

Inserting Points Uniformly at Every Instance

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Arranging a predetermined number n of points over a unit cube in the d -dimensional space or over a polyhedral region is frequently asked in many applications. The discrepancy theory concerns about deviation from a perfect uniform distribution of those points [1, 2]. What is required is a set of points that are uniformly distributed over the region. In this paper we would like to insert n points in a given region in such a way that points are distributed as uniformly as possible at every instance of inserting a point. In the sense our problem is considered as a dynamic version of the discrepancy.

Our criteria on uniformity is to minimize the gap ratio (which is the maximum gap over the minimum gap) at every point insertion. We present a preliminary result for this problem, that is, we give a linear time algorithm for finding an optimal point sequence with the maximum gap ratio bounded by $2^{\lfloor n/2 \rfloor / (\lfloor n/2 \rfloor + 1)}$ in the 1-dimensional case.

1 Problem Definition

This paper considers how to insert a predetermined number n of points into the d -dimensional unit cube as uniformly as possible. We start with a set S_0 of 2^d corner points of the unit cube and then insert n points p_1, p_2, \dots, p_n in order. For each point p_i , we measure the uniformity of the current point set $S_i = \{p_1, p_2, \dots, p_i\} \cup S_0$, which is defined by the ratio of the maximum gap over the minimum one. In the d -dimension, we define a gap by the diameter of an empty ball with its center being in the unit cube which may contain points on the surface but no point in its proper interior. The maximum gap is the diameter of the largest such empty ball with at least $d + 1$ points on the surface while the minimum gap is the diameter of the smallest empty ball with at least 2 points on the surface, that is, the distance of the closest pair of points.

Given a point sequence $P = (p_1, p_2, \dots, p_n)$, we define a ratio r_i of the maximum gap G_i over the minimum one g_i for each subsequence $P_i = (p_1, \dots, p_i)$, $i = 1, 2, \dots, n$, i.e., $r_i = G_i/g_i$. Then, the maximum ratio among r_1, \dots, r_n , denoted by R_P , is our objective function to be minimized. In other words, given an integer n , the problem is to find an optimal sequence $P^* = (p_1^*, p_2^*, \dots, p_n^*)$ of n points to be inserted into the unit cube that minimized the ratio R_{P^*} among all possible point sequences. We denote the maximum gap ratio of P^* by R_n .

2 Simple Greedy Algorithm

A simple greedy algorithm for inserting points uniformly is a so-called incremental Voronoi insertion. In this algorithm, we maintain a set of vertices which are either Voronoi vertices of a Voronoi diagram for a set of points already inserted or intersections between Voronoi edges and cube surfaces. We evaluate

each such vertex by the distance to the nearest point (site) and choose the one of the largest such distance as the next point to be inserted.

Define a point set $S_i^d = \{(x_1, x_2, \dots, x_d) \mid \text{exactly } i \text{ coordinates are either } 0 \text{ or } 1 \text{ and the remaining coordinates are } 1/2\}$ for $i < d$. For example, we have

$$\begin{aligned} S_0^3 &= \{(1/2, 1/2, 1/2)\}, \\ S_1^3 &= \{(*, 1/2, 1/2), (1/2, *, 1/2), (1/2, 1/2, *)\}, \\ S_2^3 &= \{(*, *, 1/2), (*, 1/2, *), (1/2, *, *)\}, \end{aligned}$$

where $*$ indicates 0 or 1.

The first point to be inserted must be the unique element of S_0^d , i.e., $(1/2, 1/2, \dots, 1/2)$. Then, we insert points in the set S_1^d one by one, and continue to points in $S_2^d, S_3^d, \dots, S_{d-1}^d$. Suppose we have inserted all the points in $S_0^d, S_1^d, \dots, S_{d-2}^d$ and we are now inserting the first point $p_j = (0, 0, \dots, 0, 1/2)$ in the set S_{d-1}^d . The point p_j is the mid-point of a cube edge by the definition. Thus, the shortest point-wise distance is $1/2$, that is, the minimum gap is $1/2$. Since this is the first point located on a cube edge, the empty ball centered at the next point $(0, 0, \dots, 0, 1, 1/2)$ that passes through the two points $(0, 0, \dots, 0, 0)$ and $(0, 0, \dots, 0, 1)$ remains empty. In fact, this ball is the largest empty ball. Its diameter is obviously 1. Therefore, the ratio is exactly 2.

We can also show that the maximum ratio before inserting this point is less than 2 and the maximum ratio after inserting the point until the very last point of S_{d-1}^d is at most 2. When we have inserted all the points in $S_0^d, S_1^d, \dots, S_{d-1}^d$, we can continue the same process again for 2^d sub-cubes in a recursive fashion. Thus, we can conclude that the above-mentioned approximation algorithm achieves the maximum ratio 2.

