{E \in \mathcal{E} | x \in E\}|$ the maximum degree of the hypergraph $\mathcal{H}$. This is one of the few parameters of a hypergraph which give a good bound on the discrepancy. The Beck–Fiala theorem [BF81], cf. Theorem 1.3, assures that $\mathrm{disc}(\mathcal{H}) < 2\Delta(\mathcal{H})$ holds for any hypergraph.

Beck and Fiala actually proved a more general result on the linear discrepancy of matrices. For any matrix $A = (a_{ij}) \in \mathbb{R}^{m \times n}$ denote by $\|A\|_1 := \max_{j \in [n]} \sum_{i \in [m]} |a_{ij}|$ the operator norm induced by the 1–norm on $\mathbb{R}^n$. Then $\mathrm{lindisc}(A) < 2\|A\|_1$ holds for any matrix A. For c colors we show

Theorem 2.25. *For any matrix A we have*

$$\mathrm{lindisc}(A, c) < 2\|A\|_1.$$

The following very elementary remark plays a crucial role in the proofs of the multi-color versions of both the Beck–Fiala theorem and the Barany–Grunberg theorem.

Lemma 2.26. *Let $x \in \overline{M}_c$. Assume that there is a $j' \in [c]$ such that $x_{j'} \notin \{-\frac{1}{c}, \frac{c-1}{c}\}$. Then there is a second index j'' (different from j') such that $x_{j''} \notin \{-\frac{1}{c}, \frac{c-1}{c}\}$.*

Proof. By assumption we have $cx_{j'} \notin \mathbb{Z}$. As $c \sum_{j \in [c]} x_j = 0 \in \mathbb{Z}$ by definition of $\overline{M}_c$, there exists a $j'' \in [c]$, $j' \neq j''$, such that $cx_{j''} \notin \mathbb{Z}$. In particular, $x_{j''} \notin \{-\frac{1}{c}, \frac{c-1}{c}\}$. □

Proof of Theorem 2.25. Set $\Delta := \|A\|_1$ and $\overline{A} = (\overline{a}_{ij}) := A \otimes I_c$. Note that $\Delta = \|\overline{A}\|_1$. Let $p : [n] \to \overline{M}_c$. We show $\mathrm{lindisc}(A, c, p) < 2\|A\|_1$ by constructing a $\chi : [n] \to M_c$ such that $\|A(p - \chi)\|_\infty < 2\|A\|_1$.

Set $\chi = p$. Successively we will change χ to a mapping $[n] \to M_c$. Recall that we agreed to regard p and χ as cn–dimensional vectors.

Put $J := \{j \in [cn] | \chi_j \notin \{-\frac{1}{c}, \frac{c-1}{c}\}\}$. We call the columns in J floating and the others fixed. Set $I := \{i \in [cm] | \sum_{j \in J} |\overline{a}_{ij}| > 2\Delta\}$, and call the rows from I active, the others ignored. We will ensure that during the rounding process the following conditions are fulfilled (this is clear for the start, because $\chi = p$):

(i) $(\overline{A}(p - \chi))_{|I} = 0$, i. e. all active rows have discrepancy zero.

(ii) All colors are in $\overline{M}_c$, in particular we have $\sum_{k=0}^{c-1} \chi_{cj-k} = 0$ for all $j \in [n]$.

Note that (ii) is the crucial difference to the 2–color case, where we only need a condition of type (i). This increases the number of equations investigated below and is the reason why the multi-color bound is twice the 2–color bound.

Let us assume that the rounding process is at a stage where J and I are as above and (i) and (ii) hold. If there is no floating color, i. e. $J = \emptyset$, then all χ_j, $j \in [cn]$, are in $\{-\frac{1}{c}, \frac{c-1}{c}\}$ and χ has the desired form.

Hence assume that there are still floating colors. We consider the system of equations

$$\sum_{k=0}^{c-1} \chi_{cj-k} = 0,\ j \in [n] \text{ such that } c(j-1)+k \in J \text{ for some } k \in [c]. \tag{2.4}$$

By Lemma 4.3, in every equation of (2.4) there are at least two floating variables $\chi_{j'}, \chi_{j''}$, i. e. $j', j'' \in J$. Thus (2.4) is a system of at most $\frac{1}{2}|J|$ equations.

We have

$$|J|\,\Delta \geq \sum_{j\in J}\sum_{i\in I} |\overline{a}_{ij}| = \sum_{i\in I}\sum_{j\in J} |\overline{a}_{ij}| > |I|\,2\Delta,$$

hence $|J| > 2|I|$. We conclude that the system

$$\begin{aligned} (\overline{A}\,\chi)_{|I} &= 0 \\ \sum_{k=0}^{c-1} \chi_{cj-k} &= 0,\ j \in [n] \text{ such that } c(j-1)+k \in J \text{ for some } k \in [c] \end{aligned} \tag{2.5}$$

consists of at most $|I| + \frac{1}{2}|J| < |J|$ equations and hence is under-determined (taking just the $\chi_j, j \in J$ as variables). Thus there is a non-trivial solution $x \in \mathbb{R}^J$ for (2.5). We extend x to $x_E \in \mathbb{R}^{cn}$ by

$$(x_E)_j := \begin{cases} x_j & \text{if } j \in J \\ 0 & \text{else} \end{cases}.$$

By (ii) and the definition of J, all variables $\chi_j, j \in J$ are in $]-\frac{1}{c}, \frac{c-1}{c}[$. Thus there is a $\lambda > 0$ such that at least one component of $\chi + \lambda x_E$ becomes fixed and all colors are still in $\overline{M}_c$, i. e. $\chi + \lambda x_E \in \overline{M}_c^n$. Note that $\chi + \lambda x_E$ also fulfills (i) since $(\overline{A}x_E)_{|I} = 0$. Set $\chi := \chi + \lambda x_E$. Since (i), (ii) are fulfilled for this new χ, we can continue this rounding process until all χ_j, $j \in [cn]$ are in $\{-\frac{1}{c}, \frac{c-1}{c}\}$.

We show $\|\overline{A}(p-\chi)\|_\infty < 2\Delta$. Let $i \in [cm]$. Denote by $\chi^{(0)}$ and $J^{(0)}$ the values of χ and J when the row i first became ignored. We have $\chi_j^{(0)} = \chi_j$ for all $j \notin J^{(0)}$ and $|\chi_j^{(0)} - \chi_j| < 1$ for all $j \in J^{(0)}$. Note that $\sum_{j\in J^{(0)}} |\overline{a}_{ij}| \leq 2\Delta$, since i is ignored. Thus

$$|(\overline{A}(p-\chi))_i| = |(\overline{A}(p-\chi^{(0)}))_i + (\overline{A}(\chi^{(0)}-\chi))_i| = |0 + \sum_{j\in J^{(0)}} \overline{a}_{ij}(\chi_j^{(0)} - \chi_j)| < 2\Delta.$$

This shows $\|A(p-\chi)\|_\infty < 2\|A\|_1$, and thus the claim. □

We note that our bound specialized to the case $c = 2$ is twice the bound of Beck–Fiala. This is similar to the phenomenon that also occurred in the investigation of the elementary probabilistic method. The reason in both cases is that in two colors one actually keeps the discrepancy of just one color small as discrepancy in the other color is the same.

For the c–color discrepancy we have

Corollary 2.27. $\mathrm{disc}(\mathcal{H}, c) < 2\Delta(\mathcal{H})$.

Note that this result is very similar to Theorem 2.15 which is a combination the original Beck–Fiala theorem and the recursive method. Depending on c it is a little better in some cases and a little worse in others.

2.5.2 The Theorem of Barany and Grunberg

Let $\|\cdot\|$ be any norm on $\mathbb{R}^n$. The theorem of Barany and Grunberg states that for any finite sequence $v_1, v_2, \ldots, v_k \in \mathbb{R}^n$ of vectors of norm at most 1 there are signs $\varepsilon_i \in \{-1, +1\}, i = 1, \ldots, k$ such that for all $l \in [k]$ we have

$$\left\| \sum_{i=1}^{l} \varepsilon_i v_i \right\| < 2n.$$

This seems to be similar to the Beck–Fiala theorem, but has a slightly different flavor: Here all partial sums are considered, and we may choose any norm for the input and the discrepancy. The Beck–Fiala theorem formulated in terms of a vector sequence states that for any vectors $v_1, \ldots, v_k$ of $\|\cdot\|_1$–norm at most one there are signs $\varepsilon_i \in \{-1, +1\}, i = 1, \ldots, k$ such that $\left\|\sum_{i=1}^{k} \varepsilon_i v_i\right\|_\infty < 2$. Thus neither theorem is a special case of the other.

As in the discrepancy problem of hypergraphs the signs -1 and $+1$ are a convenient way to represent a partition. From this point of view the theorem of Barany–Grunberg states that there is a 2–partition (I_1, I_2) of the set $X = \{v_1, \ldots, v_k\}$ such that for any subset $X_0 = \{v_1, \ldots, v_l\}$

$$\left\| \sum_{v \in I_j \cap X_0} v - \frac{1}{2} \sum_{v \in X_0} v \right\| < n$$

holds for both $j = 1, 2$. This motivates the following definition:

Definition (Discrepancy of vector sets and sequences). Let X be a finite set of vectors in $\mathbb{R}^n$ and $\mathcal{P} = (I_1, \dots, I_c)$ a c–partition of X. Let $\|\cdot\|$ be any norm on $\mathbb{R}^n$. We define the *discrepancy of the set* X w. r. t. $\mathcal{P}$ and $\|\cdot\|$ by

$$\operatorname{disc}(\mathcal{P}, \|\cdot\|) = \max_{j\in[c]} \left\| \sum_{v\in I_j} v - \tfrac{1}{c} \sum_{v\in X} v \right\|.$$

Given a subset $X_0 \subseteq X$ set $\mathcal{P}_{|X_0} := (I_1 \cap X_0, \dots, I_c \cap X_0)$. Let $v_1, v_2, \dots, v_k$ be a finite sequence of vectors and $\mathcal{P} = (I_1, \dots, I_c)$ be a c–partition of $\{v_1, v_2, \dots, v_k\}$. We define the *discrepancy of the sequence* $v_1, v_2, \dots, v_k$ w. r. t. $\mathcal{P}$ and $\|\cdot\|$ by

$$\operatorname{disc}((v_l)_{l\in[k]}, \mathcal{P}, \|\cdot\|) = \max_{l\in[k]} \operatorname{disc}(\mathcal{P}_{|\{v_1,\dots,v_l\}}, \|\cdot\|).$$

In this notation the Barany–Grunberg theorem states that there is a 2–partition $\mathcal{P} = (I_1, I_2)$ such that $\operatorname{disc}((v_l)_{l\in[k]}, \mathcal{P}, \|\cdot\|) < n$. We define a norm $\|\cdot\|_c$ on $\mathbb{R}^{cn}$ by

$$\|w\|_c := \max_{j\in[c]} \|w_{|\{j,j+c,\dots,j+(n-1)c\}}\|,$$

where we write $w_{|\{j,j+c,\dots,j+(n-1)c\}}$ to denote the vector $(w_j, w_{j+c}, \dots, w_{j+(n-1)c})^\top \in \mathbb{R}^n$. We can now express the discrepancies of vector sets and sequences in terms of the tensor product calculus:

Lemma 2.28. *Let $X \subseteq \mathbb{R}^n$ be a finite set of vectors and $\mathcal{P} = (I_1, \dots, I_c)$ be any c–partition of X. Let $\chi : X \to [c]$ be the corresponding coloring, i. e. for all $v \in X, l \in [c]$ we have $\chi(v) = l$ if and only if $v \in I_l$. Then the discrepancy of X w. r. t. $\mathcal{P}$ and $\|\cdot\|$ is*

$$\operatorname{disc}(\mathcal{P}, \|\cdot\|) = \left\| \sum_{v\in X} v \otimes m^{(\chi(v))} \right\|_c.$$

For vector sequences we have

$$\operatorname{disc}((v_l)_{l\in[k]}, \mathcal{P}, \|\cdot\|) = \max_{l\in[k]} \left\| \sum_{i=1}^{l} v_i \otimes m^{(\chi(v_i))} \right\|_c.$$

Proof. Remember that

$$m_j^{(\chi(v))} := \begin{cases} 1 - \frac{1}{c} & \text{if } \chi(v) = j \\ -\frac{1}{c} & \text{otherwise.} \end{cases}$$

Thus

$$\sum_{v\in X} m_j^{\chi(v)} v = \sum_{\substack{v\in X \\ \chi(v)=j}} (1 - \tfrac{1}{c}) v - \sum_{\substack{v\in X \\ \chi(v)\neq j}} \tfrac{1}{c} v = \sum_{\substack{v\in X \\ \chi(v)=j}} v - \tfrac{1}{c} \sum_{v\in X} v. \tag{2.6}$$

We have

$$\begin{aligned}
\left\| \sum_{v\in X} v \otimes m^{(\chi(v))} \right\|_c &= \max_{j\in[c]} \left\| \left(\sum_{v\in X} v\otimes m^{(\chi(v))} \right)\Big|_{\{j,j+c,\dots,j+(n-1)c\}} \right\| \\
&= \max_{j\in[c]} \left\| \sum_{v\in X} m_j^{(\chi(v))} v \right\| \\
&\overset{(2.6)}{=} \max_{j\in[c]} \left\| \sum_{v\in I_j} v - \tfrac{1}{c}\sum_{v\in X} v \right\| \\
&= \operatorname{disc}(\mathcal{P}, \|\cdot\|).
\end{aligned}$$

□

We are now ready to prove the following multi-color version of the Barany–Grunberg theorem:

Theorem 2.29. *Let $\|\cdot\|$ be any norm on $\mathbb{R}^n$ and $v_1, v_2, \dots, v_k$ be a finite sequence of vectors of norm at most 1 in $\mathbb{R}^n$. Then there is a c–partition $\mathcal{P} = (I_1, \dots, I_c)$ of $\{v_1, v_2, \dots, v_k\}$ such that*

$$\operatorname{disc}((v_l)_{l\in[k]}, \mathcal{P}, \|\cdot\|) < (c-1)n.$$

Proof. We may assume $k > n$. By Lemma 2.28 it suffices to show the existence of a coloring $\chi : [k] \to M_c$ such that $\left\| \sum_{i\in[l]} v_i \otimes \chi^{(i)} \right\|_c < (c-1)n$ for all $l \in [k]$.

As in the proof of the Barany-Grunberg theorem we give an algorithmic construction of χ. At the beginning define $A := [n]$ and $\chi_j^{(i)} := 0$ for all $i \in [k]$, $j \in [c]$. Let us call those $\chi_j^{(i)}$ where $i \in A$ and $\chi_j^{(i)} \notin \{\frac{c-1}{c}, -\frac{1}{c}\}$ *variables* and the corresponding color vector $\chi^{(i)}$ *active*. Hence at the beginning we have cn variables and n active color vectors. Furthermore all color vectors $\chi^{(i)}, i \in [k]$ are in $\overline{M_c}$ and we have $\sum_{i\in A} v_i \otimes \chi^{(i)} = 0$.

We repeat the following rounding process: Set $A_0 := \{i \in [k] | \exists j \in [c] : \chi_j^{(i)} \notin \{\frac{c-1}{c}, -\frac{1}{c}\}\}$, the set of indices of active color vectors. We try to find a nontrivial solution of the system of equations

$$\begin{aligned}
\sum_{i\in A} v_i \otimes \chi^{(i)} &= 0 \\
\sum_{j\in[c]} \chi_j^{(i)} &= 0 \quad \text{for all } i \in A_0.
\end{aligned} \tag{2.7}$$

Let n' be the number of variables and m' the rank of the system (2.7). By Lemma 2.26, each active vector contains at least two variables, so $n' \geq 2A_0$. On the other hand,

$m' \leq (c-1)n + |A_0|$, since $\sum_{j\in[c]} \chi_j^{(i)} = 0$ for all $i \in [k]$ holds at any stage of the rounding process.

If there is no nontrivial solution to (2.7), then there are at most m' variables. From $2A_0 \leq n' \leq m' \leq (c-1)n + |A_0|$ we conclude $|A_0| \leq (c-1)n$. If there are still vectors that have not been active, i. e. $A \neq [k]$, we increase the number of active vectors by setting $A := A \cup \{\max(A)+1\}$ and continue the rounding process considering the updated system (2.7). If $A = [k]$ we terminate the rounding process by changing the remaining variables to $\frac{c-1}{c}$ or $-\frac{1}{c}$ in any way such that all $\chi^{(i)}$ are in M_c.

If there is a nontrivial solution to (2.7), then we can change χ in the way that some variables become $\frac{c-1}{c}$ or $-\frac{1}{c}$ and all variables stay in $[-\frac{1}{c}, \frac{c-1}{c}]$ in the same fashion as in the proof of Beck–Fiala. Note that the invariants $\chi^{(i)} \in \overline{M_c}$ for all $i \in [k]$ and $\sum_{i\in A} v_i \otimes \chi^{(i)} = 0$ are still satisfied. Hence we can continue the rounding process.

For the analysis let $l \in [k]$. Denote by $\widetilde{\chi}^{(1)}, \ldots, \widetilde{\chi}^{(k)}$ the value of the color vectors at that stage of the rounding process when $A = [l]$ and no nontrivial solution to (2.7) can be found. Denote by $\widetilde{A_0}$ the value of A_0 at this stage. Let $\chi_f^{(1)}, \ldots, \chi_f^{(k)}$ denote the final values of the color vectors. From above we know $|\widetilde{A_0}| \leq (c-1)n$. Since $\widetilde{\chi}^{(i)} \in \overline{M_c}$ we have $\|\widetilde{\chi}^{(i)} - \chi_f^{(i)}\|_\infty < 1$ for all $i \in [l]$. Furthermore $\widetilde{\chi}^{(i)} = \chi_f^{(i)}$ holds if $i \notin \widetilde{A}_0$, since an inactive vector never becomes active again. By (2.7) we also have the equation $\sum_{i\in[l]} v_i \otimes \widetilde{\chi}^{(i)} = 0$.

Now

$$\begin{aligned}
\left\| \sum_{i\in[l]} v_i \otimes \chi_f^{(i)} \right\|_c &\leq \underbrace{\left\| \sum_{i\in[l]} v_i \otimes \widetilde{\chi}_f^{(i)} \right\|_c}_{=0 \text{ by } (2.7)} + \left\| \sum_{i\in[l]} v_i \otimes (\chi_f^{(i)} - \widetilde{\chi}^{(i)}) \right\|_c \\
&= \left\| \sum_{i\in\widetilde{A}_0} v_i \otimes (\chi_f^{(i)} - \widetilde{\chi}^{(i)}) \right\|_c \\
&= \max_{j\in[c]} \left\| \sum_{i\in\widetilde{A}_0} (v_i \otimes (\chi_f^{(i)} - \widetilde{\chi}^{(i)}))|_{\{j,j+c,\ldots,j+(n-1)c\}} \right\| \\
&= \max_{j\in[c]} \left\| \sum_{i\in\widetilde{A}_0} v_i (\chi_f^{(i)} - \widetilde{\chi}^{(i)})_j \right\| \\
&< \sum_{i\in\widetilde{A}_0} \|v_i\| \\
&\leq (c-1)n.
\end{aligned}$$

□

Note that the fact that we never have more than $(c-1)n$ active vectors has a nice interpretation in terms of online algorithms. Consider the following online vector balancing problem, stated in the language of games: Each round the first player selects a vector of norm at most one. The second player has to partition these vector into c classes. His aim its to reach a balanced partition in the end. He does not have to decide in which partition class to put the vector immediately but he can postpone his decision for up to $(c-1)n$ vectors. On the other hand, a vector assigned to a partition class can not be removed anymore.

The analysis of this game is just the proof above. The second player thus can keep the imbalance at the end of the game below $(c-1)n$.

2.6 Lower Bounds

In this section we give a general lower bound and analyze two prominent examples: Hypergraphs arising from Hadamard matrices and arithmetic progressions. We start with the c–color version of a result attributed to Lovász and Sós in [BS95]. This states that any matrix $A \in \mathbb{R}^{m \times n}$ has 2–color discrepancy at least $\operatorname{disc}(A) \geq \sqrt{\frac{n}{m}\lambda_{\min}(A^\top A)}$, where $\lambda_{\min}(\cdot)$ denotes the least eigenvalue of a matrix. For c colors we show

Theorem 2.30. *Let $A \in \mathbb{R}^{m \times n}$. Then $\operatorname{disc}(A,c) \geq \sqrt{\frac{(c-1)}{c^2}\frac{n}{m}\lambda_{\min}(A^\top A)}$.*

Proof. Let $\chi : [n] \to M_c$ be an optimal coloring with respect to c–color discrepancy. Then

$$\begin{aligned}
\operatorname{disc}(A,c) &= \|(A \otimes I_c)\chi\|_\infty \\
&\geq \frac{1}{\sqrt{cm}}\|(A \otimes I_c)\chi\|_2 \\
&\geq \frac{1}{\sqrt{cm}}\|\chi\|_2\sqrt{\lambda_{\min}((A \otimes I_c)^\top (A \otimes I_c))} \\
&\overset{2.5(iii)}{=} \frac{1}{\sqrt{cm}}\sqrt{\frac{n(c-1)}{c}}\sqrt{\lambda_{\min}((A^\top A) \otimes I_c)} \\
&\overset{2.5(v)}{=} \sqrt{\frac{(c-1)}{c^2}\frac{n}{m}}\sqrt{\lambda_{\min}(A^\top A)}.
\end{aligned}$$

□

2.6.1 Hadamard Matrices

Hypergraphs corresponding to Hadamard matrices show that Spencer's 'six standard deviations' result is best possible apart from constant factors. The following theorem extends this result to c colors. It even shows that the dependence from the number of colors detected in Theorem 2.21 is relatively tight.

Theorem 2.31. *There is a universal constant $K > 0$ such that for all $n \in \mathbb{N}$ such that there exists a Hadamard matrix of order n there is a hypergraph with n vertices and n edges having c–color discrepancy at least $K\sqrt{\frac{n}{c}}$.*

Note that it is known that there are infinitely many n such that a Hadamard matrix of order n exists. Moreover, the set of orders is dense in the sense that for all $\varepsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ there is a Hadamard matrix of order between n and $n(1+\varepsilon)$.