3 One-Dimensional Case

Our domain here is a unit interval $[0, 1]$. The two extremal points 0 and 1 are assumed to be included in the set. We can show that there is a strategy better than the incremental Voronoi insertion. Moreover, the strategy is in fact optimal.

3.1 The lower bound for R_n

We first estimate the lower bound of R_n for an n -point sequence. Let $P = (p_1, p_2, \dots, p_n)$ be a finite sequence of n points in the unit interval $[0, 1]$ such that $p_i \neq p_j$ whenever $i \neq j$. For $i = 0, \dots, n$, the points p_1, \dots, p_i partition the unit interval into $i + 1$ intervals of lengths $m_1^i, m_2^i, \dots, m_{i+1}^i$. Without loss of generality we may assume that $m_j^i \geq m_{j+1}^i$ for all $i, 0 \leq i \leq n$ and $j, 1 \leq j \leq i$. Then, the maximum and minimum gaps are given by m_1^i and m_{i+1}^i , respectively. Hence, the ratio R_P for the sequence P is

$$R_P := \max_{1 \leq i \leq n} \frac{m_1^i}{m_{i+1}^i} \quad (1)$$

Put $M^i = \{m_1^i, \dots, m_{i+1}^i\}$ and regard it as a multi-set (i.e., it may contain elements more than once). Clearly, M^{i+1} is obtained from M^i by replacing one element from M^i by two which add up to the first one. The following lemma states that if $R_P \leq 2$, then this replaced element is always the largest.

Lemma 1. *If $R_P \leq 2$, then for each $i = 0, \dots, n - 1$ there are $a, b \in [0, 1]$ such that $m_1^i = a + b$ and $M^{i+1} = \{m_2^i, \dots, m_{i+1}^i, a, b\}$ (as multi-set) and one of a and b is a smallest element of M^{i+1} .*

Proof. Assume that $M^{i+1} = M^i \setminus \{m_j^i\} \cup \{a, b\}$ for some $j, 1 \leq j \leq i + 1$ such that $m_j^i < m_1^i$ and $a + b = m_j^i$. W.l.o.g., let $b \leq a$. Then, $b \leq \frac{1}{2}m_j^i < \frac{1}{2}m_1^i$ and hence $R_P \geq m_1^{i+1}/b = m_1^i/b > 2$. If both a and b are greater than m_{i+1}^i , then again $R_P \geq m_1^i/m_{i+1}^i = (a + b)/m_{i+1}^i > 2m_{i+1}^i/m_{i+1}^i = 2$. \square

Note, however, that a priori we do not know that both a and b are not larger than m_{i+1}^i .

Lemma 2. *Given an integer $n \geq 1$, the lower bound of R_n is $2^{\lfloor n/2 \rfloor / (\lfloor n/2 \rfloor + 1)}$.*

Proof. Assume that $R_P \leq 2$ for an n -point sequence. Let first n be even. Let $j, 1 \leq j \leq \frac{n}{2} + 1$ be such that $m_j^{n/2} \in M^n$. Such a j exists, since at most $n/2$ of the elements in $M^{n/2}$ are replaced in the sequel from $M^{n/2}$ to M^n . We have

$$\frac{m_1^{n/2}}{m_j^{n/2}} \leq \frac{m_1^{n/2}}{m_{n/2+1}^{n/2}} \leq R_P.$$

Also, for each $n/2 \leq i \leq n-1$, we have $R_P \geq m_1^{i+1}/m_{i+2}^{i+1} \geq m_1^{i+1}/m_{j/2}^i$ by Lemma 1. Since $m_j^{n/2} \in M^n$,

$$R_P \geq \frac{m_1^{n/2}}{m_j^{n/2}} \geq \frac{m_1^{n/2}}{m_1^n} = \prod_{i=n/2}^{n-1} \frac{m_1^i}{m_{i+1}^{i+1}} = \prod_{i=n/2}^{n-1} \frac{m_1^i}{m_{i+2}^{i+1}} \left/ \frac{m_1^{i+1}}{m_{i+2}^{i+1}} \right. \geq \left(\frac{2}{R_P} \right)^{n/2}.$$

We conclude $R_P \geq 2^{(n/2)/(n/2+1)}$. For n odd, let $P' = (p_1, \dots, p_{n-1})$. Then, $R_P \geq R_{P'}$ by definition and $R_{P'} \geq 2^{\lfloor n/2 \rfloor / (\lfloor n/2 \rfloor + 1)}$ by the above. So, $R_n \leq 2^{\lfloor n/2 \rfloor / (\lfloor n/2 \rfloor + 1)}$. This completes the proof of Lemma 2. \square

So, we obtained the lower bound of R_n for n -point sequences. Now, what remains is for the 1-dimensional case is to give an algorithm for computing an optimal point sequence P^* .