Proof. Let $n \in \mathbb{N}$ be such that there exists a Hadamard matrix H of order n, i. e. $H \in \{+1, -1\}^{n \times n}$ and all rows of H are pairwise orthogonal. By multiplying some rows by -1 we may assume that all entries of the first column v_1 are 1. Let $v_2, \ldots, v_n$ denote the remaining columns. Set $A = \frac{1}{2}(H + J)$, where J is the $n \times n$ matrix consisting of 1s only. A is the incidence matrix of a hypergraph $\mathcal{H}$ of n edges on n vertices. We show that $\mathcal{H}$ has the desired discrepancy.

Let $\chi : [n] \to M_c$ be any coloring. Let $i \in [c]$ be such that

$$|\chi^{-1}(m^{(i)}) \setminus \{1\}| \geq \tfrac{n-1}{c}. \tag{2.8}$$

For all $j \in [c]$ set $\chi_j : [n] \to \{-\frac{1}{c}, \frac{c-1}{c}\}; k \mapsto \chi(k)_j$. Then

$$\begin{aligned} \operatorname{disc}_\chi(\mathcal{H}, c) &= \|(A \otimes I_c)\chi\|_\infty \\ &= \max_{j \in [c]} \|A\chi_j\|_\infty \\ &\geq \|A\chi_i\| \\ &\geq \tfrac{1}{\sqrt{n}}\|A\chi_i\|_2. \end{aligned}$$

(2.8) now yields

$$|\{k \in [n] \setminus \{1\} \mid \chi_i(k) = \tfrac{c-1}{c}\}| \geq \tfrac{n-1}{c}. \tag{2.9}$$

By definition of A there is a $\lambda \in \mathbb{R}$ such that $A\chi_i = \sum_{k=2}^{n} \frac{1}{2}\chi_i(k)v_k + \lambda v_1$. Since the

$v_1, \ldots, v_n$ are pairwise orthogonal, we have

$$
\begin{aligned}
\|A\chi_i\|_2 &= \sqrt{\sum_{k=2}^{n} \tfrac{1}{4}\chi_i(k)^2\|v_k\|_2^2 + \lambda^2\|v_1\|_2^2} \\
&\geq \sqrt{\sum_{k=2}^{n} \tfrac{1}{4}\chi_i(k)^2\|v_k\|_2^2} \\
&= \tfrac{1}{2}\sqrt{n}\sqrt{\sum_{k=2}^{n} \chi_i(k)^2} \\
&\geq \tfrac{1}{2}\sqrt{n}\sqrt{\tfrac{n-1}{c}\left(\tfrac{c-1}{c}\right)^2 + \tfrac{(n-1)(c-1)}{c}\left(-\tfrac{1}{c}\right)^2} \qquad \text{(by (2.9))} \\
&= \tfrac{1}{2}\sqrt{n}\sqrt{\tfrac{(n-1)(c-1)}{c^2}}.
\end{aligned}
$$

Hence $\operatorname{disc}(\mathcal{H}, c) \geq \frac{1}{2}\sqrt{\frac{(n-1)(c-1)}{c^2}}$. □

2.6.2 Arithmetic Progressions

We now give a lower bound for the multi-color discrepancies of the arithmetic progressions. See Section 2.4.4 for a short survey of this problem and an upper bound. For 2 colors, Roth [Rot64] proved the celebrated lower bound $\operatorname{disc}(\mathcal{A}_n) \geq \frac{1}{20}\sqrt[4]{n}$. A similar result is true for any number of colors. We have

Theorem 2.32. *The hypergraph of arithmetic progressions fulfills*

$$\operatorname{disc}(\mathcal{A}_n, c) \geq 0.04\,\tfrac{1}{\sqrt{c}}\,\sqrt[4]{n}.$$

Proof. We follow the approach of [BS95]. Set $k = \lfloor\sqrt{\frac{1}{6}n}\rfloor$. Let $\mathcal{E}$ be the set of arithmetic progressions of length k and difference less than $6k$ computed modulo n (hence our arithmetic progressions may be over-wrapped from n to 1 at most once). Every arithmetic progression of $\mathcal{E}$ is a union of at most two arithmetic progressions from $\mathcal{E}_n$, so the discrepancy of $\mathcal{A}_n$ is at least half the discrepancy of $([n], \mathcal{E})$.

Recall that a matrix is called circulant if the i-th row can be obtained from the first by (circular) shifting it $i-1$ times to the right. Let us enumerate the arithmetic progressions in $\mathcal{E}$ in a way that if i is not divisible by n, then $E_{i+1} = E_i + 1$ (always computed modulo n), i. e. E_{i+1} is E_i shifted right by one. Thus the incidence matrix $A = (a_{ij}) \in \{0,1\}^{6kn \times n}$ defined by $a_{ij} = 1$ if and only if $j \in E_i$ consists of $6k$ circulant sub-matrices. As sum and product of two circulant matrices is circulant again, $A^\top A$ is circulant. The eigenvectors of circulant matrices are known to be of the form $(1, \varepsilon, \varepsilon^2, \dots, \varepsilon^{n-1})^\top$, where ε is an nth root of unity. Using this one gets that the minimum eigenvalue $\lambda_{\min}(A^\top A)$ of $A^\top A$ is greater than $\frac{1}{4}k^2$.

Theorem 2.30 gives $\operatorname{disc}(([n], \mathcal{E}_n), c)^2 \geq \frac{n(c-1)}{6knc^2}\frac{1}{4}k^2 = \frac{(c-1)k}{24c^2}$. Hence

$$\begin{aligned}
\operatorname{disc}(\mathcal{A}_n, c) &\geq 0.5 \operatorname{disc}(([n], \mathcal{E}), c) \\
&\geq \sqrt{\frac{c-1}{96\sqrt{6}c^2}}\sqrt[4]{n} \\
&\geq 0.0652\sqrt{\frac{c-1}{c^2}}\sqrt[4]{n} \\
&\geq 0.04\sqrt{\tfrac{1}{c}}\sqrt[4]{n}.
\end{aligned}$$

□

We may remark that the lower bound can also be proved using harmonic analysis approach of [Weh97, DSW98]. This has been done in [DSW00], where we also proved that a low discrepancy coloring has discrepancy of the order shown above in *each* color.

Another alternative is taking a look at some very tricky notation in Roth's [Rot64] original paper. Then one finds $\operatorname{wd}(\mathcal{A}_n, 2, (p, 1-p)) = \Omega(\sqrt{p(1-p)}\sqrt[4]{n})$. Since $\operatorname{wd}(\mathcal{H}, 2, (\frac{1}{c}, 1 - \frac{1}{c})) \leq \operatorname{disc}(\mathcal{H}, c)$ holds for all hypergraphs $\mathcal{H}$, this also proves the claim. Note that our proof implicitly also uses this relation between weighted 2–color discrepancy and c–color discrepancy.

2.7 Interrelation between Different Numbers of Colors

In this section we investigate the relation of the discrepancy of one hypergraph in different numbers of colors. The preceding sections might suggest that it makes little difference which number of colors use, but this impression is false. In general there is little correlation between the discrepancies of a hypergraph in different numbers of colors. There is one exception to this rule:

Lemma 2.33. *Let $\mathcal{H} = (X, \mathcal{E})$ be any hypergraph. If c_2 divides c_1, then*

$$\operatorname{disc}(\mathcal{H}, c_2) \leq \frac{c_1}{c_2}\operatorname{disc}(\mathcal{H}, c_1).$$

Proof. Let $\chi : X \to [c_1]$ be an optimal c_1–coloring of $\mathcal{H}$. Set $q = \frac{c_1}{c_2}$ and

$$\chi_2 : X \to [c_2]; x \mapsto \left\lfloor \frac{\chi(x) - 1}{q} \right\rfloor + 1.$$

For an edge $E \in \mathcal{E}$ and a color $j \in [c_2]$ we have

$$\begin{aligned}
\mathrm{disc}_{\chi_2,j}(E) &= \left| |E \cap \chi_2^{-1}(j)| - \tfrac{1}{c_2}|E| \right| \\
&= \left| \left| E \cap \bigcup_{k=1}^{q} \chi^{-1}((j-1)q + k) \right| - \tfrac{q}{c_1}|E| \right| \\
&= \left| \sum_{k=1}^{q} \left(|E \cap \chi^{-1}((j-1)q + k)| - \tfrac{1}{c_1}|E| \right) \right| \\
&\leq \sum_{k=1}^{q} \left| |E \cap \chi^{-1}((j-1)q + k)| - \tfrac{1}{c_1}|E| \right| \\
&= \sum_{k=1}^{q} \mathrm{disc}_{\chi,(j-1)q+k}(E).
\end{aligned}$$

□

This marks the extreme case where the correlation between different numbers of colors is high. For arbitrary numbers of colors a quite different phenomenon can be observed. We investigate the opposite behavior through the following class of hypergraphs.

Fix $n, k \in \mathbb{N}$. For $l \in [k]$ set $L_l := [n] \times \{l\}$ and call these sets 'lines'. Set

$$\mathcal{E}_{nk} := \{E \subseteq [n] \times [k] | \forall l_1, l_2 \in [k] : |E \cap L_{l_1}| = |E \cap L_{l_2}|\}.$$

$\mathcal{H}_{nk} := ([n] \times [k], \mathcal{E}_{nk})$ is the (modulo isomorphism) unique hypergraph on nk vertices having k–color discrepancy zero with maximal number of edges. We determine the discrepancy of this hypergraph in any number of colors. This will show that there is little correlation apart from the case exhibited in Lemma 2.33.

Theorem 2.34. *Let* $k_0 := k \mod c$. *Set* $b_{nkc} = \left(n - \left\lfloor \frac{n}{\lceil \frac{c}{k} \rceil} \right\rfloor \right) \frac{k}{c}$. *Then*

$$b_{nk_0c} \leq \mathrm{disc}(\mathcal{H}_{nk}, c) < b_{nk_0c} + 1,$$

if $k_0 \neq 0$, *and* $\mathrm{disc}(\mathcal{H}_{nk}, c) = 0$, *if* c *divides* k.

It will be convenient to consider the case $c > kn$ (the number of color exceeds the number of vertices) separately. Any coloring χ avoids some colors, and for such a color j we have

$\mathrm{disc}_{\chi,j}([n] \times [k]) = \frac{nk}{c}$. For an occurring color j there is an edge E of size k containing at least one point colored j. Hence $\mathrm{disc}_{\chi,j}(E) \geq 1 - \frac{k}{c}$. On the other hand a coloring such that every color occurs at most once shows that we actually have $\mathrm{disc}(\mathcal{H}_{nk}, c) = \max\{\frac{nk}{c}, 1 - \frac{k}{c}\}$. As $b_{nkc} \leq \max\{\frac{nk}{c}, 1 - \frac{k}{c}\} < 1$, the case $c > kn$ is settled.

The case that c divides k is solved by Lemma 2.33, so let us assume that $k_0 \neq 0$. To prove the theorem we start with an easy observation:

Lemma 2.35. *In the notation of Theorem 2.34 we have*

$$\mathrm{disc}(H_{nk}, c) \leq \mathrm{disc}(H_{nk_0}, c).$$

Proof. Let χ_0 be an optimal c–coloring for $\mathcal{H}_{nk_0}$. Set $X_0 = [n] \times \{c \left\lfloor \frac{k}{c} \right\rfloor + 1, \ldots, k\}$ and $\sigma : X_0 \to [n] \times [k_0]; (i,l) \mapsto (i, l - c \left\lfloor \frac{k}{c} \right\rfloor)$. Then σ is a hypergraph isomorphism from $\mathcal{H}_{nk}|_{X_0}$ to $\mathcal{H}_{nk_0}$. Define a coloring $\chi : [n] \times [k] \to [c]$ by

$$\chi((i,l)) := \begin{cases} \chi_0(\sigma(i,l)) & \text{if } (i,l) \in X_0 \\ \left\lfloor \frac{l-1}{\left\lfloor \frac{k}{c} \right\rfloor} \right\rfloor + 1 & \text{else.} \end{cases}$$

Then for any edge $E \in \mathcal{E}$ and any color $j \in [c]$ we have

$$\begin{aligned}
&\mathrm{disc}_{\chi,j}(E) \\
&= \left| \left| E \cap \chi^{-1}(j) \right| - \tfrac{1}{c} \left| E \right| \right| \\
&= \left| \left| E \cap X_0 \cap \chi^{-1}(j) \right| - \tfrac{1}{c} \left| E \cap X_0 \right| + \left| (E \setminus X_0) \cap \chi^{-1}(j) \right| - \tfrac{1}{c} \left| E \setminus X_0 \right| \right| \\
&= \left| \left| \sigma(E \cap X_0) \cap \chi_0^{-1}(j) \right| - \tfrac{1}{c} \left| \sigma(E \cap X_0) \right| \right. \\
&\quad \left. + \left| E \cap ([n] \times \{(j-1) \left\lfloor \tfrac{k}{c} \right\rfloor + 1, \ldots, j \left\lfloor \tfrac{k}{c} \right\rfloor\}) \right| - \tfrac{1}{c} \left| E \cap ([n] \times [c \left\lfloor \tfrac{k}{c} \right\rfloor]) \right| \right| \\
&= \left| \left| \sigma(E \cap X_0) \cap \chi_0^{-1}(j) \right| - \tfrac{1}{c} \left| \sigma(E \cap X_0) \right| + 0 \right| \\
&= \mathrm{disc}_{\chi_0,j}(\sigma(E \cap X_0)).
\end{aligned}$$

Since $\sigma(E \cap X_0)$ is an edge of $\mathcal{H}_{nk_0}$ and χ_0 an optimal c–coloring for $\mathcal{H}_{nk_0}$, we conclude $\mathrm{disc}(\mathcal{H}_{nk}, c) \leq \mathrm{disc}(\mathcal{H}_{nk_0}, c)$. □

It would save us some problems if we could show that an optimal coloring for $\mathcal{H}_{nk}$ in the case $c < k$ has to be of the kind constructed in Lemma 2.35. Unfortunately, this is not true:

Set $k = 5$ and $c = 3$. Let n be any non-negative integer divisible by 6. Let χ be such that the color classes intersect the lines as described in the table below (this defines χ up to permutations inside the lines which have no influence on the discrepancy).

Line l	$\frac{1}{n}\left\|L_l \cap \chi^{-1}(1)\right\|$	$\frac{1}{n}\left\|L_l \cap \chi^{-1}(2)\right\|$	$\frac{1}{n}\left\|L_l \cap \chi^{-1}(3)\right\|$
1	1	0	0
2	0	1	0
3	1/6	0	5/6
4	0	1/6	5/6
5	1/2	1/2	0

We determine the discrepancy of this coloring: An edge E such that $\mathrm{disc}_{\chi,1}(E)$ is maximal has to have $\frac{5}{2}$ points and either contains all or none of the 1–colored points of line 3 and 5. Both cases yield a discrepancy of $\frac{1}{3}n$ in color 1. The situation for color 2 is the same. For color 3 the extreme edges look like this: E has either no points in color 3 and size $\frac{5}{6}n$ and thus $\mathrm{disc}_{\chi,3}(E) = \frac{5}{18}$, or E contains all points in this color, has size $\frac{25}{6}$ and thus $\mathrm{disc}_{\chi,3}(E) = \frac{5}{18}$. From Theorem 2.34 we see that χ is an optimal coloring, but χ is not of the kind of coloring used to prove Lemma 2.35. This also shows that the optimal coloring is not even unique modulo permutations of lines and permutations inside lines.

Proof of Theorem 2.34. We will investigate the case $c > k$ first, hence we have $k = k_0$.

For the upper bound we construct a coloring χ. Start with all points being uncolored. For each color j, color $\left\lfloor \frac{n}{\lceil \frac{c}{k} \rceil} \right\rfloor$ points in this color. Do so in a way that points of the same color are in the same line. This is possible as each line can hold up to $\lceil \frac{c}{k} \rceil$ such color classes. The remaining points color in any way such that all color classes differ in size by at most one.

We calculate the discrepancy of $\mathcal{H}$ with respect to χ. Let $j \in [c]$. Let E be an edge such that $d_j(E) := |E \cap \chi^{-1}(j)| - \frac{1}{c}|E|$ is maximal. Assume that there is a point in color j that is not contained in E. Let F be a set of points disjoint from E containing one point of every line, at least one of these colored j. Then

$$d_j(E \cup F) = d_j(E) + d_j(F) \geq d_j(E) + 1 - \tfrac{k}{c} > d_j(E),$$

a contradiction. Hence $\chi^{-1}(j) \subseteq E$ and $|E| \geq k \left\lfloor \frac{n}{\lceil \frac{c}{k} \rceil} \right\rfloor$ by construction. This yields

$$\mathrm{disc}_{\chi,j}(E) = \left|E \cap \chi^{-1}(j)\right| - \tfrac{1}{c}|E| \leq \left\lceil \frac{nk}{c} \right\rceil - \tfrac{k}{c} \left\lfloor \frac{n}{\lceil \frac{c}{k} \rceil} \right\rfloor . \tag{2.10}$$

Now let E be an edge such that $d_j(E) := |E \cap \chi^{-1}(j)| - \frac{1}{c}|E|$ is minimal. By a similar argument as above we see that E contains no point colored j. As $c \leq kn$, we have

$|E| \leq \left(nk - k\left\lfloor\frac{n}{\lceil\frac{c}{k}\rceil}\right\rfloor\right)$ and hence

$$\mathrm{disc}_{\chi,j}(E) = \left|\left|E \cap \chi^{-1}(j)\right| - \tfrac{1}{c}\left|E\right|\right| = \tfrac{1}{c}\left|E\right| \leq \tfrac{1}{c}\left(nk - k\left\lfloor\frac{n}{\lceil\frac{c}{k}\rceil}\right\rfloor\right). \qquad (2.11)$$

From (2.10) and (2.11) we conclude

$$\mathrm{disc}(\mathcal{H}_{nk}, c) \leq \mathrm{disc}_\chi(\mathcal{H}_{nk}) \leq \left\lceil\frac{nk}{c}\right\rceil - \tfrac{k}{c}\left\lfloor\frac{n}{\lceil\frac{c}{k}\rceil}\right\rfloor < b_{nkc} + 1.$$

For the lower bound let χ be any coloring. Let $m : [c] \to [k]$ such that $|L_l \cap \chi^{-1}(j)| \leq \left|L_{m(j)} \cap \chi^{-1}(j)\right|$ for all $l \in [k]$ and $j \in [c]$. From the pigeon-hole principle we get a line number $l \in [k]$ such that $|m^{-1}(l)| \geq \lceil\frac{c}{k}\rceil$. Let $j \in m^{-1}(l)$ such that $|L_l \cap \chi^{-1}(j)|$ is minimal. Then for all $l \in [k]$ we have $|L_l \cap \chi^{-1}(j)| \leq \left\lfloor\frac{n}{\lceil\frac{c}{k}\rceil}\right\rfloor$. Thus there is an edge of size $\left(n - \left\lfloor\frac{n}{\lceil\frac{c}{k}\rceil}\right\rfloor\right) k$ having no point in this color. We conclude $\mathrm{disc}_\chi(\mathcal{H}_{nk}) \geq \left(n - \left\lfloor\frac{n}{\lceil\frac{c}{k}\rceil}\right\rfloor\right)\frac{k}{c} = b_{nkc}$.

We now turn to the case that $c < k$ and c does not divide k. Lemma 2.35 proves the upper bound. For the lower bound let χ be any c–coloring of $\mathcal{H}_{nk}$. Let $i : [c] \times [k] \to [k]$ such that $|L_{i(j,p)} \cap \chi^{-1}(j)| \geq |L_{i(j,p+1)} \cap \chi^{-1}(j)|$ for all $j \in [c], p \in [k-1]$. Hence $i(j,p)$ denotes the index of a line with p–most points in color j.