First consider the following algorithm suggested in the lower bound proof.

- 1: Calculate $r = 2^{\lfloor n/2 \rfloor / (\lfloor n/2 \rfloor + 1)}$;
- 2: $p_1 = 1/(1+r)$;
- 3: **for** $i = 1$ to $n-2$ **do**
- 4: Let m_1^i and m_2^i be the current longest and second longest intervals, respectively;
- 5: Put a point p_{i+1} into m_1^i to partition it into two subintervals a and b so that $m_2^i / \min\{a, b\} = r$;
- 6: **end for**
- 7: Put the last point p_n so as to partition the current longest interval into two intervals of the same lengths;

This strategy always puts a point p_i so that the gap ratio is equal to $2^{\lfloor n/2 \rfloor / (\lfloor n/2 \rfloor + 1)}$ for each i . If it is possible then the sequence obtained is optimal since it coincides with the lower bound. The strategy implicitly assumes that the smaller one of the new subintervals has the minimum length among current intervals. Unfortunately, it is impossible to keep the ratio. The reason is as follows. Let a_i and b_i ($a_i > b_i$) be new subintervals generated after the i -th insertion. Then, $M^1 = \{a_1, b_1\} = \{\frac{r}{r+1}, \frac{1}{r+1}\}$ and $M^2 = \{b_1, a_2, b_2\} = \{\frac{1}{r+1}, \frac{r-1}{r}, \frac{1}{r(r+1)}\}$. We insert p_3 into M^2 . Note that the maximum interval length in M^2 depends on the number of points to be inserted. If $b_1 \geq a_2$, (the case of $r \leq \frac{1+\sqrt{5}}{2}$), then $b_3 = \frac{a_2}{r} = \frac{r-1}{r^2}$ and $a_3 = \frac{1}{r+1} - b_3 = \frac{1}{r^2(r+1)}$. Since $a_3 < b_3$ for $r > \sqrt{2}$, $r_3 = a_2/a_3 = r(r^2-1) > R_n$. This suggests that if n is large enough, say $n > 3$, the assumption of above strategy does not hold. On the other hand, if $b_1 < a_2$ (the case of $r > \frac{1+\sqrt{5}}{2}$), then $r_2 = \frac{a_2}{b_2} = r^2 - 1$, and $2 < R_n^2 - 1$ for $n \geq 8$. Therefore, we cannot obtain an optimal point sequence P^* by the above strategy.

Observation 1. *The first ratios should be strictly less than $2^{\lfloor n/2 \rfloor / (\lfloor n/2 \rfloor + 1)}$, and moreover, these ratios are never decidable until the last intervals are determined.*

This Observation 1 suggests that an optimal point sequence of corresponding interval lengths should be determined in a bottom-up fashion, that is, from the last interval to the first one.

3.2 An optimal point insertion strategy

A rough sketch of our strategy is as follows. Let (p_1, p_2, \dots, p_n) be a point sequence to be inserted in the unit interval $x_1 = [0, 1]$. We maintain all intervals generated during n insertions, and we denote by x_j the interval induced by $p_{\lfloor n/2 \rfloor}$. Hereafter, we denote the j -th interval by x_j , and unify x_j and its length $|x_j|$. Each point $p_i, i = 1, \dots, n$, is inserted into the current largest interval x_i to split it into two new subintervals x_{2i} and x_{2i+1} with $x_{2i} + x_{2i+1} = x_i$. An important observation here is that we can determine the point p_i so that it results in a sorted sequence $(x_{i+1}, x_{i+2}, \dots, x_{2i}, x_{2i+1})$ of intervals in the decreasing order of their lengths. The process is terminated when the last point p_n is inserted to have a sequence $(x_{n+1}, x_{n+2}, \dots, x_{2n+1})$.