Assume first that there exists a $j \in [c]$ such that $\sum_{p=1}^{\lceil\frac{k}{c}\rceil}\left|L_{i(j,p)} \cap \chi^{-1}(j)\right| \leq \lfloor\frac{k}{c}\rfloor n + \left\lfloor\frac{n}{\lceil\frac{c}{k_0}\rceil}\right\rfloor$. If $\left|L_{i(j,\lceil\frac{k}{c}\rceil)} \cap \chi^{-1}(j)\right| < \left\lfloor\frac{n}{\lceil\frac{c}{k_0}\rceil}\right\rfloor$, then there is an edge E such that $|E| = k\left(n - \left\lfloor\frac{n}{\lceil\frac{c}{k_0}\rceil}\right\rfloor\right)$ and $|E \cap \chi^{-1}(j)| \leq \lfloor\frac{k}{c}\rfloor\left(n - \left\lfloor\frac{n}{\lceil\frac{c}{k_0}\rceil}\right\rfloor\right)$, hence $\mathrm{disc}_{\chi,j}(E) \geq b_{nk_0c}$. If $\left|L_{i(j,\lceil\frac{k}{c}\rceil)} \cap \chi^{-1}(j)\right| \geq \left\lfloor\frac{n}{\lceil\frac{c}{k_0}\rceil}\right\rfloor$, then there is an edge E such that $|E \cap L_l| = n - \left|L_{i(j,\lceil\frac{k}{c}\rceil)} \cap \chi^{-1}(j)\right|$ for all $l \in [k]$

and $|E \cap \chi^{-1}(j)| \leq \lfloor \frac{k}{c} \rfloor n + \left\lfloor \frac{n}{\lceil \frac{c}{k_0} \rceil} \right\rfloor - \lceil \frac{k}{c} \rceil |L_{i(j, \lceil \frac{k}{c} \rceil)} \cap \chi^{-1}(j)|$. In this case also

$$\begin{aligned}
\mathrm{disc}_{\chi,j}(E) &\geq \tfrac{k}{c}\left(n - \left|L_{i(j,\lceil \frac{k}{c} \rceil)} \cap \chi^{-1}(j)\right|\right) - \lfloor \tfrac{k}{c} \rfloor n \\
&\quad - \left\lfloor \frac{n}{\lceil \frac{c}{k_0} \rceil} \right\rfloor + \lceil \tfrac{k}{c} \rceil \left|L_{i(j,\lceil \frac{k}{c} \rceil)} \cap \chi^{-1}(j)\right| \\
&= \tfrac{k_0}{c} n + \left(1 - \tfrac{k_0}{c}\right) \left|L_{i(j,\lceil \frac{k}{c} \rceil)} \cap \chi^{-1}(j)\right| - \left\lfloor \frac{n}{\lceil \frac{c}{k_0} \rceil} \right\rfloor \\
&\geq \left(n - \left\lfloor \frac{n}{\lceil \frac{c}{k_0} \rceil} \right\rfloor\right) \tfrac{k_0}{c} = b_{nk_0c}
\end{aligned}$$

holds. So let us assume from now on

$$\sum_{l=p}^{\lceil \frac{k}{c} \rceil} |L_{i(j,p)} \cap \chi^{-1}(j)| > \left\lfloor \frac{k}{c} \right\rfloor n + \left\lfloor \frac{n}{\lceil \frac{c}{k_0} \rceil} \right\rfloor \quad \text{for all } j \in [c]. \tag{2.12}$$

Assume that there exists a color $j \in [c]$ such that $\left|L_{i(j,\lfloor \frac{k}{c} \rfloor)} \cap \chi^{-1}(j)\right| \leq n - \left\lfloor \frac{n}{\lceil \frac{c}{k_0} \rceil} \right\rfloor - 1$. From (2.12) we conclude $\left|L_{i(j,\lceil \frac{k}{c} \rceil)} \cap \chi^{-1}(j)\right| \geq 2 \left\lfloor \frac{n}{\lceil \frac{c}{k_0} \rceil} \right\rfloor + 2 := m$ and $k_0 \leq \frac{1}{3}c$. Thus there is an edge E such that $|E \cap L_l| = m$ for all $l \in [k]$ and $|E \cap \chi^{-1}(j)| \geq \lceil \frac{k}{c} \rceil m$. Our assumptions yield $\frac{1}{\lceil \frac{c}{k_0} \rceil} \geq \frac{3k_0}{4c}$. If $\left\lceil \frac{c}{k_0} \right\rceil$ does not divide n, we have

$$\begin{aligned}
\mathrm{disc}_{\chi,j}(E) &\geq \left\lceil \frac{k}{c} \right\rceil m - \frac{km}{c} \\
&= 2 \left\lceil \frac{n}{\lceil \frac{c}{k_0} \rceil} \right\rceil \left(1 - \frac{k_0}{c}\right) \\
&= \left(2 - \frac{k_0}{c}\right) \left\lceil \frac{n}{\lceil \frac{c}{k_0} \rceil} \right\rceil - \frac{k_0}{c} - \left\lfloor \frac{n}{\lceil \frac{c}{k_0} \rceil} \right\rfloor \frac{k_0}{c} \\
&\geq \left(2 - \frac{k_0}{c}\right) \frac{n}{\lceil \frac{c}{k_0} \rceil} + \left(2 - \frac{k_0}{c}\right) \frac{1}{\lceil \frac{c}{k_0} \rceil} - \frac{k_0}{c} - \left\lfloor \frac{n}{\lceil \frac{c}{k_0} \rceil} \right\rfloor \frac{k_0}{c} \\
&\geq (2 - \tfrac{1}{3})\tfrac{3}{4} \frac{nk_0}{c} + \frac{k_0}{2c} - \frac{3k_0^2}{4c^2} - \left\lfloor \frac{n}{\lceil \frac{c}{k_0} \rceil} \right\rfloor \frac{k_0}{c} \\
&\geq \left(n - \left\lfloor \frac{n}{\lceil \frac{c}{k_0} \rceil} \right\rfloor \right) \frac{k_0}{c} = b_{nk_0c}.
\end{aligned}$$

The calculation for the case that $\left\lceil \frac{c}{k_0} \right\rceil$ is a divisor of n is about the same, a little easier actually, as we do not have to care about the fractional parts.

It remains to look at the case that for all $j \in [c]$ we have $\left| L_{i(j,\lfloor \frac{k}{c} \rfloor)} \cap \chi^{-1}(j) \right| > n - \left\lfloor \frac{n}{\lceil \frac{c}{k_0} \rceil} \right\rfloor - 1$. In this case $i_{|[c]\times[\lfloor \frac{k}{c} \rfloor]}$ is injective and $i([c] \times [\lfloor \frac{k}{c} \rfloor]) \cap i([c] \times \{\lceil \frac{c}{k} \rceil\}) = \emptyset$. Thus i maps $[c] \times \{\lceil \frac{c}{k} \rceil\}$ into a set of size k_0. From the pigeon–hole principle we conclude that there is a line L_l such that $i(j, \lceil \frac{c}{k} \rceil) = l$ for $\left\lceil \frac{c}{k_0} \right\rceil$ different colors j. Again from the pigeon–hole principle we see that for at least one of these colors we have $|L_{i(j,\lceil \frac{k}{c} \rceil)} \cap \chi^{-1}(j)| \leq \frac{n}{\lceil \frac{c}{k_0} \rceil}$ contradicting (2.12). This ends the proof of Theorem 2.34. $\square$

From Lemma 2.3 we know that any hypergraph on nk points has c–color discrepancy at most $\lceil \frac{nk}{c} \rceil (1 - \frac{1}{c})$. The theorem above states that $\mathcal{H}_{nk}$ has c–color discrepancy $\mathrm{disc}(\mathcal{H}_{nk}, c) \geq \frac{nk_0}{c}(1 - \frac{k_0}{2c})$. This shows that there is little correlation between the discrepancies in different colors (as $\mathrm{disc}(\mathcal{H}_{nk}, k) = 0$).

Chapter 3

Linear and Hereditary Discrepancy

3.1 The Problem: History and Results

In this chapter we investigate a classical problem in the field of 2–color discrepancies. As already mentioned in Chapter 1 there is a non-trivial relation between the linear and the hereditary discrepancy of a hypergraph (and more generally, a matrix). Results of Beck and Spencer [BS84a] and Lovász, Spencer and Vesztergombi [LSV86] show that any real matrix $A \in \mathbb{R}^{m \times n}$ satisfies

$$\mathrm{lindisc}(A) \leq 2\,\mathrm{herdisc}(A).$$

Recall that this result was crucial for the recursive method in Chapter 2.

In [Spe87], Spencer improves this slightly to $\mathrm{lindisc}(A) \leq 2(1 - 2^{-2^n})\,\mathrm{herdisc}(A)$. He also provides a matrix A satisfying $\mathrm{lindisc}(A) = 2(1 - \frac{1}{n+1})\,\mathrm{herdisc}(A)$ and introduces the problem of closing this gap. For a whole bunch of these matrices, namely a characterization of all totally unimodular matrices satisfying this equality, see Theorem 4.2.

The main result of this chapter is:

Theorem 3.1. *Let A be any $m \times n$ matrix. Set $q := \lfloor \log_2(m) \rfloor + 1$. Then*

$$\mathrm{lindisc}(A) \leq 2\left(1 - 2^{-q}\right)\mathrm{herdisc}(A).$$

In particular, $\mathrm{lindisc}(A) \leq 2(1 - \frac{1}{2m})\,\mathrm{herdisc}(A)$.

If A is the incidence matrix of a hypergraph (i. e. $A \in \{0,1\}^{m \times n}$), then we may assume $m \leq 2^n - 1$, as each two rows can be assumed different and different from $(0, 0, \ldots, 0)$. In the language of hypergraphs this just means that no edge occurs twice and that the empty edge can be ignored. This yields $2\,(1 - 2^{-q}) \leq 2\,(1 - 2^{-n})$ and

Corollary 3.2. *For* $A \in \{0,1\}^{m \times n}$ *we have*

$$\operatorname{lindisc}(A) \leq 2(1-2^{-n}) \operatorname{herdisc}(A).$$

So for hypergraphs in particular, and matrices with $m \ll 2^{2^n}$ in general, our result is a considerable improvement of the known results; the improvement being better the sparser the matrix is. The result is not far from the optimum: The example of [Spe87] has

$$\operatorname{lindisc}(A) \quad = \quad 2\left(1-\frac{1}{m}\right) \operatorname{herdisc}(A),$$

while the Theorem 3.1 gives

$$\operatorname{lindisc}(A) \quad \leq \quad 2\left(1-\frac{1}{m+1}\right) \operatorname{herdisc}(A)$$

for $m = 2^l - 1, l \in \mathbb{N}$.

The case of totally unimodular matrices is settled in Chapter 4 using a different approach.

3.2 The Proof

Our proof uses the original proof of Beck and Spencer [BS84a], which we state here for convenience.

Original Proof: Let $p \in [-1,1]^n$. We will construct an $\varepsilon \in \{-1,1\}^n$ such that $\|A(p-\varepsilon)\|$ is small. Define $a^{(0)} \in [0,1]^n$ by $a_j^{(0)} := \frac{1}{2}(p_j+1)$ for all $j \in [n]$. As

$$p \mapsto \min_{\varepsilon \in \{-1,1\}^n} \|A(p-\varepsilon)\|_\infty$$

is a continuous function and $\{\sum_{i=1}^n x_i 2^{-i} \mid n \in \mathbb{N}, x \in \{0,1\}^n\}$ is dense in $[0,1]$, we may assume that there is $k \in \mathbb{N}$ such that $a_j^{(0)} 2^k \in \mathbb{Z}$ for all $j \in [n]$. We are going to round the vector $a^{(0)}$ successively to a vector of shorter binary expansion until we have a $0,1$ vector.

Suppose that for some $l \in \{0,\ldots,k-1\}$, the $(a_j^{(l)})_{j=1,\ldots,n}$ are already defined and satisfy $a_j^{(l)} 2^{k-l} \in \mathbb{Z}$ for all $j \in [n]$. Set $X := \{j \in [n] \mid a_j^{(l)} 2^{k-l} \text{ odd}\}$, the set of all j such that the binary expansion of $a_j^{(l)} 2^{k-l}$ ends in 1 (these are the components of $a^{(l)}$ that need to be rounded). Find $\varepsilon^{(l)} : X \to \{-1,+1\}$ such that

$$d_i^{(l)} := \sum_{j \in X} \varepsilon_j^{(l)} a_{ij} \in [-\operatorname{herdisc}(A), \operatorname{herdisc}(A)]$$

for all $i \in [m]$.

Define

$$a_j^{(l+1)} := \begin{cases} a_j^{(l)} + 2^{-(k-l)}\varepsilon_j^{(l)} & \text{if } j \in X \\ a_j^{(l)} & \text{otherwise.} \end{cases}$$

Then we have $a_j^{(l+1)} 2^{k-l-1} \in \mathbb{Z}$ for all $j \in [n]$ and

$$\sum_{j\in[n]} a_{ij}(a_j^{(l+1)} - a_j^{(l)}) = 2^{-(k-l)} d_i^{(l)}$$

for all $i \in [m]$. That means we rounded the $a^{(l)}$ to $a^{(l+1)}$ in such a way that $\|A(a^{(l+1)} - a^{(l)})\|_\infty$ is small.

Having defined $a_j^{(l)}$ for all $j \in [n], l \in \{0, \ldots, k\}$ we set $\varepsilon_j := 2a_j^{(k)} - 1$ (this is in $\{-1, 1\}$) and have

$$\begin{aligned} \|A(\varepsilon - p)\|_\infty &= 2\left\|A(a^{(k)} - a^{(0)})\right\|_\infty \\ &= 2\left\|\sum_{l\in[k]} A(a^{(l)} - a^{(l-1)})\right\|_\infty \\ &\le \left\|\sum_{l\in[k]} 2^{-k+l} d^{(l)}\right\|_\infty \\ &\le 2\,\mathrm{herdisc}(A), \end{aligned}$$

where the last inequality follows from the definition of the $d^{(l)}, l \in [k]$ and the triangle inequality. □

The key idea now is based on the following simple observation: if we replace $\varepsilon^{(l)}$ by $-\varepsilon^{(l)}$, we get $-d^{(l)}$ instead of $d^{(l)}$. By choosing signs for the $\varepsilon^{(l)}$, $l \in [k]$ in a clever way and not using the triangle inequality, we improve the above result.

Note that if we change the sign of one $\varepsilon^{(l)}$, this leads to a different $a^{(l+1)}$ and thus may change all the subsequently determined variables. Therefore we have to decide the signs 'on-line'. This might be described best in the language of games. Consider the following two-player perfect information game:

The Game: At the start of the game the vector $v \in \mathbb{R}^m$ is zero. One round of the game is: Player A gives a vector $w \in [-1, 1]^m$ and Player B then chooses a sign $\delta \in \{-1, 1\}$. The vector v is then updated to $v := 0.5v + \delta w$. The game is played for a fixed number k of rounds. Player A aims to maximize $\|v\|_\infty$ while B wants to keep the norm down. What is the maximum number c that A can reach?

This is the game we are playing (as Player B against the algorithm as Player A) when deciding on the signs of the $\varepsilon^{(l)}$, $l \in \{0, \dots, k-1\}$. In the game we normalized the w to be in $[-1,1]^m$ (while above we have $d_i^{(l)} \in \{-\operatorname{herdisc}(A), \dots, \operatorname{herdisc}(A)\}$), but it is clear that this just changes c to $c \operatorname{herdisc}(A)$ as an upper bound. So we have the following general result.

Lemma 3.3. *If c is the maximum value Player A can reach in the above described game, then* $\operatorname{lindisc}(A) \le c \operatorname{herdisc}(A)$.

We complete the proof by determining this constant c.

Lemma 3.4. *The maximum value Player A can reach is $c = 2\,(1 - 2^{-q})$.*

Proof. We investigate the following strategy for Player B. Whatever vectors $w^{(1)}, \dots, w^{(k-q)}$ Player A chooses in the first $k-q$ rounds, pick $\delta^{(1)}, \dots, \delta^{(k-q)} := 1$ (any other choice would do, too). Set $w := \sum_{j=1}^{k-q} 2^{-k+j} w^{(j)}$. Choose the next sign $\delta^{(k-q+1)}$ in such a way that the number of components $i \in [m] =: X_1$ such that $\operatorname{sgn}(w_i)$ and $\operatorname{sgn}(\delta^{(k-q+1)} w_i^{(k-q+1)})$ are different is maximal. Set $X_2 := \{i \in [m] \mid \operatorname{sgn}(w_i) = \operatorname{sgn}(\delta^{(k-q+1)} w_i^{(k-q+1)})\}$. Next choose $\delta^{(k-q+2)} \in \{-1, 1\}$ such that the number of components $i \in X_2$ such that $\operatorname{sgn}(w_i)$ and $\operatorname{sgn}(\delta^{(k-q+2)} w_i^{(k-q+2)})$ are different is maximal. Set $X_3 := \{i \in X_2 \mid \operatorname{sgn}(w_i) = \operatorname{sgn}(\delta^{(k-q+2)} w_i^{(k-q+2)})\}$. Continue in this fashion until $\delta^{(k)}$ and X_q are determined.

Note that $|X_{j+1}| \le \left\lfloor \frac{|X_j|}{2} \right\rfloor$ for all $j \in [q-1]$, which gives $|X_q| < 1$, i.e. $X_q = \emptyset$. So for every component i there is a $j \in \{k-q+1, \dots, k\}$ such that w_i and $\delta^{(q)} w_i^{(q)}$ have different signs. The worst case is the one where all $\delta^{(j)} w_i^{(j)}$, $j \in \{k-q+1, \dots, k\}$ are 1 (or -1) and w_i is zero. This gives us

$$\begin{aligned}
\left\| \sum_{j=1}^{k} 2^{-k+j} \delta^{(j)} w^{(j)} \right\|_\infty &= \left\| \sum_{j=k-q+1}^{k} 2^{-k+j} \delta^{(j)} w^{(j)} + w \right\|_\infty \\
&\le \sum_{z=0}^{q-1} 2^{-z} \\
&= 2\left(1 - 2^{-q}\right).
\end{aligned}$$

Player B can not do any better, as the following strategy for A reveals. Let r denote the biggest power of 2 that is less than or equal to m (so $r = 2^{q-1}$). Choose the first $k-q$ vectors as zero. The last q vectors choose like this: components greater than r are always set zero (for instance). For an index $i = 1 + \sum_{j=0}^{q-2} x_j 2^j \le r$, $x_0, \dots, x_{q-2} \in \{0, 1\}$ and a

$p \in \{0, \dots, q-1\}$ set $w_i^{(k-p)} := 2x_p - 1$. Here is an example for $m = 5$:

$$\begin{aligned} w^{(j)} &= (0,0,0,0,0) \text{ for } i \in [k-q] \\ w^{(k-2)} &= (-1,-1,-1,-1,0) \\ w^{(k-1)} &= (-1,-1,+1,+1,0) \\ w^{(k)} &= (-1,+1,-1,+1,0). \end{aligned}$$

Whatever signs $\delta^{(j)}, j \in [k]$ are chosen, there will always be a component $i \in [r]$ such that $w_i^{(k-q+1)} = \dots = w_i^{(k)}$ and thus

$$\sum_{j=1}^{k} 2^{-k+j} w_i^{(j)} = w_i^{(k)} \sum_{j=0}^{q-1} 2^{-j} = w_i^{(k)} 2 \left(1 - 2^{-q}\right).$$

This proves Lemma 3.4 and thus the theorem. □

At this point we should remark that Lemma 3.4 is a special case of Theorem 7.1. For the reader's convenience we still gave the proof here.

3.3 Discussion

All of the above is constructive in the following sense. Let A and p be given as above. Assume that computing a coloring with discrepancy not greater than h is possible for every submatrix A in time $\mathcal{O}(f(A))$ (discrepancy is an NP–hard problem, so we can not skip this assumption). Then we have an algorithm rounding p to ε in time at most $\mathcal{O}(k \max\{n, m, f(A)\})$ such that

$$\|A(p - \varepsilon)\|_\infty \leq 2^{-k} \|A\|_\infty + 2\left(1 - 2^{-q}\right) h,$$

where $\|A\|_\infty := \sup_{\|x\|_\infty = 1} \|Ax\|_\infty$ and k is the maximum binary length we use to express the p_i in our algorithm. In particular, if $H = \operatorname{herdisc}(A)$ we get a polynomial time algorithm solving the lattice approximation problem with approximation error at most $2 \operatorname{herdisc}(A)$ by choosing k such that $2^{-k}\|A\|_\infty \leq \frac{\operatorname{herdisc}(A)}{2m}$, e. g. $k \geq \log_2(2mn)$. At this point we should remark that when talking about algorithmic complexity we always assume that all elementary computations can be done with arbitrary precision in constant time.

There are some ideas which we did not know how to use in the general case. They might be useful in restricted situations. Recall that the $d_i^{(l)}$ are discrepancies of hyperedges of induced subgraphs under an optimal coloring. We assumed all $d_i^{(l)}$ to take the worst possible value in $[-\operatorname{herdisc}(A), \operatorname{herdisc}(A)]$, but in general an optimal coloring just creates a few badly colored hyperedges. This is true in particular if herdisc(A) is large. Note that Spencer's example has herdisc$(A) = 1$. I would suspect that the bound can be improved at least slightly for larger values of herdisc(A).

Another point is that in general not all induced subgraphs do have herdisc(A) as discrepancy. This might be used also in a different ε-choosing strategy: Choose the $\varepsilon^{(l)}$ in such a way that the resulting $a^{(l+1)}$ represents a hypergraph with small discrepancy.

Finally, let us remark that the opposite relation is also unclear. It is even not known whether the hereditary discrepancy can be bounded in term of the linear discrepancy at all. Looking at this from the other side, the question is how far one can decrease the linear discrepancy of a hypergraph by adding vertices. Matoušek [Mat00] recently showed that one cannot decrease it below 2 for some hypergraphs. Beyond this nothing is known.

Chapter 4

Linear Discrepancy of Totally Unimodular Matrices

In this chapter we investigate the linear discrepancy of totally unimodular matrices. This problem has attracted attention in several special cases. We give an efficient algorithm that solves the lattice approximation problem for totally unimodular matrices optimally (recall from Section 1.2.3 that the linear discrepancy of a matrix is the worst-case approximability of the lattice defined by it). We also give a sharp upper bound on this linear discrepancy and characterize the extremal cases.

4.1 History and Results

Let $A \in \mathbb{R}^{m \times n}$ be any real matrix and $p \in [0,1]^n$. Recall from Section 1.2.3 the definition of the $0,1$ linear discrepancy of A with respect to p:

$$\operatorname{lindisc}_{01}(A,p) := \min_{z \in \{0,1\}^n} \|A(p-z)\|_\infty.$$

The $0,1$ linear discrepancy of A is $\operatorname{lindisc}_{01}(A) := \max_{p \in [0,1]^n} \operatorname{lindisc}(A,p)$. In this chapter we prefer to use the $0,1$ notion of linear discrepancy instead of the usual one. This is due to the fact that this notion put more emphasis on the relation to integer linear programs and rounding techniques which will be the heart of our proof. As both notions just differ by the constant factor of 2, we lose nothing by switching the notions.

An $m \times n$ matrix A is called *totally unimodular* if each square submatrix has determinant $-1, 0$ or 1. In particular, $A \in \{-1,0,1\}^{m \times n}$. Totally unimodular matrices arise naturally in several areas (cf. [Hof79]). For example, incidence matrices of bipartite graphs are totally unimodular, as well as network matrices.

The discrepancy problem for totally unimodular matrices is well-understood. Their discrepancy is at most one (see also Theorem 2.1). By definition, submatrices of totally unimodular matrices are totally unimodular, hence the discrepancy of all submatrices is at most one as well. The beautiful theorem of Ghouila-Houri [GH62] states that also the converse holds:

Theorem (Ghouila-Houri, 1962). *A matrix is totally unimodular if and only if it has hereditary discrepancy at most one.*

This solves the hereditary discrepancy problem (and in particular the discrepancy problem) for totally unimodular matrices. For the linear discrepancy of totally unimodular matrices a sharp upper bound was missing so far.