Now, let us describe how to determine the point sequence. It is divided into two subsequences at $k = \lfloor n/2 \rfloor$. For the first half (p_1, \dots, p_k) , the current longest interval x_i is unevenly partitioned into the new two subintervals x_{2i} and x_{2i+1} , so that $x_{2i} > x_{2i+1}$ and $x_i = x_{2i} + x_{2i+1}$. Since we are trying to achieve a ratio strictly less than 2, the ratio x_{i+1}/x_{2i+1} must be strictly less than 2. For the remaining points (p_{k+1}, \dots, p_n) , the current longest interval x_i is partitioned evenly into two new subintervals x_{2j} and x_{2j+1} so that $x_{2j} = x_{2j+1} = x_j/2$ and x_{j+1}/x_{2j+1} is strictly less than 2, or equal to R_n . This is because the intervals x_{2i} and $x_{2i+1}, i = k+1, \dots, n$, will never be subdivided during the remaining insertion. Since minimum gaps are maximized by evenly partitioning, it minimizes the maximum gap ratios.

More concretely, we first compute the target ratio $R_n = 2^{k/(k+1)}$ where $k = \lfloor n/2 \rfloor$, and a magic number $y_1 = (2^{l-k} + 2 \sum_{i=2}^{k+1} \frac{R_n^{i-1}}{2^{i-1}})^{-1}$, where $l = \lceil n/2 \rceil$. Then, we fix the last $2k+2$ interval;

$$\begin{aligned} x_{2l} = x_{2l+1} = y_1 & \quad \text{if } n \text{ is odd.} \\ x_{2l+1} = y_1 & \quad \text{if } n \text{ is even.} \\ x_{2(l+1)} = x_{2(l+1)+1} = \frac{R_n}{2} y_1, \\ x_{2(l+2)} = x_{2(l+2)+1} = \left(\frac{R_n}{2}\right)^2 y_1, \\ & \quad \vdots \\ x_{2(l+k)} = x_{2(l+k)+1} = \left(\frac{R_n}{2}\right)^k y_1. \end{aligned}$$

The remaining intervals can be determined so that $x_i = x_{2i} + x_{2i+1}, i = k, k-1, \dots, 2, 1$. This strategy can be summarized in the following pseudo code.

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1:  $R_n = 2^{\lfloor n/2 \rfloor / (\lfloor n/2 \rfloor + 1)}$ ;
2:  $y_1 = \left(2^{\lceil n/2 \rceil - \lfloor n/2 \rfloor} + 2 \sum_{i=2}^{\lfloor n/2 \rfloor + 1} \frac{R_n^{i-1}}{2^{i-1}}\right)^{-1}$ ;
3: if  $n$  is odd then  $x_{2\lfloor n/2 \rfloor} = x_{2\lfloor n/2 \rfloor + 1} = y_1$ ;
4: else  $x_{2\lfloor n/2 \rfloor + 1} = y_1$ ; endif
5: for  $i = 1$  to  $\lfloor n/2 \rfloor$  do
6:    $x_{2(\lfloor n/2 \rfloor + 1)} = x_{2(\lfloor n/2 \rfloor + 1) + 1} = \left(\frac{R_n}{2}\right)^i \cdot y_1$ ;
7: end for
8: for  $i = \lfloor n/2 \rfloor$  downto 1 do
9:    $x_i = x_{2i} + x_{2i+1}$ ;
10: end for
11: Compute the point sequence  $P$  from the interval sequence  $(x_{i+1}, x_{i+2}, \dots, x_{2i}, x_{2i+1})$ ;

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3.3 The configuration of an insertion

Before showing the optimality and correctness of our strategy, we introduce a notation, in order to simplify the arguments for proof. For any point sequence P , the every i -th insertion can be represented

by a rooted tree structure with size $2i + 1$, where the root is the unit segment x_1 . For i -th insertion, each node corresponds to an interval generated by p_1, \dots, p_i , and the interval sequence $(x_{i+1}, \dots, x_{2i+1})$ is obtained from the leaves in old order. The two nodes are connected by an edge, if one node is induced from the other node by inserting a point. More formally, there are two edges start for the node of x_i , $i = 1, \dots, \lfloor \frac{n}{2} \rfloor$, to the nodes of x_{2i} and x_{2i+1} , respectively. This implies that every inner node is already subdivided. Figure 1 shows an example of the configuration tree for $n = 4$. The grey nodes are leaf nodes. We can see that the intervals corresponding to leaf nodes subdivide the unit segment.

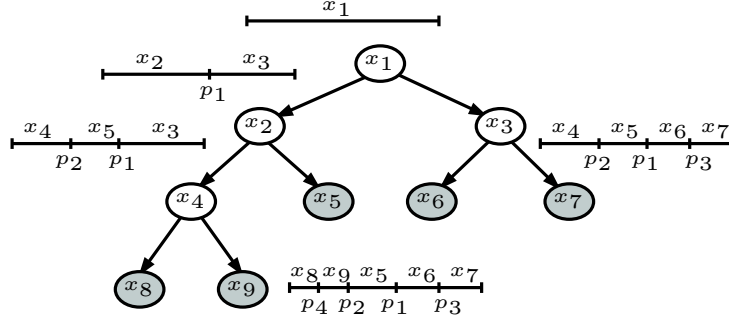


Fig. 1. The configuration for a point sequence $P = (p_1, p_2, p_3, p_4)$.