Using the well-known result due to Beck and Spencer [BS84a] and Lovász, Spencer and Vesztergombi [LSV86] that

$$\operatorname{lindisc}_{01}(A) \leq \operatorname{herdisc}(A)$$

holds for any matrix A, immediately we have $\operatorname{lindisc}(A) \leq 1$ for a totally unimodular matrix A. We refer to Chapter 3 for a detailed analysis of this problem. The current best result in this direction yields a bound $\operatorname{lindisc}(A) \leq 1 - 2^{(-\lfloor \log_2(m) \rfloor - 1)}$.

Spencer conjectures that even $\operatorname{lindisc}_{01}(A) \leq (1 - \frac{1}{n+1}) \operatorname{herdisc}(A)$ holds for any A. This would yield $\operatorname{lindisc}_{01}(A) \leq 1 - \frac{1}{n+1}$ for totally unimodular matrices, but Spencer's over 15 years old conjecture seems far from being proven. It is backed up by the fact that Spencer [Spe87] provides an example of a matrix A such that $\operatorname{lindisc}_{01}(A) = (1 - \frac{1}{n+1}) \operatorname{herdisc}(A)$. As this A is totally unimodular, it also shows that $\operatorname{lindisc}_{01}(A) \leq 1 - \frac{1}{n+1}$ is the best possible general upper bound for the linear discrepancy of totally unimodular matrices.

For a special class of totally unimodular matrices, Peng and Yan [PY00] used a combinatorial approach. A matrix A is called *strongly unimodular*, if it is totally unimodular and if each matrix obtained from A by replacing a single non-zero entry by zero is also totally unimodular. Peng and Yan show that for a strongly unimodular $0, 1$ matrix A,

$$\operatorname{lindisc}_{01}(A) \leq 1 - 3^{-\frac{n+1}{2}}$$

holds. They use a decomposition lemma due to Crama, Loebl and Poljak [CLP92], which states that such a matrix is, roughly speaking, the union of incidence matrices of digraphs. In the same paper Peng and Yan show an upper bound of $1 - \frac{1}{n+1}$ for strongly unimodular $0, 1$ matrices which have at most two non-zeros in every row. An alternative proof extending this result to $-1, 0, 1$ matrices was given in [Doe00b].

All these results can be transferred into efficient algorithms solving the lattice approximation problem with approximation error at most the claimed bound. They do not, however, guarantee a better approximation in cases where $\operatorname{lindisc}_{01}(A, p)$ is smaller, i. e. a better approximation exists.

In this paper we do not follow the approach via the hereditary discrepancy, nor do we use any structure theory for totally unimodular matrices. Instead, we consider suitable linear programs and apply the theorem of Hoffman and Kruskal. This yields two types of results: A theoretical one bounding the approximation error of an optimal solution and characterizing the critical cases, and a practical one, namely an efficient algorithm solving the lattice approximation problem optimally.

For the theoretical aspect we have:

Theorem 4.1. *Let $A \in \mathbb{R}^{m \times n}$ be a totally unimodular matrix and $p \in [0,1]^n$. Then there is an $z \in \{0,1\}^n$ such that*

$$\|A(p-z)\|_\infty \le \min\{1 - \tfrac{1}{n+1}, 1 - \tfrac{1}{m}\}.$$

In particular, $\operatorname{lindisc}_{01}(A) \le 1 - \frac{1}{n+1}$.

This result is sharp, as the example due to Spencer proves. Theorem 4.1 shows that Spencer's conjecture $\operatorname{lindisc}_{01}(A) \le (1 - \frac{1}{n+1}) \operatorname{herdisc}(A)$ holds for totally unimodular matrices. As a side product, our approach yields a characterization of all totally unimodular matrices such that $\operatorname{lindisc}_{01}(A) = 1 - \frac{1}{n+1}$.

Theorem 4.2. *Let A be an $m \times n$ totally unimodular matrix. Then* $\operatorname{lindisc}_{01}(A) = 1 - \frac{1}{n+1}$ *holds if and only if there is a collection of $n+1$ rows of A such that each n thereof are linearly independent. If* $\operatorname{lindisc}_{01}(A,p) = 1 - \frac{1}{n+1}$ *for some $p \in [0,1]^n$, then $p_i \in \{\frac{1}{n+1}, \dots, \frac{n}{n+1}\}$ for all $i \in [n]$.*

Thus all such 'extreme' matrices contain a matrix resembling Spencer's example and possibly some additional rows which have no influence on the linear discrepancy.

This is the theoretical analysis. As mentioned we are also able to solve the lattice approximation problem optimally.

Theorem 4.3. *There is an algorithm that computes for any totally unimodular matrix $A \in \mathbb{R}^{m \times n}$ and $p \in [0,1]^n$ an optimal solution x for the lattice approximation problem, i. e. an $x \in \{0,1\}^n$ such that $\|A(p-x)\|_\infty = \operatorname{lindisc}_{01}(A,p)$. The complexity of this algorithm is $\mathcal{O}(\log m)$ times the complexity of finding an extremal point of a polytope in $\mathbb{R}^n$ described by $2(m+n)$ linear constraints or proving its emptiness.*

4.2 Definitions and Notation

For a real number $r \in \mathbb{R}$ write $\lfloor r \rfloor := \max\{z \in \mathbb{Z} | z \le r\}$ for the largest integer not greater than r, and $\lceil r \rceil := \min\{z \in \mathbb{Z} | z \ge r\}$ for the smallest integer not being less than r. Set $\{r\} := r - \lfloor r \rfloor$, the fractional part of r.

Let $b \in \mathbb{R}^m$. We assume the above notation lifted to vectors in the natural way, e. g. $\lfloor b \rfloor := (\lfloor b_i \rfloor)_{i \in [m]}$. Part of our strategy will be to round those components of b to the nearest integer which are already very close to an integer. For $d \in [0, \frac{1}{2}]$ we define $I^-(b,d) := \{i \in [m] | \{b_i\} < d\}$, the set of indices such that b_i is less than d above the nearest integer (and hence a candidate for being rounded down), and $I^+(b,d) := \{i \in [m] | 1 - \{b_i\} < d\}$, the set of indices such that b_i is less than d below the nearest integer. Set $I(b,d) := I^-(b,d) \cup I^+(b,d)$. Let $r(b,d) \in \mathbb{R}^m$ denote the vector resulting from rounding the components with index in $I(b,d)$ to the nearest integer, i. e. for all $i \in [m]$ we have

$$r(b,d)_i = \begin{cases} \lfloor b_i \rfloor & \text{if } i \in I^-(b,d) \\ \lceil b_i \rceil & \text{if } i \in I^+(b,d) \\ b_i & \text{else} \end{cases} .$$

The total error of this rounding is described by

$$e(b,d) := \|r(b,d) - b\|_1 = \sum_{i \in I^-(b,d)} (b_i - \lfloor b_i \rfloor) + \sum_{i \in I^+(b,d)} (\lceil b_i \rceil - b_i).$$

Let $g(b)$ denote the maximum value of $d \in [0, \frac{1}{2}]$ such that $e(b,d) < 1$ (the maximum exists, since $d \mapsto e(b,d)$ is left-continuous). For a matrix $A \in \mathbb{R}^{m \times n}$ set $g(A) := \max_{p \in [0,1]^n} g(Ap)$.

Lemma 4.4. *Let $b \in \mathbb{R}^m$ and $d \in [0, \frac{1}{2}]$. Then*

(i) $e(b,d) < |I(b,d)|\, d \leq md$,

(ii) $g(b) \geq \frac{1}{m}$.

In particular $g(A) \geq \frac{1}{m}$ holds for any $m \times n$ matrix A.

Proof. We have

$$\begin{aligned} e(b,d) &= \sum_{i \in I^-(b,d)} (b_i - \lfloor b_i \rfloor) + \sum_{i \in I^+(b,d)} (\lceil b_i \rceil - b_i) \\ &< \sum_{i \in I^-(b,d)} d + \sum_{i \in I^+(b,d)} d \\ &= |I(b,d)|\, d \leq md. \end{aligned}$$

In particular $e(b, \frac{1}{m}) < 1$. Thus $g(b) \geq \frac{1}{m}$ by definition. □

These bounds are sharp. The vector $(d - \varepsilon)\mathbf{1}_m, \varepsilon > 0$ shows that (i) does not allow any further improvement, and $b = \frac{1}{m}\mathbf{1}_m$ is an example for $g(b) = \frac{1}{m}$.

4.3 Solving the Lattice Approximation Problem

In this section we present an algorithm that solves the lattice approximation problem for totally unimodular matrices efficiently and optimally.

Polyhedra

Our proof is self-contained apart from the well-known theorem of Hoffman and Kruskal [HK56]. This states that the set of feasible solutions of a linear program is an integral polyhedron, if the constraint matrix is totally unimodular and the right-side vector is integral. Hence in this case the existence of optimal solutions implies that there are also integral optimal solutions. All polyhedra in this work will be bounded and thus compact (of course everything is finite-dimensional). Hence the existence of optimal solutions is ensured if the polyhedron is non-empty.

Let A be a totally unimodular $m \times n$ matrix and $p \in [0,1]^n$. Set $b = Ap$. For all $d \in [0, \frac{1}{2}]$ define

$$P_d := \{x \in [0,1]^n \mid \lfloor \mathrm{rd}(b,d) \rfloor \leq Ax \leq \lceil \mathrm{rd}(b,d) \rceil \}.$$

P_d is an integral polyhedron by [HK56]. We first observe

Lemma 4.5. *For all $d \in [0, \frac{1}{2}[, x \in P_0$ we have*

$$x \in P_d \iff \|b - Ax\|_\infty \leq 1 - d.$$

Proof. As a compact polyhedron is the convex hull of its extremal points we may assume x to be an extremal point of P_d. Thus x is integral. Let $i \in [m]$. Note first that $\lfloor b_i \rfloor \leq \lfloor \mathrm{rd}(b,d)_i \rfloor$ and $\lceil \mathrm{rd}(b,d)_i \rceil \leq \lceil b_i \rceil$. Thus $(Ax)_i$ can take at most two values, namely $\lfloor b_i \rfloor$ and $\lceil b_i \rceil$.

If $\{b_i\} \notin [0,d[\cup]1-d,1]$, then is does not matter which of these values is taken as $|b_i - (Ax)_i| \leq 1 - d$ holds in both cases. This is different if $\{b_i\} \in [0,d[\cup]1-d,1]$. Taking the wrong value would yield an approximation error of more than $1-d$. Fortunately, $\mathrm{rd}(b,d)_i$ is integral if $\{b_i\} \in [0,d[\cup]1-d,1]$ by definition. Moreover, $\mathrm{rd}(b,d)_i$ equals the closer of the values $\lfloor b_i \rfloor$ and $\lceil b_i \rceil$. Thus we have $(Ax)_i = \mathrm{rd}(b,d)_i$ and $|b_i - (Ax)_i| \leq \frac{1}{2} \leq 1 - d$.

The second implication is proved similarly. □

The Algorithm

We claim that the following algorithm solves the lattice approximation problem for totally unimodular matrices:

(i) Set
$D := \{d \in [0, \frac{1}{2}[\mid \exists i \in [m] : \{b_i\} \in \{d, 1-d\}\} \cup \{\frac{1}{2}\}$.

(ii) Using a binary search strategy determine the largest $d \in D$ such that $P_d \neq \emptyset$.

(iii) Find an extremal point x of this P_d.

Correctness

Let d, x be the output of the algorithm. As P_d is integral, $x \in \{0,1\}^n$. From Lemma 4.5 we have $\|b - Ax\|_\infty \leq 1 - d$. Let $y \in \{0,1\}^n$ such that $\text{lindisc}_{01}(A, p) = \|b - Ay\|_\infty$, that is, y is an optimal approximation. Assume first that $l := \text{lindisc}_{01}(A, p) > \frac{1}{2}$. Then $P_{1-l} \neq \emptyset$, as $y \in P_{1-l}$ by Lemma 4.5. Since $\|b - Ay\|_\infty = l$, there is an $i \in [m]$ such that $|b_i - (Ay)_i| = l$. As Ay is integral, $b_i \in \{l, 1-l\}$ and $1 - l \in D$. From the maximality of d we deduce $1 - l \leq d$. From $\|b - Ax\|_\infty \leq 1 - d \leq l$ and the optimality of y we conclude $\|b - Ax\|_\infty = \text{lindisc}_{01}(A, p)$.

Now let us consider the case that $\text{lindisc}_{01}(A, p) \leq \frac{1}{2}$. Then for each $i \in [m]$ we have

$$\begin{array}{lcl} (Ay)_i = \lfloor b_i \rfloor & \iff & \{b_i\} < \frac{1}{2} \\ (Ay)_i = \lceil b_i \rceil & \iff & \{b_i\} > \frac{1}{2}. \end{array} \tag{4.1}$$

In particular, $\|b - Ay\|_\infty = \max_{i \in [m]} \min\{\{b_i\}, 1 - \{b_i\}\}$. We also find that $y \in P_{\frac{1}{2}}$. Thus $d = \frac{1}{2}$ and (4.1) holds as well with y replaced by x. Therefore we also have $\|b - Ax\|_\infty = \max_{i \in [m]} \min\{\{b_i\}, 1 - \{b_i\}\}$. Thus $\|b - Ax\|_\infty = \text{lindisc}_{01}(A, p)$ again.

This proves that x is an optimal solution of the lattice approximation problem corresponding to A and p.

Complexity

It remains to show that our algorithm is efficient. We would not like to discuss any linear programming theory here. We simply assume that linear programs can be solved efficiently and refer to any book on linear programming (e. g. [Chv83]) for a discussion of that problem. Nor do we want to discuss any problems concerned with exact number representations and complexities of elementary calculations. We therefore assume that all elementary calculations can be done in constant time with perfect accuracy.

Computing the set D requires m steps, namely checking whether $\{b_i\}$ or $1 - \{b_i\}$ should be included in D for each $i \in [m]$. This also shows $|D| \leq m + 1$. For the binary search we need to sort the elements of D by their size which has worst-case complexity $\mathcal{O}(m \log m)$ (cf. e. g. [CLR90]). Finally, up to $\lceil \log_2(m+1) \rceil$ times a linear system of $2(m+n)$ constraints has to be solved to decide emptiness of the corresponding polytope and to compute the extremal point of the final P_d. Note that if the linear systems are solved using the simplex method any solution already is an extremal point of the polytope. Summarizing we see that solving the linear systems dominates the other steps of the algorithm in terms of complexity. This finally proves Theorem 4.3.

4.4 Approximability and Linear Discrepancy

In this section we prove Theorem 4.1, that is, we analyze how bad an optimal approximation can be in the worst case. The proofs are is independent of the preceding section.

Lemma 4.6. *Let $A \in \mathbb{R}^{m\times n}$ be a totally unimodular matrix and $p \in [0,1]^n$. Then*

$$\operatorname{lindisc}_{01}(A,p) \le 1 - g(Ap).$$

In the language of Section 4.3, Lemma 4.6 claims that $P_{g(Ap)} \neq \emptyset$, but this approach is misleading. Instead we consider the polytope P_0 together with a suitable objective function.

Proof. Set $b := Ap$. Let $P := \{x \in [0,1]^n | \lfloor b \rfloor \le Ax \le \lceil b \rceil\}$. As A is totally unimodular, P is an integral polyhedron (this is [HK56]). Define $f : P \to \mathbb{R}$ by

$$f(x) = \sum_{i \in I^-(b,g(b))} ((Ax)_i - \lfloor b_i \rfloor) + \sum_{i \in I^+(b,g(b))} (\lceil b_i \rceil - (Ax)_i). \tag{4.2}$$

for all $x \in P$. Thus $f(x)$ is the total error inflicted by rounding Ax in that way that was used to get $r(b,g(b))$ from b. By definition, f is non-negative. We first show that for all $x \in P \cap \mathbb{Z}^n$ we have

$$f(x) < 1 \iff \forall i \in [m] \begin{cases} \{b_i\} < g(b) \Rightarrow (Ax)_i = \lfloor b_i \rfloor \\ \{b_i\} > 1 - g(b) \Rightarrow (Ax)_i = \lceil b_i \rceil \end{cases}. \tag{4.3}$$

Suppose that $f(x) < 1$ for $x \in P \cap \mathbb{Z}^n$. As A and x are integral, $f(x) \le 0$, and since f is non-negative, $f(x) = 0$. Since all parts of the sum in (4.2) are non-negative, they are all zero. Hence $(Ax)_i = r(b,g(b))_i$ for all $i \in I(b,g(b))$. This is the right hand-side of (4.3). On the other hand, if the right hand-side of (4.3) is fulfilled, we have $f(x) = 0$ by (4.2). Thus (4.3) holds.

Consider the linear optimization problem

$$\min_{x \in P} f(x).$$

p is a feasible solution and $f(p) = e(Ap, g(b)) = e(b, g(b)) < 1$. Hence there is an optimal solution x^* such that $f(x^*) < 1$. As P is integral, we may assume $x^* \in \mathbb{Z}^n$.

Let us compute $\|A(p - x^*)\|_\infty = \|b - Ax^*\|_\infty$. Let $i \in [m]$. If $i \in I^-(b, g(b))$, then $(Ax^*)_i = \lfloor b_i \rfloor$ by (4.3). Hence $|b_i - (Ax^*)_i| < g(b)$. Similarly for $i \in I^+(b, g(b))$. Thus we may assume $i \in [m] \setminus I(b, g(b))$, i. e. $b_i \in [\lfloor b_i \rfloor + g(b), \lceil b_i \rceil - g(b)]$. As $(Ax^*)_i \in \{\lfloor b_i \rfloor, \lceil b_i \rceil\}$ due to $x^* \in P$, we conclude $|b_i - (Ax^*)_i| \leq 1 - g(b)$. □

Lemma 4.6 is sharp in the worst-case, as this example due to Spencer shows: Set $m := n+1$. Let $A \in \{0,1\}^{m \times n}$ denote the $m \times n$ matrix with

$$a_{ij} = \begin{cases} 1 & \text{if } i = j \text{ or } i = n+1 \\ 0 & \text{else} \end{cases} .$$

Set $p = \frac{1}{m}\mathbf{1}_n$. It is easy to see that any $z \in \{0,1\}^n$ fulfills $\|A(p - z)\|_\infty \geq 1 - \frac{1}{m}$: If any $z_j, j \in [n]$ equals 1, then $(A(p - z))_j = p_j - z_j = \frac{1}{m} - 1$. Otherwise we have $(A(p - z))_{n+1} = n\frac{1}{m} = 1 - \frac{1}{m}$.

Lemma 4.4 and 4.6 yield

Corollary 4.7. *Let $A \in \mathbb{R}^{m \times n}$ be a totally unimodular matrix. Then*

$$\operatorname{lindisc}_{01}(A) \leq 1 - g(A).$$

In particular $\operatorname{lindisc}_{01}(A) \leq 1 - \frac{1}{m}$.

We now refine the result of Lemma 4.6 and finally prove Theorem 4.1. Before doing so let us remark that a weaker bound in terms of n follows from a purely combinatorial argument. If $A \in \mathbb{R}^{m \times n}$ is totally unimodular and any two rows of A are linearly independent, then

$$m \leq \binom{n}{2} + \binom{n}{1} \tag{4.4}$$

holds. We would show (4.4) using the connection between the VC–dimension and the primal shatter function of hypergraphs, but may-be more direct ways are possible as well. Unfortunately, (4.4) is sharp. To prove Theorem 4.1, we therefore need a different approach.

Proof of Theorem 4.1. Let $p \in [0,1]^n$ and $b := Ap$. Denote the rows of A by $a_1, \ldots, a_m$. Set $I := I(b, \frac{1}{n+1}) = \{i \in [m] \mid |b_i - \operatorname{rd}(b_i, \frac{1}{2})| < \frac{1}{n+1}\}$. Let us call these rows 'critical' for the moment, because they are the ones where a rounding error of more than $1 - \frac{1}{n+1}$ can occur when using the approach of the previous section.

We proceed by showing that it is enough to consider at most n critical rows. Let $I_0 \subseteq I$ be chosen such that $\{a_i | i \in I_0\}$ is a basis for the vector space generated by all critical rows.

In particular, $|I_0| \leq n$. Let A_0 denote the matrix obtained from A by deleting all critical rows except $a_i, i \in I_0$. Then $A_0 p = b_{|([m]\setminus I)\cup I_0} =: b_0$. From Lemma 4.4 and $I(b_0, \frac{1}{n+1}) = I_0$ we conclude $e(b_0, \frac{1}{n+1}) < \frac{n}{n+1}$. Hence $g(A_0 p) \geq \frac{1}{n+1}$. By Lemma 4.6 there is a $z \in \{0,1\}^n$ such that $\|A_0(p-z)\|_\infty < 1 - \frac{1}{n+1}$. In particular, for all $i \in I_0$ we have

$$|a_i \cdot (p-z)| < \tfrac{1}{n+1}, \tag{4.5}$$

where $\cdot$ denote the usual inner product on $\mathbb{R}^n$.

We end the proof by showing that this z also fulfills $\|A(p-z)\|_\infty \leq 1 - \frac{1}{n+1}$. Let $j \in I \setminus I_0$. As I_0 is a basis for the vector space generated by all critical rows, there are $\lambda_i, i \in I_0$ such that $a_j = \sum_{i \in I_0} \lambda_i a_i$. Since A is totally unimodular, Cramer's rule implies $\lambda_i \in \{-1, 0, 1\}$ for all $i \in I_0$. Now

$$|a_j \cdot (p-z)| \leq \sum_{i \in I_0} |\lambda_i a_i \cdot (p-z)| \leq n \tfrac{1}{n+1}$$

by (4.5). □

The proof above yields some more information, which we can use for a characterization of those totally unimodular matrices A that fulfill $\operatorname{lindisc}_{01}(A) = 1 - \frac{1}{n+1}$.

4.5 A Characterization

The proof above yields some more information, which we now use for a characterization of totally unimodular matrices that have $\operatorname{lindisc}_{01}(A) = 1 - \frac{1}{n+1}$.

Proof of Theorem 4.2. Let A be a totally unimodular matrix satisfying $\operatorname{lindisc}_{01}(A) = 1 - \frac{1}{n+1}$. Choose $p \in [0,1]^n$ such that $\operatorname{lindisc}_{01}(A, p) = 1 - \frac{1}{n+1}$. Set $b := Ap$ and $I := \{i \in [m] \,|\, |b_i - \operatorname{rd}(b_i, \frac{1}{2})| \leq \frac{1}{n+1}\}$. Note that $I = I(b, d)$ for some $d > \frac{1}{n+1}\}$. For any $J \subseteq [m]$ define V_J to be the vector space generated by the rows $a_j, j \in J$. Let I_0 be a minimal subset of I such that $V_{I_0} = V_I$. In particular, the rows $a_i, i \in I_0$ form a basis of V_I. If there is an $i \in I_0$ such that $\{b_i\} \notin \{\frac{1}{n+1}, 1 - \frac{1}{n+1}\}$, or if $|I_0| < n$, then by mimicking the proof above we get a $z \in \{0,1\}^n$ such that $\|A(p-z)\|_\infty \leq \max\{1 - d, \sum_{i \in I_0} |b_i - \operatorname{rd}(b_i, \frac{1}{2})|\} < 1 - \frac{1}{n+1}$. We conclude $|I_0| = n$ and $\{b_i\} \in \{\frac{1}{n+1}, 1 - \frac{1}{n+1}\}$ for all $i \in I_0$, and hence also for all $i \in I$. From Lemma 4.4 we get $|I| \geq n+1$ (otherwise $g(b) \geq d$, and Lemma 4.6 yields a contradiction).

From the fact that A is totally unimodular, we know that each $a_i, i \in I \setminus I_0$, can be expressed in the form $a_i = \sum_{j \in I_0} \lambda_j a_j$ with some $\lambda_j \in \{-1, 0, 1\}, j \in I_0$. Let us assume that for each $i \in I \setminus I_0$ there is such an expression $a_i = \sum_{j \in I_0} \lambda_j a_j$ such that at least one of the $\lambda_j, j \in I_0$ is zero. Then by mimicking the proof of Theorem 4.1 above (using this I_0), we find a $z \in \{0,1\}^n$ such that $|a_i \cdot (p-z)| = \frac{1}{n+1}$ for all $i \in I_0$ and $|a_i \cdot (p-z)| \leq \frac{n-1}{n+1}$ for all

$i \in I \setminus I_0$. This is again a contradiction to our choice of p. Hence there is an $i \in I \setminus I_0$ such that $a_i = \sum_{j \in I_0} \lambda_j a_j$ with some (by the way unique) $\lambda_j \in \{-1, 1\}, j \in I_0$. In particular, any n of the rows with index in $I_0 \cup \{i\}$ are linearly independent.

Let A' and b' denote the restrictions of A and b on the rows with index in I_0. Then A' is non-singular, and thus p is already determined by $A'p = b'$. As $(n+1)b' \in \{1, n\}^n$ was shown in the first paragraph, we have $(n+1)p \in \mathbb{Z}^n$ (the inverse of a totally unimodular matrix is totally unimodular, and thus integral). Clearly, none of the $p_i, i \in [n]$ is 0 or 1 — otherwise we may just put $z_i = p_i$ reducing the dimension of the problem by one. Hence all $p_i, i \in [n]$ are in $\{\frac{1}{n+1}, \ldots, \frac{n}{n+1}\}$ as claimed.

Now let A be such that there are $n+1$ rows each n thereof being linearly independent. Without loss of generality we may assume these to be the rows $a_1, \ldots, a_{n+1}$. As above there are $\lambda_1, \ldots, \lambda_n \in \{-1, 1\}$ such that $a_{n+1} = \sum_{i \in [n]} \lambda_i a_i$. Define $b' \in \mathbb{R}^n$ by

$$b'_i := \begin{cases} \frac{1}{n+1} & \text{if } \lambda_i = 1 \\ 1 - \frac{1}{n+1} & \text{else} \end{cases}$$

for all $i \in [n]$. Let A_0 denote the matrix consisting of the rows $a_1, \ldots, a_n$ only. As A_0 has full rank, the system $A_0 x = b'$ has a unique solution x. Since A_0 is totally unimodular and $(n+1)b' \in \mathbb{Z}^n$, $(n+1)x$ is also integral. Set $p = \{x\}$ and $b = Ap$. Then $\{b_i\} = \{b'_i\}$ for $i \in [n]$. We claim that any $z \in \{0, 1\}^n$ fulfills $\|A(p - z)\|_\infty \geq 1 - \frac{1}{n+1}$. Let us assume $|a_i \cdot (p - z)| < 1 - \frac{1}{n+1}$ for all $i \in [n]$ (otherwise we are done). Then $\lambda_i a_i \cdot (p - z) = \frac{1}{n+1}$ holds for all $i \in [n]$ by definition of b'. Thus $a_{n+1} \cdot (p - x) = \sum_{i \in [n]} \lambda_i a_i \cdot (p - x) = n\frac{1}{n+1}$. This proves the claim. □

It is a trivial consequence of the definition of the linear discrepancy that if a matrix B consists of some rows of the matrix A, then $\text{lindisc}_{01}(B) \leq \text{lindisc}_{01}(A)$. In the light of Theorem 4.2 it makes sense to call a totally unimodular $m \times n$ matrix critical, if $m = n+1$ and $\text{lindisc}_{01}(A) = 1 - \frac{1}{n+1}$. Theorem 4.2 then states that a totally unimodular $m \times n$ matrix has linear discrepancy $1 - \frac{1}{n+1}$ if and only if it contains a critical one. The reasoning above also shows that for critical matrices A, there are just two different p such that $\text{lindisc}_{01}(A, p) = 1 - \frac{1}{n+1}$ holds, namely the one constructed, call it $p^{(1)}$, and $p^{(2)} := 1 - p^{(1)}$.

4.6 The Multi-Color Case

A natural question in this work is how the previous results extend to multi-color discrepancies. One should expect that the tensor product calculus allows a similar expression, and that this yields a bound of, say, $1 - \frac{1}{(c-1)n+1}$.

Working with the matrix $A \otimes X$, where $X \in \{0, 1\}^{c \times (c-1)}$ is defined by $x_{ij} = 1$ if and only if $i = j$ or $i = c$, seems to do the job. Possibly adding all rows containing a single 1 to A

(this does not change the unimodularity of A) ensures that we end up with a valid coloring. So where is the problem?

The problem is that the tensor product of two totally unimodular matrices needs not to be totally unimodular. This is surprising, but can not be helped. We give a counter-example:

Let $n = 3$ and $c = 3$. Let A denote the incidence matrix of the hypergraph of intervals on $[n]$, i. e. $I_n = ([n], \{[a, b] \cap [n] \mid a, b \in \mathbb{N}\})$. Since $A \otimes X$ and $X \otimes A$ can be transformed into another by row and column permutations, both matrices have the same set of determinants of submatrices. Now $X \otimes A$ has the block structure

$$\begin{pmatrix} A & 0 \\ 0 & A \\ A & A \end{pmatrix}.$$

Let us thinks of $X \otimes A$ as the incidence of a hypergraph on the vertex set $\overline{1}, \ldots, \overline{n}, \underline{1}, \ldots, \underline{n}$ (corresponding to the columns in this order). This hypergraph contains the edges $\{\overline{1}, \overline{2}\}$, $\{\overline{2}, \overline{3}\}$, $\{\overline{3}, \underline{3}\}$, $\{\underline{1}, \underline{2}, \underline{3}\}$ and $\{\overline{1}, \underline{1}\}$ (see Figure 4.1). Hence the induced subhypergraph on the vertex set $\{\overline{1}, \overline{2}, \overline{3}, \underline{1}, \underline{3}\}$ contains an odd cycle. This shows (either directly or via the Theorem of Ghouila-Houri) that $A \otimes X$ contains a submatrix of determinant not in $\{-1, 0, 1\}$.

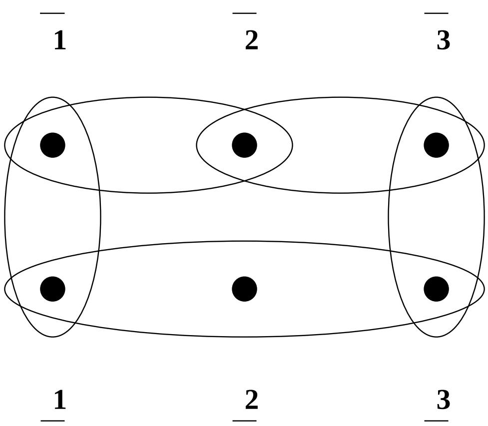

Figure 4.1: Some hyperedges

This does not yet show that we do not have a multi-color analogue of the 2–color results in this section, but makes it easier to understand the difficulties. The following examples shows that the multi-color linear discrepancy of a totally unimodular matrix can exceed 1:

Assume n to be sufficiently large and let $\mathcal{H}$ denote the hypergraph of intervals in $[n]$ again. We show that already the weighted discrepancy of $\mathcal{H}$ is at least $1 + \frac{1}{9}$ (it is actually larger, but our objective is just to show that it is larger than 1). Put $p = (\frac{5}{6}, \frac{1}{9}, \frac{1}{18})$. Let $\chi : [n] \to [3]$ be an optimal coloring with respect to this weight. Let $x \in [n]$ be such that $\chi(x) = 2$ and $x \leq n - 10$. An x like this exists as 10 successive vertices avoiding color 2 already form an edge having discrepancy $1 + \frac{1}{9}$. Now easy counting shows that all of the vertices $x, \ldots, x + 10$ have to be colored 1 except of exactly one of the vertices $x + 5, x + 6$ or $x + 7$ (otherwise the discrepancy of χ is at least $1 + \frac{1}{6}$). If this exceptional vertex y is not colored 2, then the edge $\{x + 1, \ldots, x + 10\}$ has no vertex colored in color 2 and thus has discrepancy $\frac{10}{9}$ in color 2. If $\chi(y) = 2$, then $\{x, \ldots, y\}$ contains at most 8 vertices two of which are colored 2. This again yields a discrepancy of at least $1 + \frac{1}{9}$.

These examples show that neither the methods nor the results of the previous sections admit a straight-forward extension to higher numbers of colors. This seems to be an interesting problem for further research.

Chapter 5

Random Colorings Respecting the Structure

The only efficient approach to find a low discrepancy coloring in the general case involves random colorings. Alon and Spencer [AS00] investigated random colorings obtained by independently flipping a coin for each vertex to decide its color (Theorem 1.1). In this chapter we analyze a different kind of random colorings. This yields better bounds for the general case, as it reduces the number and size of the hyperedges to be considered. It also allows to prescribe that some subsets of the vertex set have to be colored perfectly. This is used successfully to exploit additional structural knowledge on the hypergraph.

5.1 The Basic Idea

Suppose that we have the following: A partition $\mathcal{P} = \{P_1, \ldots, P_r\}$ of the vertex set and colorings $\chi_i : P_i \to \{-1, +1\}$ such that $|\chi_i(E \cap P_i)| \leq 1$ holds for all edges $E \in \mathcal{E}$. We will shortly see that this is less artificial than it may seem at first. For each $i \in [r]$ 'flip a coin', i. e. independently and uniformly choose a sign $\varepsilon_i \in \{-1, +1\}$. Let $\chi : X \to \{-1, +1\}$ denote the union of the $\varepsilon_i \chi_i$, that is, $\chi(x) = \varepsilon_i \chi_i(x)$ if $x \in P_i$.

For a hyperedge $E \in \mathcal{E}$ set $I_E := \{i \in [r] | |\chi_i(E \cap P_i) \neq 0\}$. Then $\chi_i(E \cap P_i) \in \{-1, +1\}$ for all $i \in I_E$. The Chernoff bound (exactly in the same way as in Alon and Spencer's result) now yields $P(|\chi(E)| > \lambda) < e^{-\frac{\lambda^2}{2|I_E|}}$ instead of $P(|\chi(E)| > \lambda) < e^{-\frac{\lambda^2}{2|E|}}$ for the coin-flip coloring of [AS00]. Thus we replaced the cardinality of E by the possibly smaller number of P_i such that $\chi_i(E \cap P_i) \neq 0$.

A second effect is that we might reduce the number of edges to be considered. Set $E_{\mathcal{P}} := \bigcup\{E \cap P_i \,|\, \exists i \in [r] : \chi_i(E \cap P_i) \neq 0\}$ for all $E \in \mathcal{E}$. Then $\chi(E) = \chi(E_{\mathcal{P}})$. Depending on the partition $\mathcal{P}$ and the colorings χ_i, the mapping $E \mapsto E_{\mathcal{P}}$ is not injective, that means we need to consider fewer edges. Since the number of edges influences the discrepancy just in a logarithmic factor, this effect is less important than the size reduction described in the previous paragraph.

Summarizing we have:

Theorem 5.1. *Let* $s_0 := \max_{E\in\mathcal{E}} |E_{\mathcal{P}}|$ *and* $m_0 := |\{E_{\mathcal{P}} | E \in \mathcal{E}\}|$. *Then*

$$\operatorname{disc}(\mathcal{H}) \leq \sqrt{2s_0 \ln(2m_0)}.$$

5.2 Arbitrary Hypergraphs

Let us assume that n is even. We now show that we may apply the method described above to any hypergraph. This yields a constant factor improvement over the usual random coloring bound. Let $\mathcal{P}$ be any matching on V, that is, $\mathcal{P}$ consist of $\frac{n}{2}$ disjoint sets each holding 2 vertices. For each $P = \{v_1, v_2\} \in \mathcal{P}$ independently 'flip a coin' to decide a color for v_1 and the opposite color for v_2. Let χ denote the resulting (random) coloring. If a hyperedge $E \in \mathcal{E}$ contains a matching edge $P \in \mathcal{P}$, we have $\chi(E) = \chi(E \setminus P)$, since P contains one point in each color. Therefore $|E_{\mathcal{P}}| \leq \frac{n}{2}$, and we immediately derive the bound $\operatorname{disc}(\mathcal{H}, \chi) > \sqrt{n \ln(2m)}$ with probability less than one. If n is not even we may just add one additional vertex contained in no hyperedge. Now the number of vertices is even, and both the original and this hypergraph have the same discrepancy. Thus we have:

Theorem 5.2. *For all hypergraphs $\mathcal{H}$ having n vertices and m edges,*

$$\operatorname{disc}(\mathcal{H}) \leq \sqrt{(n+1)\log(2m)}.$$

Note that this works for any matching! In particular by choosing a suitable matching we can enforce several subsets of V to be perfectly balanced. All we need is to choose a matching which also induces a matching on these subsets. For example any disjoint family of subsets of V can be ensured a perfect coloring in addition to the discrepancy guarantee for the hyperedges.

Another aspect is that we might be able to choose a matching such that many matching edges are contained in hyperedges of $\mathcal{H}$. This then reduces the relevant size $|E_{\mathcal{P}}|$ of the hyperedges and further reduces the discrepancy.

Taking a random matching we generate a random 2–coloring having color-classes of equal size. From the perfect symmetry we easily deduce that each such coloring has the same probability to be chosen. Analyzing this random experiment is much harder, as we can not use the Chernoff inequality any more. Chvátal [Chv79] and Uhlmann [Uhl66] showed that for this random experiment similar inequalities hold. They also yield Theorem 5.2, but the proofs of Chvátal and Uhlmann are quite complicated compared to the ones of the Chernoff bound.

An extension of this method to the multi-color case is straightforward. Instead of taking a matching we partition the vertex set into sets of size c and choose arbitrary random colorings using all c colors on each of these sets. Again we have that sets completely contained in an edge reduce the relevant size of the edge. Unfortunately, now it is much less likely that one of these sets is contained in an edge. In particular, we may end up with edges that have relevant size $\frac{c-1}{c}n$.

A second disadvantage is hidden in the analysis. As one of these sets may have more than a single point in an edge without becoming irrelevant, we are not anymore in the situation that the relevant vertices of an edge are colored independently. This requires again deeper results for the analysis of the random experiment. We are not interested in this for the simple reason that the recursive method of Section 2.4 provides a much stronger result: There we gain a factor of $\sqrt{\frac{1}{c}}$, which clearly outnumbers everything we could derive with this section's ideas. This seems to be another example in which the 2-color behavior is different from the multi-color one.

5.3 Exploiting the Structure

A major difficulty using the probabilistic method in discrepancy theory so far was to use structural information in the design of the random experiment. One idea applied to the hypergraph of arithmetic progressions is to decompose the hyperedges into so-called canonical sets ([Bec81]). One tries to find a small number of sets such that each hyperedge is the disjoint union of few of these sets. Then a random coloring has low discrepancy on these sets (as there are not too many of them). As each hyperedge is the union of few canonical sets, this coloring is also good for the hypergraph itself. This was the very basic idea only, usually the canonical sets have some extra properties like small vertex degree. This can be exploited in designing the random experiment.

Here we show a different way to use structural knowledge that works well for the higher-dimensional boxes: Let us consider the hypergraph of d–dimensional boxes in $[n]^d$, that is $\mathcal{H}_n^d = ([n]^d, \{S_1 \times \cdots \times S_d | S_i \subseteq [n]\})$. This is (simply?) the d–fold cartesian product of the complete hypergraph $([n], 2^{[n]})$ on n points (see Chapter 6 for the definition of product hypergraphs). The usual probabilistic argument yields a bound of

$$\begin{aligned}
\operatorname{disc}(\mathcal{H}_n^d) &\leq \sqrt{2n^d \ln(2\, 2^{nd})} \\
&= \sqrt{2\ln 2}\, n^{\frac{d+1}{2}} \sqrt{d}\,(1+o(1)) \\
&\approx 1.18 n^{\frac{d+1}{2}} \sqrt{d}\,(1+o(1)).
\end{aligned}$$

We show:

Theorem 5.3. *For all $n, d \in \mathbb{N}$,*

$$\operatorname{disc}(\mathcal{H}_n^d) \le 1.05\, 2^{-d/2}(n+1)^{\frac{d+1}{2}}\sqrt{d}.$$

Proof. Assume n to be even. Set $\mathcal{P} := \{\prod_{i\in[d]}\{2x_i-1, 2x_i\} | x_1, \cdots, x_d \in [\frac{n}{2}]\}$, that is, we split the n^d–cube into 2^d–cubes in a rather canonical way. The coloring corresponding to each such small cube shall be such that adjacent corners always receive the opposite color. More formally, a vertex is colored $+1$ if and only if an even number of its coordinates is even. Let χ again be the random coloring obtained from independently taking these colorings or their inverse colorings. Let E be a hyperedge of $\mathcal{H}_n^d$ and $P \in \mathcal{P}$. As both E and P are boxes, so is $E \cap P$. From the definition of χ we see that any subbox S of P such that $|S| \neq 1$ fulfills $\chi(S) = 0$. As $E_{\mathcal{P}}$ has at most one vertex in each box, we conclude $|E_{\mathcal{P}}| \le 2^{-d}n^d$.

Now let us count the number of the relevant hyperedges. Let $E = S_1 \times \cdots \times S_d$. Assume that for some $i \in [d]$ and $x \in [\frac{n}{2}]$ we have $\{2x-1, 2x\} \subseteq S_i$. Then no box $\prod_{i\in[d]}\{2x_i-1, 2x_i\}$ such that $x_i = x$ intersects E in exactly one vertex. Thus $E_{\mathcal{P}} = (S_1 \times \cdots \times (S_i \setminus \{2x-1, 2x\}) \times \cdots \times S_d)_{\mathcal{P}}$. By induction we see that $\pi : E \mapsto E_{\mathcal{P}}$ is a projection $\mathcal{E} \to \mathcal{E}$. Therefore we need to counts its fixed points only. We just exhibited that a necessary condition for this is

$$\forall i \in [d]\, \forall x \in [\tfrac{n}{2}] : |S_i \cap \{2x-1, 2x\}| \le 1.$$

For each $i \in [d], x \in [\frac{n}{2}]$ we therefore have exactly three possibilities: $S_i \cap \{2x-1, 2x\}$ is empty or $\{2x-1\}$ or $\{2x\}$. This makes at most $3^{nd/2}$ fixed points.

We want to show a further reduction: Note that for all $i \in [d]$,

$$\gamma_i : \mathcal{E} \to \mathcal{E}; S_1 \times \cdots \times S_i \times \cdots \times S_d \mapsto S_1 \times \cdots \times ([n] \setminus S_i) \times \cdots \times S_d$$

is a fixed-point-free bijection of $\mathcal{E}$ that leaves the set $\mathcal{E}_{\mathcal{P}}$ of reduced hyperedges invariant and preserves discrepancy: We have $\chi(E) = \chi(\gamma_i(E))$ for all hyperedges E. In particular, the group $\mathbb{Z}_2^d$ acts on $\mathcal{E}$ and $\mathcal{E}_{\mathcal{P}}$ in such a way that all orbits have length 2^d. As all elements of an orbit have the same discrepancy with respect to χ it is enough to consider just one representative from each orbit. This reduces the number of relevant hyperedges by another factor of 2^d. we finally have

$$\begin{aligned}\operatorname{disc}(\mathcal{H}, \chi) &\le \sqrt{2\, 2^{-d}n^d \ln(2\, 2^{-d}3^{nd/2})} \\ &= \sqrt{\ln 3}\, 2^{-d/2} n^{\frac{d+1}{2}}\sqrt{d} \\ &\le 1.05\, 2^{-d/2} n^{\frac{d+1}{2}}\sqrt{d}.\end{aligned}$$

The case of odd n is solved by the observation that all hyperedges of $\mathcal{H}_n^d$ are also hyperedges of $\mathcal{H}_{n+1}^d$. Therefore $\operatorname{disc}(\mathcal{H}_n^d) \le \operatorname{disc}(\mathcal{H}_{n+1}^d)$. □

We should remark that the size reduction yields a change in the order of magnitude in terms of d, namely the additional $2^{-d/2}$ factor, whereas counting the relevant edges only improves the constant by about 11%.

Chapter 6

Discrepancy of Products of Hypergraphs

In this chapter another aspect of classical 2–color discrepancy is touched: The discrepancy of cartesian products of arithmetic progressions. This is a natural extension of the discrepancy problem for ordinary arithmetic progressions. For the latter we refer to Section 2.4.4. The higher dimensional problem has been investigated by P. Wehr [Weh97] in her dissertation.

For the d–dimensional case she proved a lower bound of order $n^{\frac{d}{4}}$ by a clever approach using discrete Fourier transforms. This also greatly simplified Roth's proof of the one-dimensional case. Unfortunately, her upper bound was off the lower one by a polylogarithmic factor. It seem to be very difficult to extend the method of Matoušek and Spencer [MS96] to arbitrary dimensions. Therefore she had to use the slightly inferior method of an earlier paper by Beck [Bec81]. Fortunately, a very general approach with a decent algebraic flavor solves the problem.

This chapter is organized as follows. First we give a general upper bound for the discrepancy of products of hypergraphs. Using this result we solve the discrepancy problem for the hypergraph of multi-dimensional arithmetic progressions and show that P. Wehr's lower bound is asymptotically sharp. In the next two sections we investigate some further product hypergraphs. This shows that our upper bound in the general case can be arbitrarily bad. We also see that products of very simple hypergraphs can be surprisingly hard to analyze. In the last section we show that for the similar construction of symmetric direct products the discrepancy is bounded by the discrepancy of one factor regardless of the number of factors.

6.1 Direct Products of Hypergraphs

Let $\mathcal{G} = (X, \mathcal{E})$ and $\mathcal{H} = (Y, \mathcal{F})$ be hypergraphs. Define the direct product of $\mathcal{G}$ and $\mathcal{H}$ by

$$\mathcal{G} \times \mathcal{H} := (X \times Y, \{A \times B | A \in \mathcal{E}, B \in \mathcal{F}\}).$$

Multi-color discrepancy is almost sub-multiplicative:

Theorem 6.1. *For any $c \in \mathbb{N}$ and any two hypergraphs $\mathcal{G}$ and $\mathcal{H}$ we have*

$$\operatorname{disc}(\mathcal{G} \times \mathcal{H}, c) \leq c \operatorname{disc}(\mathcal{G}, c) \operatorname{disc}(\mathcal{H}, c).$$

Proof. Pick a Latin square $Q = (q_{ij})_{i,j \in [c]}$ of dimension c, i. e. $Q \in [c]^{c \times c}$ such that every row and column contains every number of the set $[c]$ exactly once.[1] As Q is a Latin square we may define a permutation π_i of $[c]$ for every $i \in [c]$ by the following rule: $\pi_i(j)$ is the unique $k \in [c]$ such that $q_{jk} = i$.

Choose optimal colorings $\chi_{\mathcal{G}}$ and $\chi_{\mathcal{H}}$ of $\mathcal{G}$ and $\mathcal{H}$ respectively, i. e. $\operatorname{disc}(\mathcal{G}, \chi_{\mathcal{G}}) = \operatorname{disc}(\mathcal{G}, c)$ and $\operatorname{disc}(\mathcal{H}, \chi_{\mathcal{H}}) = \operatorname{disc}(\mathcal{H}, c)$. Define $\chi : X \times Y \to [c]$ by

$$\chi(x, y) := q_{\chi_{\mathcal{G}}(x)\chi_{\mathcal{H}}(y)}$$

for all $x \in X, y \in Y$.

Let $A \in \mathcal{E}, B \in \mathcal{F}$. Set

$$\begin{aligned} a_i &= |\chi_{\mathcal{G}}^{-1}(i) \cap A| - \frac{|A|}{c}, \\ b_i &= |\chi_{\mathcal{H}}^{-1}(i) \cap B| - \frac{|B|}{c} \end{aligned}$$

for all $i \in [c]$. Then we have

$$\sum_{i=1}^{c} a_i = 0 = \sum_{i=1}^{c} b_i. \tag{6.1}$$

[1] Note that for every $c \in \mathbb{N}$ there is a Latin square of dimension c: Let $*$ be any group multiplication on $[c]$. Then $q_{ij} := i * j$ defines a Latin square.

This yields

$$\begin{aligned}
|\chi^{-1}(i) \cap (A \times B)| &= \sum_{j=1}^{c} |\chi_G^{-1}(j) \cap A| \, |\chi_H^{-1}(\pi_i(j)) \cap B| \\
&= \sum_{j=1}^{c} \left(a_j + \frac{|A|}{c} \right) \left(b_{\pi_i(j)} + \frac{|B|}{c} \right) \\
&= \sum_{j=1}^{c} a_j b_{\pi_i(j)} + \frac{|B|}{c} \sum_{j=1}^{c} a_j + \frac{|A|}{c} \sum_{k=1}^{c} b_k + c \frac{|A||B|}{c^2} \\
&= \sum_{j=1}^{c} a_j b_{\pi_i(j)} + \frac{|A \times B|}{c} \qquad \text{by (6.1).}
\end{aligned}$$

As $|a_i| \leq \operatorname{disc}(\mathcal{G}, c)$ and $|b_i| \leq \operatorname{disc}(\mathcal{H}, c)$, we have

$$\left| |\chi^{-1}(i) \cap (A \times B)| - \frac{|A \times B|}{c} \right| = \left| \sum_{j=1}^{c} a_j b_{\pi_i(j)} \right| \leq c \operatorname{disc}(\mathcal{G}, c) \operatorname{disc}(\mathcal{H}, c).$$

This proves the theorem. □

For two colors, $\operatorname{disc}(\mathcal{G}, 2) = 2 \operatorname{disc}(\mathcal{G})$ yields

Theorem 6.2. *Discrepancy is sub-multiplicative, i. e.*

$$\operatorname{disc}(\mathcal{G} \times \mathcal{H}) \leq \operatorname{disc}(\mathcal{G}) \operatorname{disc}(\mathcal{H}).$$

6.2 Multi-Dimensional Arithmetic Progressions

We now turn to the problem of cartesian products of arithmetic progressions. For the one-dimensional case see Section 2.4.4. Wehr [Weh97] defines

Definition 6.3. A d–dimensional arithmetic progression in $[n]^d$ is the cartesian product of d arithmetic progressions in $[n]$.

From Theorem 6.2 we easily deduce

Corollary 6.4. *Let $\mathcal{A}_n^d$ be the hypergraph of d–dimensional arithmetic progressions on $[n]^d$. Then $\operatorname{disc}(\mathcal{A}_n^d) \leq C^d n^{\frac{d}{4}}$ for an absolute constant $C > 0$. In consequence, we have*

$$\operatorname{disc}(\mathcal{A}_n^d) = \Theta_d(n^{\frac{d}{4}})$$

for any fixed $d \in \mathbb{N}$.

Proof. It follows from Definition 6.3 that the hypergraph of d–dimensional arithmetic progressions is nothing else than the d–fold direct product of the hypergraph $\mathcal{A}_n$ of one–dimensional arithmetic progressions. Using the upper bound for the one-dimensional case of Matoušek and Spencer [MS96], Theorem 6.2 implies the upper bound. The lower is taken from Wehr [Weh97]. □

Let us remark that a multi-color result of $\operatorname{disc}(\mathcal{A}_n^d, c) = \Theta_{c,d}(n^{\frac{d}{4}})$ was shown in Doerr, Srivastav and Wehr [DSW00]. More precisely, a lower bound of $\operatorname{disc}(\mathcal{A}_n^d, c) \geq \frac{\sqrt{c-1}}{c}\pi^{-d}n^{\frac{d}{4}}$ was proven there. Combining Theorem 2.23 and 6.1 we derive an upper bound of $(C')^d c^{0.84d-1} n^{\frac{d}{4}}$. The extra c^{d-1} inflicted from d times applying Theorem 6.1 greatly increases the gap between the two bounds. Compared to the one-dimensional case we do not know if the discrepancy decreases for larger numbers of colors. One way to solve this problem would be to find upper bounds for the discrepancies of the induced subgraphs of the higher dimensional arithmetic progressions. Then the recursive method of Section 2.4 could be applied directly to the higher-dimensional arithmetic progressions. Unfortunately, bounding the discrepancies of the induced subgraphs of the higher dimensional arithmetic progressions turns out to be very difficult. We have to leave this as an open problem.

The discrepancy problem of d–dimensional arithmetic progressions motivates the investigation of discrepancies of direct products of hypergraphs in general, and also proved its usefulness. We thus continue this line of research.

6.3 Lower Bounds

A natural problem arising from Theorem 6.2 is to decide if or to what extent the discrepancy of $\mathcal{G} \times \mathcal{H}$ can be smaller than the product $\operatorname{disc}(\mathcal{G}) \operatorname{disc}(\mathcal{H})$. The case of arithmetic progressions might suggest equality, but this is not the case, as the following examples show:

Example 1: The hypergraph of two–element subsets of a three–element set $\mathcal{G} = ([3], \binom{[3]}{2})$ has discrepancy two (one color class has at least two elements, i. e. it contains a monochromatic two–set). The direct product $\mathcal{G} \times \mathcal{G}$ can be colored is a way that there is no monochromatic 2×2 rectangle: $\chi(i,i) := 1$ and $\chi(i,j) := -1$ for $i,j \in [3], i \neq j$. Hence $\operatorname{disc}(\mathcal{G} \times \mathcal{G}) \leq 2 < 4 = \operatorname{disc}(\mathcal{G})^2$. (Easy to see if we visualize $\mathcal{G} \times \mathcal{G}$ like that: The vertices form a 3×3–grid, the hyperedges consist of the corners of the rectangles having horizontal (and vertical) edges. All these rectangles have one or two points on the diagonal of the grid, thus having discrepancy two or zero with respect to χ.)

Looking at examples like this one might ask whether the discrepancy of a direct product is at least the discrepancy of its factors, or in an even weaker form we ask, whether the discrepancy of a direct product of two hypergraphs of nonzero discrepancy has discrepancy greater than 0. In general this is not true:

Example 2: Let $\mathcal{G}$ be the hypergraph

$$([7], \{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{2,3,4,5,6,7\}\})$$

as depicted in Figure 6.1 (straight lines representing hyperedges).

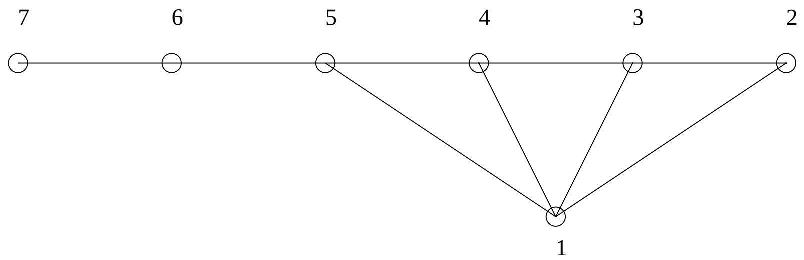

Figure 6.1: Example 2

$\mathcal{G}$ does not have discrepancy 0; if so, the points $2,3,4$ and 5 were in the same color class leaving the edge $E = \{2,3,4,5,6,7\}$ imbalanced.

The hypergraph $\mathcal{G} \times \mathcal{G}$ however has discrepancy 0. The coloring depicted in Figure 6.2 does the job.

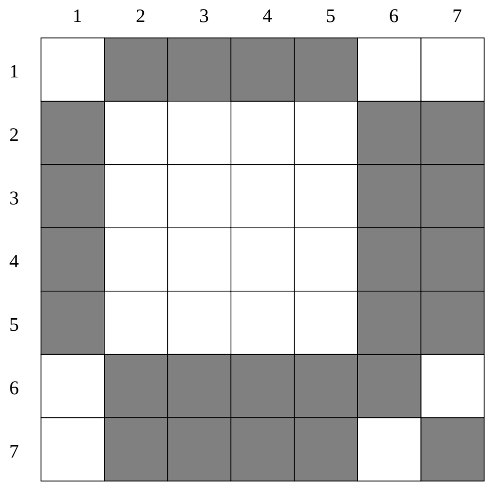

Figure 6.2: A coloring of $\mathcal{G} \times \mathcal{G}$ with discrepancy 0

6.4 Discrepancy of Boxes in $[R]^2$

We have just seen that general methods to derive bounds for the discrepancy of direct products of hypergraphs from the discrepancies of the factors do not exist. We thus started looking at some examples. The surprising result is that even for very well-understood hypergraphs it can be quite difficult to analyze the discrepancy of their product. To give some impression of the difficulties, let us estimate the discrepancy of

$$\mathcal{B}_{Rm} := \left([R], \binom{[R]}{m}\right) \times \left([R], \binom{[R]}{m}\right) = ([R] \times [R], \{A \times B | A, B \subseteq [R], |A| = |B| = m\}),$$

the hypergraph of 2–dimensional m–boxes in $[R]^2$ for some $R, m \in \mathbb{N}$. This is my part of a joint work [ADS99] with G. Agnarson (Reykjavik) and T. Schoen (Kiel) on Ramsey and discrepancy aspects of these hypergraphs. To ease the comparison with the Ramsey type results of this paper I kept the original notation which differs slightly from the one the reader might have gotten used to by now.

Put $d_{Rm} = \operatorname{disc}(\mathcal{B}_{Rm})$. In [ADS99] it was shown that for $m2^m \lesssim R$ we have a monochromatic m–box, that is, $d_{Rm} = m^2$. This marks the extreme case occurring when m is very small compared to R. Here we are interested in the case when m is larger compared to R. It turns out that also for relatively large m there is always a 'badly' colored m–box (of discrepancy $\Theta(m^{\frac{3}{2}})$, if $m < \frac{1}{2}R$ is a constant fraction of R.) The precise results are collected in the Theorems 6.5 and 6.6 below. We start with an upper bound.

Theorem 6.5 (Upper bound). *$d_{Rm} \leq 2m^{\frac{3}{2}}\sqrt{\log\left(\frac{eR}{m}\right)}$ for all $m \leq R$.*

Proof. We use the basic probabilistic bound of Theorem 1.1. Since $m! > 2(\frac{m}{e})^m$ we have

$$\begin{aligned} d_{Rm} &\leq \sqrt{2m^2 \log\left(2\binom{R}{m}^2\right)} \\ &\leq \sqrt{2m^2 \log\left(\left(\frac{eR}{m}\right)^{2m}\right)} \\ &= 2m^{\frac{3}{2}}\sqrt{\log\left(\frac{eR}{m}\right)}. \end{aligned}$$

□

The ideas of Chapter 5 do not help very much is this setting. Using the randomized chess board coloring of Section 5.3 reduces the number of relevant hyperedges slightly. The size reduction only works for larger hyperedges, i. e. $m > \frac{1}{2}R$. Our lower bound also shows that there is not too much room for improvement:

Theorem 6.6 (Lower bound). *For $m \le \frac{1}{2}R$ we have*

$$d_{Rm} \ge m^{\frac{3}{2}} \left(1 - \tfrac{2m}{R-1}\right) \sqrt{1 - \tfrac{m}{R-1}} - 3m.$$

More specifically, if $m \le \frac{1}{2}(R-1)^{\frac{2}{3}}$ we have

$$d_{Rm} \ge \min \left\{ m^{\frac{3}{2}} \sqrt{\tfrac{2}{5} \log_2 \left(\tfrac{R}{2m^{\frac{3}{2}}} \right)} - 4m, \tfrac{1}{3}(m-1)^2 - 4m \right\}.$$

To prove the theorem, we need the following lemma.

Lemma 6.7. *Let $d \le m \le R$ be given. Among all colorings $f : [R] \to \{-1, 1\}$ the number of m–subsets of $[R]$ having discrepancy at most d is maximal if and only if the color classes of f deviate in size by at most one.*

Proof. Set $D := \{i \in [-d, d] | m + i \text{ even}\}$. Obviously the number of m–subsets of $[R]$ having discrepancy at most d with respect to a given coloring f depends only on the sizes of the color classes. Hence for $R_1, R_2 \le R$ such that $R_1 + R_2 = R$

$$\delta(R_1, R_2) := \sum_{i \in D} \binom{R_1}{\frac{1}{2}(m-i)} \binom{R_2}{\frac{1}{2}(m+i)}$$

is the number of these sets. Assume $1 \le R_1 \le R_2$. Using induction is suffices to show $\delta(R_1, R_2) > \delta(R_1 - 1, R_2 + 1)$. Set $d_0 = \max D$. Then we have

$$\begin{aligned}
&\delta(R_1, R_2) - \delta(R_1 - 1, R_2 + 1) \\
&= \sum_{i \in D} \left[\binom{R_1 - 1}{\frac{1}{2}(m-i) - 1} + \binom{R_1 - 1}{\frac{1}{2}(m-i)} \right] \binom{R_2}{\frac{1}{2}(m+i)} \\
&\quad - \sum_{i \in D} \binom{R_1 - 1}{\frac{1}{2}(m-i)} \left[\binom{R_2}{\frac{1}{2}(m+i) - 1} + \binom{R_2}{\frac{1}{2}(m+i)} \right] \\
&= \sum_{i \in D} \binom{R_1 - 1}{\frac{1}{2}(m-i) - 1} \binom{R_2}{\frac{1}{2}(m+i)} - \sum_{i \in D} \binom{R_1 - 1}{\frac{1}{2}(m-i)} \binom{R_2}{\frac{1}{2}(m+i) - 1} \\
&= \binom{R_1 - 1}{\frac{1}{2}(m-d_0) - 1} \binom{R_2}{\frac{1}{2}(m+d_0)} - \binom{R_1 - 1}{\frac{1}{2}(m+d_0)} \binom{R_2}{\frac{1}{2}(m-d_0) - 1} > 0.
\end{aligned}$$

□

Now let us prove Theorem 6.6.

Proof. Let R and $d \le m \le \frac{1}{2}R$ be given. Let $f : [R]^2 \to \{-1,+1\}$ be any coloring.

Our general approach is the following: For a row $\{i\} \times [R] \subseteq [R]^2$ estimate the number of sets $\{i\} \times B_0$ of size m that have discrepancy $|f(\{i\} \times B_0)|$ at least d. Call n_d the minimum possible number of these sets among all colorings $f : [R]^2 \to \{-1,+1\}$. Without loss of generality we may assume that for any f there are at least $\frac{1}{2}n_d$ such sets with $f(\{i\} \times B_0) \ge d$. We will find conditions that imply $\frac{1}{2}n_d R > (m-1)\binom{R}{m}$. We conclude from the pigeon–hole principle that in that case there are m different numbers $i_k, k \in [m]$ and an m–set $B_0 \subseteq [R]$, such that $f(\{i_k\} \times B_0) \ge d$ for all $k \in [m]$. Thus $\{i_1, \dots, i_m\} \times B_0$ is an m–box having discrepancy at least md.

Let us first assume that R and m are both even. We will deal with the other values of m and R at the very end of this proof. Since a set M of even size always has an even discrepancy $d = |f(M)|$ with regard to all colorings f, we only need to consider even $d \ge 2$. By Lemma 6.7 we may assume that both color classes of f have size $\frac{R}{2}$.

We will estimate n_d in two ways. Which one is better depends on whether $m \le \frac{1}{2}R$, or more specifically $m \le \frac{1}{2}R^{\frac{2}{3}}$.

First case: We start with the easier and more general case $m \le \frac{1}{2}R$. Clearly we have

$$\begin{aligned} n_d &= \binom{R}{m} - \sum_{i=-\frac{d}{2}+1}^{\frac{d}{2}-1} \binom{\frac{R}{2}}{\frac{m}{2}-i}\binom{\frac{R}{2}}{\frac{m}{2}+i} \\ &\ge \binom{R}{m} - (d-1)\binom{\frac{R}{2}}{\frac{m}{2}}^2 . \end{aligned}$$

It suffices therefore to find the largest d such that

$$\tfrac{1}{2}R\left(\binom{R}{m} - (d-1)\binom{\frac{R}{2}}{\frac{m}{2}}^2\right) > (m-1)\binom{R}{m},$$

or equivalently $d - 1 < \left(1 - \frac{2(m-1)}{R}\right)\binom{R}{m}\binom{\frac{R}{2}}{\frac{m}{2}}^{-2}$. Define

$$g(n,m) := \frac{\sqrt{n}}{\sqrt{2\pi m(n-m)}}\left(\frac{n}{m}\right)^m \left(\frac{n}{n-m}\right)^{n-m}. \tag{6.2}$$

From a sharp version of the Stirling formula $\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n+1}} < n! < \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n}}$ due to Robbins [Rob55] we see that for all $1 \le m \le \frac{1}{2}n$ we have $e^{-\frac{1}{6m}}g(n,m) \le \binom{n}{m} \le g(n,m)$.

Using this we get

$$\frac{\left(1-\frac{2(m-1)}{R}\right)\binom{R}{m}}{\binom{\frac{R}{2}}{\frac{m}{2}}^2} \geq \frac{\left(1-\frac{2(m-1)}{R}\right)e^{-\frac{1}{6m}}g(R,m)}{g(\frac{R}{2},\frac{m}{2})^2}$$
$$= \sqrt{\frac{\pi m}{2}}\, e^{-\frac{1}{6m}}\left(1-\frac{2(m-1)}{R}\right)\sqrt{1-\frac{m}{R}}.$$

We may assume $m \geq 10$ as otherwise our bound is negative and there is nothing to prove. Hence a sufficient condition for the existence of an m–box of discrepancy md is that $d-1 \leq \frac{6}{5}\sqrt{m}(1-\frac{2m}{R})\sqrt{1-\frac{m}{R}}$. Choosing d even and maximal we get $d \geq \frac{6}{5}\sqrt{m}(1-\frac{2m}{R})\sqrt{1-\frac{m}{R}}-1$ and hence $d_{Rm} \geq \frac{6}{5}m^{\frac{3}{2}}(1-\frac{2m}{R})\sqrt{1-\frac{m}{R}}-m$.

Second case: We will now consider the more specific case when $m \leq \frac{1}{2}R^{\frac{2}{3}}$. In this situation it is advantageous to estimate

$$n_d \geq 2\binom{\frac{R}{2}}{\frac{m-d}{2}}\binom{\frac{R}{2}}{\frac{m+d}{2}}.$$

We use the elementary fact that for $t > 1$ the function $t \mapsto (1-\frac{1}{t})^t$ is monotone increasing and tends to $\frac{1}{e}$. In particular we have $\frac{1}{4} \leq (1-\frac{1}{t})^t < \frac{1}{2}$ for all $t \geq 2$. We have therefore

$$\mathbf{a}(x) := \left(\frac{x^2}{x^2-d^2}\right)^{\frac{x}{2}} \geq 2^{\frac{d^2}{2x}} \text{ for } x > d > 0, \tag{6.3}$$

$$\mathbf{b}(x) := \left(\frac{x-d}{x+d}\right)^{\frac{d-1}{2}} \geq 4^{-\frac{d(d-1)}{x+d}} \text{ for } x > 3d > 0. \tag{6.4}$$

Assume that $m > 3d$. We now find a condition on d that will imply $\frac{1}{2}n_d > (m-1)\binom{R}{m}$. Using the Stirling formula we get

$$\frac{\frac{1}{2}n_d}{(m-1)\binom{R}{m}} > \frac{R\binom{\frac{R}{2}}{\frac{m-d}{2}}\binom{\frac{R}{2}}{\frac{m+d}{2}}}{(m-1)\binom{R}{m}}$$
$$\geq \frac{Re^{-\frac{1}{6\frac{m-d}{2}}}g(\frac{R}{2},\frac{m-d}{2})e^{-\frac{1}{6\frac{m+d}{2}}}g(\frac{R}{2},\frac{m+d}{2})}{(m-1)g(R,m)}$$
$$= \frac{\sqrt{R(R-m)}\cdot cR\sqrt{m}\cdot\mathbf{a}(m)\mathbf{b}(m)\mathbf{a}(R-m)\mathbf{b}(R-m)}{(R-m+d)(m-1)(m+d)},$$

where $c = \sqrt{\frac{2}{\pi}}\, e^{\frac{-2m}{3(m^2-d^2)}}$.

By (6.3) and (6.4) we have $\mathbf{a}(x)\mathbf{b}(x) \geq 2^{\frac{d^2}{2x}} 4^{-\frac{d(d-1)}{x+d}} > 2^{\frac{-3d^2}{2x}}$ for $x > 3d$. Since $m > 3d$ we have $\frac{\sqrt{R(R-m)}}{R-m+d} > 1$ and $\frac{\sqrt{m}}{(m-1)(m+d)} > \frac{3}{4} m^{-\frac{3}{2}}$. Hence we get

$$\frac{\frac{1}{2} n_d}{(m-1)\binom{R}{m}} > \frac{3}{4} c \frac{R}{m^{\frac{3}{2}}} 2^{\frac{-3d^2}{2}\left(\frac{1}{m}+\frac{1}{R-m}\right)}.$$

We may assume $m \geq 14$. Hence $c = \sqrt{\frac{2}{\pi}}\, e^{\frac{-2m}{3(m^2-d^2)}} \geq \sqrt{\frac{2}{\pi}}\, e^{-\frac{3}{4m}} \geq \frac{3}{4}$ and $\frac{1}{m} + \frac{1}{R-m} \leq \frac{9}{10}\frac{1}{m}$. Therefore we get

$$\frac{\frac{1}{2} n_d}{(m-1)\binom{R}{m}} > \frac{9}{16} \frac{R}{m^{\frac{3}{2}}} 2^{-\frac{5d^2}{3m}}.$$

Thus our condition $\frac{1}{2} n_d R > (m-1)\binom{R}{m}$ is fulfilled when we have $m > 3d$ and $\frac{9}{16}\frac{R}{m^{\frac{3}{2}}} 2^{-\frac{5d^2}{3m}} \geq 1$, that is to say, that for all even d such that $d < \frac{m}{3}$ and $d \leq \sqrt{\frac{3}{5} m \log_2\left(\frac{9}{16}\frac{R}{m^{\frac{3}{2}}}\right)}$, there exists an m–box of discrepancy md. Hence there is an even d such that $d \geq \sqrt{\frac{3}{5} m \log_2\left(\frac{9}{16}\frac{R}{m^{\frac{3}{2}}}\right)} - 2$ or $d \geq \frac{m}{3} - 2$. Therefore we have $d_{Rm} \geq \min\left\{ m^{\frac{3}{2}} \sqrt{\frac{3}{5}\log_2\left(\frac{9}{16}\frac{R}{m^{\frac{3}{2}}}\right)} - 2m, \frac{m^2}{3} - 2m \right\}$.

Until now we have only considered R and m to be even. We will now consider general values of R and m. If $f : [R]^2 \to \{-1, +1\}$ is a coloring, then by restriction, we get a coloring $[R-1]^2 \to \{-1, +1\}$, and hence we have $d_{Rm} \geq d_{R-1\,m}$. Also note that an $(m-1)$–box of discrepancy d_{Rm-1} in $[R]^2$ will yield an m–box of discrepancy at least $d_{Rm-1} - (2m-1)$, simply by adding a vertical and horizontal line to our $(m-1)$–box. Hence $d_{Rm} \geq d_{Rm-1} - 2m + 1$. Therefore we have

$$d_{Rm} \geq \frac{6}{5}\left(2\lfloor\tfrac{m}{2}\rfloor\right)^{\frac{3}{2}}\left(1 - \frac{2m}{R-1}\right)\sqrt{1 - \frac{m}{R-1}} - 3m, \tag{6.5}$$

for all $m \leq \frac{1}{2}R$ and

$$d_{Rm} \geq \min\left\{ (m-1)^{\frac{3}{2}} \sqrt{\frac{3}{5}\log_2\left(\frac{9}{16}\frac{R-1}{m^{\frac{3}{2}}}\right)} - 4m, \frac{(m-1)^2}{3} - 4m \right\}.$$

for all $m \leq \frac{1}{2}(R-1)^{\frac{2}{3}}$. From $m \geq 10$ in the first case and $m \geq 14$ in the second we get the theorem. □

Theorem 6.8. *For all $R \in \mathbb{N}$ the discrepancy d_R of the hypergraph of all square boxes in $[R]^2$ is bounded by*

$$\frac{1}{15} R^{\frac{3}{2}} - \frac{4}{5} R \leq d_R \leq \frac{3}{4}(R+1)^{\frac{3}{2}}.$$

Proof. Note that the lower bound is trivial for $R \leq 144$. For the remaining values taking $m = \frac{1}{4}(R-1)$ in equation (6.5) of the proof of Theorem 6.6 is enough.

The upper is derived from Theorem 5.3. □

6.5 Symmetric Direct Products

We return to another general construction of hypergraphs. Let $\mathcal{H} = (X, \mathcal{E})$ denote a hypergraph. Set $\mathcal{E}^d_{\text{sym}} := \{E^d | E \in \mathcal{E}\}$. We call $\mathcal{H}^d_{\text{sym}} := (X^d, \mathcal{E}^d_{\text{sym}})$ the d–fold *symmetric direct product* of $\mathcal{H}$. This construction was investigated in Wehr's dissertation, where she also proved the theorem below. We give a very short algebraic proof here. All one needs is to note that the 2–element group $\mathbb{Z}_2$ acts on the symmetric product in a way that all hyperedges are left invariant and that the set of fixed points is just the diagonal.

Theorem 6.9.

$$\operatorname{disc}(\mathcal{H}^d_{\text{sym}}) \leq \operatorname{disc}(\mathcal{H}).$$

Proof. Let $D := \{(x, \ldots, x) | x \in X\}$ be the diagonal of X^d. For $x \in X^d \setminus D$ set

$$a(x) := \min\{i | x_i \neq x_{i+1}\}.$$

Define $f : X^d \to X^d$ by

$$(f(x))_i := \begin{cases} x_{a(x)+1} & \text{if } x \notin D, i \leq a(x) \\ x_1 & \text{if } x \notin D, i = a(x)+1 \\ x_i & \text{otherwise} \end{cases}$$

for all $i \in [d], x \in X^d$. Right from the definition, we see that $f(f(x)) = x$ holds for all $x \in X^d$, so f is a bijection. For all $x \in X^d \setminus D$ the f–orbit $O_f(x)$ of x has order 2 and consists of x and $f(x)$. Further we have

$$\{x_i | i \in [d]\} = \{(f(x))_i | i \in [d]\},$$

and thus f leaves the hyperedges of $\mathcal{H}^d_{\text{sym}}$ invariant.

Pick an optimal coloring $\chi_{\mathcal{H}}$ of $\mathcal{H}$. Choose a system R of representatives of the f–orbits in $V^d \setminus D$, i. e. for all $x \in V^d \setminus D$ either x or $f(x)$ lies in R. Define $\chi : V^d \to \{-1, 1\}$ by

$$\chi(x) := \begin{cases} -1 & \text{if } x \in R \\ \chi_{\mathcal{H}}(v) & \text{if } x = (v, \ldots, v) \in D \\ 1 & \text{otherwise} \end{cases} .$$

Let $E \in \mathcal{E}$. From the properties of f and R we deduce $|E^d \cap R| = |f(E^d \cap R)| = |f(E^d) \cap f(R)| = |E^d \cap (X^d \setminus D \setminus R)| = |E^d \setminus D \setminus R|$. So we have

$$\sum_{x \in E^d} \chi(x) = \sum_{x \in E^d \cap R} -1 + \sum_{x \in E^d \cap D} \chi(x) + \sum_{x \in E^d \cap f(R)} 1 = \sum_{v \in E} \chi_{\mathcal{H}}(v),$$

and this proves the theorem. □

Lower bound for this construction seem difficult again. For the general case nothing can be said, as is obvious from Example 2 above. In the special case of arithmetic progressions, it is not clear to us how to use the Fourier analysis approach of [Weh97, DSW98].

Chapter 7

Vector Balancing Games

In this chapter we study a particular class of vector balancing games. In contrast to previous works by several authors we will assume that decisions in earlier rounds become less and less important as the play continues. We therefore investigate the following game: Initially the position vector $p \in \mathbb{R}^d$ is zero. Each round the first player chooses a vector $x \in \mathbb{R}^d$ having $||x||_\infty \leq 1$. The second player then chooses a sign $\varepsilon \in \{-1, +1\}$ and the position vector is updated $p := \frac{1}{q}p + \varepsilon x$. The parameter $q > 1$ is fixed for the game and quantifies the effect of aging. The pay-off for the first player shall be $||p||_\infty$.

It turns out that the optimal strategies differ depending on whether the players know the number of rounds played or not. We study both cases separately. We restrict ourselves to the case that $q \geq 2$. We determine the exact value of the game where the number of rounds is fixed and give nearly tight bounds for the continuous version.

The investigation of this type of games was motivated by the fact that they naturally appeared in the proof of Theorem 3.1.

7.1 Introduction to Vector Balancing Games

A *vector balancing game* in $d \geq 2$ dimensions is a two-player game. Each round the first player chooses a vector x from some given set $X \subseteq \mathbb{R}^d$. The second player then chooses a sign $\varepsilon \in \{-1, +1\}$ and the position vector p, initially set to zero, is changed to $p + \varepsilon x$. The first player's aim is to maximize $||p||$ for some given norm $|| \cdot ||$, while the second player tries to minimize this quantity. We call $||p||$ the pay-off for the first player.

The idea of such games is to represent the on-line version of vector balancing problems. A *vector balancing problem* consists of a set X of vectors and the task is to partition this set into two classes X_1 and X_2 such that the sums $\sum_{x \in X_i} x$ over all vectors in each class are roughly equal, that is, their difference is small with respect to some norm. It is convenient to represent the 2–partition by a mapping $\varepsilon : X \to \{-1, +1\}$ such that $\varepsilon(x) = 1$ holds if and only if $x \in X_1$. With this setting we can express the imbalance $\sum_{x \in X_1} x - \sum_{x \in X_2} x$ of the partition simply by $\sum_{x \in X} \varepsilon(x)x$. Taking the vectors of X as column vectors of a matrix we see that the vector balancing problem is equivalent to the discrepancy problem for matrices[1] (and hence is a generalization of the discrepancy problem for hypergraphs).

The additional difficulty in an on-line vector balancing problem is that one does not know the set of vectors at the beginning but one gets to know them one by one and has to decide on a sign without knowing the next one. As common for most on-line problems we translate this problem into the language of games and thus get a vector balancing game as just described. The name 'Pusher-Chooser' game is also commonly used for these types of games.

Previous Results

Several forms of vector balancing games have been studied. They differ in the set of vectors available to the first player and the norm that is used to determine the pay-off. An other variant is to allow a buffer of some size where the second player can store some vectors and thus postpone the decision on the respective signs. We mention some results of the different types:

Continuous Games. In this variant the first player may choose any vector with norm at most one, i. e. $X = \{x \in R^d | \, \|x\| \leq 1\}$, and the pay-off is measured using the same norm. For the Euclidean or 2–norm it is easy to see that both players have strategies ensuring that $\|p\|_2 \geq \sqrt{n}$ respectively $\|p\|_2 \leq \sqrt{n}$ holds after n rounds. We say that the value of this game is $\sqrt{n}$.

If the $\|\cdot\|_2$ –norm is replaced by the maximum norm $\|\cdot\|_\infty$, Spencer [Spe77] gave an upper bound of $\sqrt{2n \ln(2d)}$. For the d round game he proved a lower bound of $\sqrt{d \log d}(1 - o(1))$ in [Spe87].

Discrete Games. For games with finite set X Barany [Bar79] found a complete solution. His result implies that in the case $X = \{0, 1\}^d$ and $\|\cdot\| = \|\cdot\|_\infty$ the value of the game played sufficiently many rounds is 2^{d-2}.

Games with Buffer. The first result allowing a buffer is due to Barany and Grunberg [BG81] who showed that given a buffer of size d the second player can keep $\|p\|$ below $2d$ no matter what norm is used (the same norm is required in the definition of $X = \{x \in R^d | \, \|x\| \leq 1\}$ and the pay-off). This was improved by Peng and Yan [PY98]. They show that a buffer

[1] We defined the discrepancy of matrices by $\min_{\chi \in \{-1,+1\}^n} \|A\chi\|_\infty$, but of course the maximum norm can be replaced by other norms and the resulting discrepancy problem is still interesting.

of size $d-1$ suffices. They also remark that for the 2–norm allowing a buffer of less that $d-1$ vectors gives no improvement to the no-buffer case.

7.2 Vector Balancing Games with Time Constraints

In this chapter we introduce a new aspect. We assume that a decision made in the past (i. e. in an earlier round of the game) is less important than a newer one. We represent this by the different update rule $p := \frac{1}{q}p + \varepsilon x$ for some parameter $q > 1$. We restrict ourselves to the maximum norm, i. e. the first player chooses vectors x such that $\|x\|_\infty \leq 1$ and the pay-off is measured using $\|\cdot\|_\infty$ on p too.

Immediately we see that the pay-off is bounded by $\frac{q}{q-1}$. This is due to the update rule which rescales the importance of decisions in the past relative to the current one. A different approach working with absolute values is the following: In round i the first player chooses a vector $x^{(i)}$ with norm at most q^{i-1} and the the second player updates the position vector either to $p := p + x^{(i)}$ or $p := p - x^{(i)}$. The values of an n round game then differs from ours approach by a factor of q^{n-1}. Hence we lose nothing by investigating the first approach which we find more natural.

The game is somewhat different depending on whether the aging parameter q is at least 2 or not. In the first case the aging aspect is very dominant. Thus the strategies are completely different from the ones in the game without aging. If $1 < q < 2$ the aging is less important requiring different strategies again. We restrict ourselves to the case $q \geq 2$.

Contrary to the no-buffer games described above the maximum value for $\|p\|_\infty$ does not necessarily occur after the last round[2]. This motivates the distinction of two versions of the game. First, the value of $\|p\|_\infty$ after the last round is the pay-off for the first player, and second, the maximum value of $\|p\|_\infty$ occurred in play is the pay-off for the first player. We call the two versions the *fixed end version* and *open end version* respectively.

We show that the fixed end version of the game has value $\frac{q-q^{-\lfloor \log_2 d \rfloor+1}}{q-1}$ if at least $\log_2 d$ rounds are played (otherwise replace $\lfloor \log_2 d \rfloor$ by the number of rounds). For the open end version we will analyze a second game similar to Spencer's tenure game and show that the first player can get a pay-off of at least $\frac{q-2q^{-\lfloor \log_2 d \rfloor - \lfloor \log_2 \log_2 d \rfloor + 1}}{q-1}$ (again assuming sufficiently many rounds played) while the second player has a strategy keeping $\|p\|_\infty$ below $\frac{q}{q-1} - q^{-\log_2 d - \log_2 \log_2 d - 4}$ throughout the game. This shows that it is indeed helpful for both players to know the exact number of rounds.

[2]Actually, Spencer's proof for the upper bound in [Spe77] also requires the second player to know the number of rounds. Olson [Ols85] later gave a strategy that does not need this information and still yields the same bound (apart from constants).

7.3 The Fixed End Game

In this section we analyze the version of the vector balancing game with aging where both player know the number n of rounds played. For clarity we restate the rules of this game which we denote by G_{ndq}.

Rules of G_{ndq}*:* Initially the position vector $p \in \mathbb{R}^d$ is set zero. A round of the game consists of three steps:

(i) Player A chooses a vector $x \in \mathbb{R}^d$ such that $\|x\|_\infty \leq 1$,

(ii) Player B chooses a sign $\varepsilon \in \{-1, +1\}$,

(iii) the position vector is updated: $p := \frac{1}{q}p + \varepsilon x$.

The game is played for n rounds. The value $\|p\|_\infty$ at the end of the game is the pay-off for player A, i. e. Player A aims to maximize $\|p\|_\infty$ and Player B to minimize this quantity. The maximum pay-off Player A can enforce is called the value $v(G_{ndq})$ of the game.

A particular sequence of moves respecting the rules of the game is called an instance of the game. Formally, it is a pair $I = ((x^{(i)})_{i\in[n]}, (\varepsilon^{(i)})_{i\in[n]})$ such that $\|x^{(i)}\|_\infty \leq 1$ and $\varepsilon^{(i)} \in \{-1, +1\}$ for all $i \in [n]$. From the definition we see that the pay-off for this instance is $\sum_{i\in[n]} q^{n-i}\varepsilon^{(i)}x^{(i)}$.

In the following let us assume that $q \geq 2$. In the analysis of this case (see the proof of Theorem 7.1 below) we exhibit a surprising phenomenon. It turns out that only the last $\lfloor \log_2 d \rfloor$ moves are important (for this reason the players need to know the number of rounds). Any value $\|p\|_\infty$ that Player A might have reached up to round $n - \lfloor \log_2 d \rfloor$ will not only not help him, but even be contraproductive. Hence up to this point the players will pursue the opposite aims. The optimal strategy for player A is to select $x^{(i)} = 0$ for $i \in [n - \lfloor \log_2 d \rfloor]$ and thus minimizing $\|p\|_\infty$.

Theorem 7.1. *Assume $q \geq 2$ and $n, d \in \mathbb{N}$. Set $r := \min\{n, \lfloor \log_2 d \rfloor\}$. The value of the game G_{ndq} is*

$$v(G_{ndq}) = \frac{q - q^{-r+1}}{q - 1}.$$

Proof. Player A can follow this strategy: Choose the first $n-r$ vectors as zero ($x^{(i)} := 0$ for all $i \in [n-r]$). The last r vectors choose like this: Components with index greater than 2^r are always set zero (for instance). For an index $i = 1+\sum_{j=0}^{r-1} a_j 2^j \leq 2^r$, $a_0, \ldots, a_{r-1} \in \{0, 1\}$

and a $p \in \{0, \ldots, r-1\}$ set $x_i^{(n-p)} := 2a_p - 1$. Here is an example for $d = 5$:

$$\begin{aligned} w^{(j)} &= (0,0,0,0,0) \text{ for } i \in [n-q] \\ w^{(n-2)} &= (-1,-1,-1,-1,0) \\ w^{(n-1)} &= (-1,-1,+1,+1,0) \\ w^{(n)} &= (-1,+1,-1,+1,0). \end{aligned}$$

Whatever signs $\varepsilon^{(j)}, j \in [n]$ are chosen, there will always be a component $i \in [2^r]$ such that $x_i^{(n-r+1)} = \ldots = x_i^{(n)}$ and thus

$$\sum_{j=1}^{n} q^{-n+j} x_i^{(j)} = x_i^{(n)} \sum_{j=0}^{r-1} q^{-j} = x_i^{(n)} \frac{1 - q^{-r}}{1 - \frac{1}{q}}.$$

Hence the pay-off in this instance is $\frac{1-q^{-r}}{1-\frac{1}{q}}$.

We investigate the following strategy for Player B. Assume $r < n$, as otherwise any choice of moves for Player B does the job. Whatever vectors $x^{(1)}, \ldots, x^{(n-r)}$ Player A chooses in the first $n-r$ rounds, pick $\varepsilon^{(1)}, \ldots, \varepsilon^{(n-r)} := 1$ (any other choice would do, too). Set $p := \sum_{j=1}^{n-r} q^{-n+j} x^{(j)}$. Choose the next sign $\varepsilon^{(n-r+1)}$ in such a way that the number of components $i \in [d] =: X_1$ such that $\operatorname{sgn}(p_i)$ and $\operatorname{sgn}(\varepsilon^{(k-r+1)} x_i^{(k-r+1)})$ are different is maximal. Set $X_2 := \{i \in X_1 \mid \operatorname{sgn}(p_i) = \operatorname{sgn}(\varepsilon^{(k-r+1)} x_i^{(n-r+1)}) \neq 0\}$. Next choose $\varepsilon^{(n-r+2)} \in \{-1, 1\}$ such that the number of components $i \in X_2$ such that $\operatorname{sgn}(p_i)$ and $\operatorname{sgn}(\varepsilon^{(n-r+2)} x_i^{(n-r+2)})$ are different is maximal. Set $X_3 := \{i \in X_2 \mid \operatorname{sgn}(p_i) = \operatorname{sgn}(\varepsilon^{(n-r+2)} x_i^{(n-r+2)}) \neq 0\}$. Continue in this fashion until $\varepsilon^{(n)}$ and X_r are determined.

Note that $|X_{j+1}| \leq \left\lfloor \frac{|X_j|}{2} \right\rfloor$ for all $j \in [r-1]$, which gives $|X_r| < 1$, i.e. $X_r = \emptyset$. So for every component i there is a $j \in \{n-r+1, \ldots, n\}$ such that p_i and $\varepsilon^{(j)} x_i^{(j)}$ have different signs. The worst case — and here $q \geq 2$ comes into play — is the one where for one component $i \in [d]$ all $\varepsilon^{(j)} x_i^{(j)}$, $j \in \{n-r+1, \ldots, n\}$ are 1 (or -1) and p_i is zero. For the pay-off at the end of the game we have

$$\begin{aligned} \left\| \sum_{j=1}^{n} q^{-n+j} \varepsilon^{(j)} x^{(j)} \right\|_\infty &= \left\| \sum_{j=n-r+1}^{n} q^{-n+j} \varepsilon^{(j)} x^{(j)} + p \right\|_\infty \\ &\leq \sum_{z=0}^{r-1} q^{-z} \\ &= \frac{1 - q^{-r}}{1 - \frac{1}{q}}. \end{aligned}$$

This ends the proof. □

7.4 The Open End Game

In this section we investigate the version of the game where the players do not know the number of rounds played. Therefore we consider the case that the game is played on and on. The pay-off for Player A shall be the supremum over all values of $\|p\|_\infty$ that occurred during play.[3] We denote this game by $G_{\infty dq}$.

An instance for $G_{\infty dq}$ is a pair $I = ((x^{(i)})_{i\in\mathbb{N}}, (\varepsilon^{(i)})_{i\in\mathbb{N}})$ such that $\|x^{(i)}\|_\infty \leq 1$ and $\varepsilon^{(i)} \in \{-1,+1\}$ for all $i \in \mathbb{N}$. From the definition we see that the pay-off for this instance is $\sup_{n\in\mathbb{N}} \sum_{i\in[n]} q^{-n+i}\varepsilon^{(i)}x^{(i)}$. Let us assume again $q \geq 2$. This will not be necessary for the proofs, but the results get uninteresting for q too close to 1. We show

Theorem 7.2. *The value of the game $G_{\infty dq}$ satisfies*

$$\frac{q - 2q^{-\lfloor\log_2 d\rfloor - \lfloor\log_2\log_2 d\rfloor + 1}}{q-1} \leq v(G_{\infty dq}) \leq \frac{q}{q-1} - q^{-\log_2 d - \log_2\log_2 d - 4}.$$

Similarly to the fixed end version of the game we will also work with strategies of 'changing signs'. As the players do not know the number of moves, the analysis is much harder. A first strategy for player B is to enforce a 'change of signs' in every block B_k of rounds $(k-1)\log_2 d + 1, \ldots, k\log_2 d$. This is equivalent to saying that Player B should play according to his strategy of section 7.3 assuming all rounds $k\log_2 d, k \in \mathbb{N}$ to be last rounds. This might though lead to a subsequence of rounds $S = \{k\log_2 d+2, \ldots, (k+2)\log_2 d-1\}$ where all values $\varepsilon^{(i)}x_j^{(i)}, i \in S$ have the same sign different from zero for some component $j \in [d]$. We show that Player B can do better: He has a strategy such that for every component $j \in [d]$ and any set S of roughly $\log_2 d + \log_2\log_2 d$ successive rounds there are $i_1, i_2 \in S$ such that $\operatorname{sgn}(\varepsilon^{(i_1)}x_j^{(i_1)}) \neq \operatorname{sgn}(\varepsilon^{(i_2)}x_j^{(i_2)})$. This gives a much better bound for the value of the game $G_{\infty dq}$ as the next lemma shows.

Lemma 7.3. *Let $r \in \mathbb{N}$.*

(i) Suppose that for every sequence $(x^{(k)})_{k\in\mathbb{N}}$ Player B can choose his moves $(\varepsilon^{(k)})_{k\in\mathbb{N}}$ in such a way that for every $k \in \mathbb{N}$ and every component $i \in [d]$ not all numbers $\varepsilon^{(k)}x_i^{(k)}, \ldots, \varepsilon^{(k+r)}x_i^{(k+r)}$ have the same sign.

Then $v(G_{\infty dq}) \leq \frac{q}{q-1} - q^{-r}$.

(ii) Suppose that Player A has a strategy using $\{-1,1\}$ vectors only that enforces that for some $k \in \mathbb{N}$, $i \in [d]$ all numbers $\varepsilon^{(k)}x_i^{(k)}, \ldots, \varepsilon^{(k+r-1)}x_i^{(k+r-1)}$ have the same sign different from zero. Then $v(G_{\infty dq}) \geq \frac{q-2q^{-r+1}}{q-1}$.

[3] From the viewpoint of games this is just one side of the coin as we do not study the best situation Player B can reach (say after some initial rounds). The reason is the connection between the game and the corresponding vector balancing problem. The supremum over all $\|p\|_\infty$ corresponds to the worst-case imbalance that might occur in the corresponding balancing process. The latter is the quantity we are interested in.

Proof. Let $(x^{(k)})_{k\in\mathbb{N}}$ be a sequence of vectors in $\mathbb{R}^d$ of norm at most 1 and $(\varepsilon^{(k)})_{k\in\mathbb{N}}$ be a sequence in $\{-1,+1\}$.

Assume first that for every $k \in \mathbb{N}$ and every component $i \in [d]$ not all of the numbers $\varepsilon^{(k)}x_i^{(k)},\dots,\varepsilon^{(k+r)}x_i^{(k+r)}$ have the same sign. Let $k \in \mathbb{N}$ and $p := \sum_{j\in[k+r]} q^{-k-r+j}\varepsilon^{(j)}x^{(j)}$ denote the position vector at the end of round $k+r$. Let $i \in [d]$. As not all of the numbers $\varepsilon^{(k)}x_i^{(k)},\dots,\varepsilon^{(k+r)}x_i^{(k+r)}$ have the same sign, the worst-case is clearly the one where $x_i^{(k)}$ is zero and other numbers are all 1 or all -1. Thus we have

$$|p_i| = \left|\sum_{j\in[k+r]} q^{-k-r+j}\varepsilon^{(j)}x_i^{(j)}\right| \le \sum_{\substack{j\in[k+r]\\ j\neq k}} q^{-k-r+j}\left|\varepsilon^{(j)}x_i^{(j)}\right| \le \sum_{\substack{j\in[k+r]\\ j\neq k}} q^{-k-r+j} < \frac{q}{q-1} - q^{-r}.$$

Assume now that all vectors are $\{-1,1\}$ vectors and there is a $k \in \mathbb{N}$ and a component $i \in [d]$ such that all numbers $\varepsilon^{(k)}x_i^{(k)},\dots,\varepsilon^{(k+r-1)}x_i^{(k+r-1)}$ have the same sign. Let $p := \sum_{j\in[k+r-1]} q^{-k-r+1+j}\varepsilon^{(j)}x^{(j)}$ denote the position vector at the end of round $k+r-1$. We have

$$\begin{aligned}
|p_i| &= \left|\sum_{j\in[k+r-1]} q^{-k-r+1+j}\varepsilon^{(j)}x_i^{(j)}\right| \\
&\ge \left|\sum_{j=k}^{k+r-1} q^{-k-r+1+j}\varepsilon^{(j)}x_i^{(j)}\right| - \left|\sum_{j=1}^{k-1} q^{-k-r+1+j}\varepsilon^{(j)}x_i^{(j)}\right| \\
&\ge \sum_{j=0}^{r-1} q^{-j} - \sum_{j=1}^{k-1} q^{-k-r+1+j} \\
&> \frac{q^{-r}-1}{\frac{1}{q}-1} + q^{-r}\frac{1}{\frac{1}{q}-1} = \frac{q-2q^{-r+1}}{q-1}.
\end{aligned}$$

This completes the proof. □

Note that an upper bound of $\frac{q}{q-1} - \frac{2q^{-r+1}}{q-q^{-r}}$ can be shown without much more effort (one needs to use the assumption of changing signs not only on the last $r+1$ vectors but on every group of $r+1$ vectors). A similar argument improves the lower bound slightly.

In the remainder of this section we determine the correct value for r such that both players have strategies as in Lemma 7.3. To do so we analyze the following game C_d for $d \in \mathbb{N}$: Given are d piles $p_1,\dots,p_d$ of say cards that initially hold one card each. A round of this game consists of the three steps

(i) Player A chooses a set $S \subseteq [d]$ of piles,

(ii) Player B either removes all cards from the piles in S or all cards from the piles in $[d] \setminus S$. Formally, B chooses a set $T \in \{S, [d] \setminus S\}$ and sets $p_i := 0$ for all $i \in T$.

(iii) One card is placed on every pile ($p_i := p_i + 1$ for all $i \in [d]$).

The game is played infinitely many rounds. The pay-off for Player A is the maximum value of $\|p\|_\infty$ that occurred during play. This game is similar to the tenure game Spencer investigated in [Spe94]. Instead of step (iii) there a card is added only to those piles p_i that have $p_i \neq 0$. Hence the number of active piles reduces in play and finally is zero. The tenure game has a nice solution: Both players do their best if they choose their moves such that $\sum_{i \in [d]} 2^{p_i}$ is maximized respectively minimized. A function like this, that is, assigning real numbers to states of the game in such a manner that maximizing respectively minimizing this function are good strategies for the players, is called *potential function*. For the game C_d a potential function is much harder to find as all piles keep active throughout the game.

To motivate the investigation of this game we first show the connection between the games C_d and $G_{\infty dq}$.

Lemma 7.4. *Suppose that the value of C_d equals r, i. e. Player B has a strategy such that no pile ever contains more than r cards and Player A can enforce such a situation. Then Player B has a strategy as in Lemma 7.3(i) and Player A has a strategy as in Lemma 7.3(ii). Hence the value of C_d determines the value of $G_{\infty dq}$ almost completely.*

Proof. Let $I_G = ((x^{(j)})_{j \in \mathbb{N}}, (\varepsilon^{(j)})_{j \in \mathbb{N}})$ be an instance of the game $G_{\infty dq}$. We call an instance $I_C = ((S^{(j)})_{j \in \mathbb{N}}, (T^{(j)})_{j \in \mathbb{N}})$ of C_d corresponding to I_G if

(i) $\forall j \in \mathbb{N} : S^{(j)} = \{i \in [d] \,|\, \operatorname{sgn}(\varepsilon^{(j)} x_i^{(j)}) = \operatorname{sgn}(x_i^{(j+1)}) \neq 0\}$,

(ii) $\forall j \in \mathbb{N} : T^{(j)} = S^{(j)} \iff \varepsilon^{(j+1)} = -1$.

Suppose I_G and I_C as above and corresponding. Then for all $i \in [d]$ and $k \in \mathbb{N}$ we have

$$i \in T^{(k)} \iff \left| \operatorname{sgn}(\varepsilon^{(k)} x_i^{(k)}) + \operatorname{sgn}(\varepsilon^{(k+1)} x_i^{(k+1)}) \right| \leq 1.$$

Note that the right-hand side just means that $\varepsilon^{(k)} x_i^{(k)}$ and $\varepsilon^{(k+1)} x_i^{(k+1)}$ have different signs or are both zero. Hence for a component $i \in [d]$ and $r \in \mathbb{N}$ we have that all numbers $\varepsilon^{(k)} x_i^{(k)}, \ldots, \varepsilon^{(k+r-1)} x_i^{(k+r-1)}$ have the same non-zero sign if and only if $i \notin T^{(k)} \cup \ldots \cup T^{(k+r-2)}$. This is equivalent to the fact that the position vector $p^{(k+r-2)}$ after round $k+r-2$ in C_d fulfills $p_i^{(k+r-2)} \geq r$. Thus it is enough to show that Player B can choose signs in game $G_{\infty dq}$ such that the corresponding instance of C_d has value at most r and Player A can choose $\{-1, 1\}$ vectors such that the corresponding instance of C_d has value at least r.

The following strategy for Player B does the job. For the first move in $G_{\infty dq}$ Player B may choose any sign $\varepsilon^{(1)}$. After A's second move set $S^{(1)} = \{i \in [d] | \operatorname{sgn}(\varepsilon^{(1)} x_i^{(1)}) = \operatorname{sgn}(x_i^{(2)}) \neq 0\}$. Choose $T^{(1)} \in \{S^{(1)}, [d] \setminus S^{(1)}\}$ according to the strategy that keeps the position vector in C_d at norm at most r. If $T^{(1)} = S^{(1)}$ set $\varepsilon^{(2)} = -1$ and $\varepsilon^{(2)} = 1$ otherwise. Continue like this for all rounds of the game. It is clear that $I_G = ((x^{(j)})_{j\in\mathbb{N}}, (\varepsilon^{(j)})_{j\in\mathbb{N}})$ and $I_C = (p, (S^{(j)})_{j\in\mathbb{N}}, (T^{(j)})_{j\in\mathbb{N}})$ are corresponding instances such that at no time any pile in I_C gets higher than r.

The following strategy serves for Player A. For the first move in $G_{\infty dq}$ Player A may choose any $\{-1, +1\}$ vector $x^{(1)}$. Set $p = (1, \dots, 1)^\top \in \mathbb{R}^d$. Let $S^{(1)}$ be a choice of A in C_d following the strategy that enforces a pile of height r once in the game. Define $x^{(2)} \in \mathbb{R}^d$ by

$$x_i^{(2)} := \begin{cases} \varepsilon^{(1)} x_i^{(1)} & \text{if } i \in S \\ -\varepsilon^{(1)} x_i^{(1)} & \text{else} \end{cases} .$$

If Player B chooses $\varepsilon^{(2)} = -1$, set $T^{(1)} = S^{(1)}$, else set $T^{(1)} = [d] \setminus S^{(1)}$. Update the position vector p of the card game as required by the rules. Continue like this for the rest of the game. As Player A is following his strategy, there will once be a pile of height r. By definition the instance of the vector game and the card game are corresponding. Thus there are $k \in \mathbb{N}$, $i \in [d]$ such that all numbers $\varepsilon^{(k)} x_i^{(k)}, \dots, \varepsilon^{(k+r-1)} x_i^{(k+r-1)}$ have the same sign different from zero. This shows that A's strategy is as required by Lemma 7.3(ii). $\square$

To complete the proof of Theorem 7.2 we bound the value $v(C_d)$ of the card game C_d.

Lemma 7.5. *For $d \geq 3$ the value of the card game C_d satisfies*

$$\lfloor \log_2 d \rfloor + \lfloor \log_2 \log_2 d \rfloor \leq v(C_d) \leq \frac{\log_2 d + \log_2 \log_2 d}{\log_2 \left(2 - \frac{1}{\log_2 d}\right)} + 1 \leq \log_2 d + \log_2 \log_2 d + 4.$$

For $d = 2$ we have $v(C_d) = 2$.

Proof. We first observe that the order of the piles is irrelevant, hence we may describe the position vector by an expression $x_1^{n_1} x_2^{n_2} \dots x_l^{n_l}$ meaning that there are n_i piles each holding x_i cards for all $i \in [l]$. Similarly, a move by Player A (which is a subset S of the index set $[d]$) can be described by such an expression (again it is not important which of the possibly several piles of same size are in S).

We start with a strategy for Player A. Let us assume first that d is a power of 2. It is clear that Player A can force a position vector $x_1^{n_1} x_2^{n_2} \dots x_l^{n_l}$ where all $n_i, i \in [l]$ are even, to change to the position vector $(x_1 + 1)^{\frac{n_1}{2}} (x_2 + 1)^{\frac{n_2}{2}} \dots (x_l + 1)^{\frac{n_l}{2}} 1^{\frac{d}{2}}$ by choosing the set $x_1^{\frac{n_1}{2}} x_2^{\frac{n_2}{2}} \dots x_l^{\frac{n_l}{2}}$. Repeated application of this strategy on the initial position vector of 1^d leads to the position $(\log_2 d + 1)^1 (\log_2 d)^1 (\log_2 d - 1)^2 (\log_2 d - 2)^4 \dots 2^{\frac{d}{4}} 1^{\frac{d}{2}}$.

We call a partial position (which is just the restriction of p to a subset of $[d]$) a logarithmic ladder of type $L(s,e,l)$ for some $s,e,l \in \mathbb{N}, s \geq l$ if it equals $s^e(s-1)^{2e}(s-2)^{4e}\dots(s-l+1)^{2^{(l-1)}e}$. In this notation we just showed that Player A can enforce a logarithmic ladder $L(\log_2 d, 1, \log_2 d)$. We now show that Player A can enforce a logarithmic ladder $L(s+1,1,\lfloor\frac{l}{2}\rfloor)$ from a position containing a logarithmic ladder $L(s,1,l)$. A's first move is $S = L(s,1,\lfloor\frac{l}{2}\rfloor)$. If B chooses S, then (among other piles) a logarithmic ladder $L(s-\lfloor\frac{l}{2}\rfloor, 2^{\lfloor\frac{l}{2}\rfloor}, \lceil\frac{l}{2}\rceil)$ remains and is updated to $L(s+1-\lfloor\frac{l}{2}\rfloor, 2^{\lfloor\frac{l}{2}\rfloor}, \lceil\frac{l}{2}\rceil)$ in step (iii) of this round. By the equi-partition argument from above (applied $\lfloor\frac{l}{2}\rfloor$ times) Player A can enforce a logarithmic ladder $L(s+1,1,\lceil\frac{l}{2}\rceil)$. If B chooses the complement of S, then S is simply updated to $L(s+1,1,\lfloor\frac{l}{2}\rfloor)$.

Applying this logarithmic ladder partition strategy $\lfloor\log_2\log_2 d\rfloor$ times on $L(\log_2 d, 1, \log_2 d)$, Player A can reach a logarithmic ladder $L(\log_2 d + \lfloor\log_2\log_2 d\rfloor, 1, 1)$ which is nothing more than a pile of size $\log_2 d + \lfloor\log_2\log_2 d\rfloor$. This proves the lower bound for d a power of 2. If d is not a power of 2, Player A fixes a set of $2^{\lfloor\log_2 d\rfloor}$ piles, plays on these piles according to the strategy just described and ignores the remaining piles.[4]

For the upper bound we have a potential function strategy: Set $\lambda := \frac{2\log_2 d - 1}{\log_2 d}$ and $v : \mathbb{N} \to \mathbb{R}; i \mapsto \lambda^{i-1}$. Set $v(p) := \sum_{i\in[d]} v(p_i)$ for a position vector $p \in \mathbb{N}^d$. We analyze the strategy for Player B to choose that one of the alternatives which minimizes $v(p)$ for the resulting position vector p. Write $p \circ A$ for the position vector resulting from p if Player B chooses the set A of piles to be emptied. We have $v(p \circ A) = |A| + \sum_{i\in[d]\setminus A} v(p_i+1) = |A| + \sum_{i\in[d]\setminus A} \lambda v(p_i)$. We claim that Player B can ensure $v(p) \leq d\log_2 d$ throughout the game. This is clear for the first move, so let us assume that we are in some round such that the position vector p at the start of this round fulfills $v(p) \leq d\log_2 d$. Let $S \subseteq [d]$ denote A's move and T be one of the alternatives S and $[d]\setminus S$ which minimizes $v(p \circ T)$.

[4] Joel Spencer (private communication in summer 2000) noted that the last two paragraphs can be replaced by a potential function argument. Having reached the position $(\log_2 d+1)^1(\log_2 d)^1(\log_2 d - 1)^2(\log_2 d - 2)^4 \dots 2^{\frac{d}{4}} 1^{\frac{d}{2}}$, Player A may use his strategy from the tenure game. This means trying to maximize the potential function $v(p) = \sum_{i\in[d]} 2^{p_i}$. This involves the so-called splitting lemma showing that he can split the piles into two groups having similar potential. See [Spe94] for the details. As a result of this approach, the lower bound improves to $\lfloor\log_2 d + \log_2(2+\log_2 d)\rfloor$. This is not so important for our purposes, but both beautiful and a great step towards the determination of the exact value of this game (if this is possible at all).

Then

$$\begin{aligned}
v(p \circ T) &\leq \tfrac{1}{2}(v(p \circ S) + v(p \circ ([d] \setminus S))) \\
&= \tfrac{1}{2}(|S| + \sum_{i \in [d] \setminus S} \lambda v(p_i) + |[d] \setminus S| + \sum_{i \in S} \lambda v(p_i)) \\
&= \tfrac{1}{2}(d + \lambda v(p)) \\
&\leq \tfrac{1}{2}(d + \frac{2 \log_2 d - 1}{\log_2 d} d \log_2 d) \\
&= d \log_2 d.
\end{aligned}$$

This proves that B can keep $v(p) \leq d \log_2 d$. Hence $\|p\|_\infty \leq \frac{\log_2 d + \log_2 \log_2 d}{\log_2\left(2 - \frac{1}{\log_2 d}\right)} + 1$. The last inequality follows from some calculus. The case $d = 2$ is solved by a moment's thought. $\square$

The proof of Theorem 7.2 follows from Lemma 7.3 to 7.5.

7.5 Summary and Outlook

This chapter is a first investigation of vector balancing games that contain some temporal aspects. We restricted ourselves to the case $q \geq 2$ and the maximum norm. We determined the precise value of the game where the players know the number of rounds and we gave close bounds for the game where they do not know have this information. Knowing the number of rounds definitely helps the players.

There are different games with some aging aspect thinkable. Let us have a look at some other variants:

For the $\|\cdot\|_2$–norm game, an additional aging assumption does not change the character of the game. The non-aging case simply generalizes to

Theorem 7.6. *Let $q \geq 1$. The vector balancing game G_{ndq} where the maximum norm is replaced with the Euclidean norm has value*

$$\sqrt{\sum_{i=0}^{n-1} \frac{1}{q^{2i}}}.$$

It makes no difference whether the players know the number of rounds or not.

Another variant is to allow a buffer. From the point of view of application though it seems a little strange that on the one hand time plays an important role (aging) and on the other decisions can be postponed (buffer concept). Some results also give the impression that buffers and aging don't go well: For the fixed end version G_{ndq} with $q \geq 2$ it is easy to see that a buffer does not change the value of the game (this also leads to a lower bound for the open ended game).

Similarly, for the Euclidean game from above a buffer of size less than $d-1$ does not help. For a buffer of any size $\sqrt{\sum_{i=0}^{d-1} \frac{1}{q^{2i}}}$ is a lower bound. Hence for q not too close to 1 the effect of a buffer can not be very big.

We thus get the impression that buffers and also different norms might not lead to very exciting results. For the discrete version of vector balancing games (i. e. the first player may choose his vectors from a finite set) though it is not clear right now what happens if an aging aspect is added to the rules. This could be a difficult problem as (to our knowledge) there are no results on non-aging discrete games that take into account the number of rounds played.

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