Therefore, the above strategy constructs eventual configuration tree, and then obtains the rough partition of the unit segment. So, each interval length corresponding a leaf is calculated using the magic number y_1 . Since each inner node has exactly two children and both interval are known, every interval length is also determined in the direction from leaves toward root.

3.4 The optimality and correctness

Finally, we show that the maximum gap ratio R_P of the point sequence P computed by our strategy is equal to R_n . The magic number y_1 plays very important roles to optimize R_P , and so it is made neatly. How to determine y_1 and the optimality requires Lemma 3. And Lemma 4 shows the correctness of the strategy.

Lemma 3. *If any set of intervals $\{x_{i+1}, \dots, x_{2i+2}, x_{2i+3}\}$ are sorted in non-increasing order with respect to their lengths, then the above strategy achieves the maximum gap ratio $R_P = 2^{\lfloor n/2 \rfloor / (\lfloor n/2 \rfloor + 1)}$.*

Proof. Let y_i denote the length of $x_{2(l+i)+1}$ for $i = 0, 1, \dots, k$, where $k = \lfloor n/2 \rfloor$ and $l = \lceil n/2 \rceil$. Note that the node x_l has interval y_1 as one of the children in the tree configuration. Now, we assume the gap ratio r_i is defined by $\frac{x_{i+1}}{x_{2i+1}} = \frac{x_{i+1}}{y_i}$ for $(l+i)$ -th insertion. By Lemma 4, this definition for r_i does not induce an inconsistency. From this fact, the minimum interval is y_i and the maximum interval is $x_{2(l+i)} + x_{2(l+i)+1} = 2y_{i+1}$, for $(l+i)$ -th insertion ($1 \leq i < k$). For the last insertion, the minimum interval is y_{k+1} and the maximum interval is y_1 . Therefore, the gap ratios r_i for $i = l, l+1, \dots, l+k$, are described as follow,

$$\begin{aligned}
 r_l &= \frac{x_{l+1}}{x_{2l+1}} = \frac{x_{2l+2} + x_{2l+3}}{x_{2l+1}} = \frac{2y_2}{y_1}, \\
 r_{l+1} &= \frac{x_{l+2}}{x_{2l+3}} = \frac{x_{2l+4} + x_{2l+5}}{x_{2l+3}} = \frac{2y_3}{y_2}, \\
 &\vdots \\
 r_{n-1} &= r_{l+k-1} = \frac{x_{l+k}}{x_{2(l+k-1)+1}} = \frac{x_{2(l+k)} + x_{2(l+k)+1}}{x_{2(l+k-1)+1}} = \frac{2y_{k+1}}{y_k}, \\
 r_n &= r_{l+k} = \frac{x_{l+k+1}}{x_{2(l+k)+1}} = \frac{y_1}{y_{k+1}}.
 \end{aligned}$$

Since $x_{2i+3} > x_{4i+2}$ for $i \leq l-1$, $r_i = \frac{x_{i+1}}{x_{2i+1}} = \frac{x_{2i+2}+x_{2i+3}}{x_{4i+2}+x_{4i+3}} \leq \frac{x_{2i+2}}{x_{4i+3}} = r_{2i+1}$. This implies $R_n = \max\{r_l, r_{l+1}, \dots, r_n\} \geq \max\{r_1, r_2, \dots, r_{l-1}\}$. Thus, R_n is minimized when

$$\begin{aligned} R_n &= (r_l \cdot r_{l+1} \cdots r_{l+k})^{\frac{1}{k+1}} \\ &= \left(\frac{2y_2}{y_1} \cdot \frac{2y_3}{y_2} \cdots \frac{2y_{k+1}}{y_k} \cdot \frac{y_1}{y_{k+1}} \right)^{\frac{1}{k+1}} = 2^{\frac{k}{k+1}} = 2^{\lfloor \frac{n}{2} \rfloor / (\lfloor \frac{n}{2} \rfloor + 1)}. \end{aligned}$$

□

Since every $r_i = R_n$, we have $y_i = \frac{2}{R_n} y_{i+1}$ for $i = 1, \dots, k$, and $y_{k+1} = \frac{y_1}{R_n}$. Moreover, since $y_1 = (\frac{2}{R_n})^{i-1} y_{i+1}$ for $i = 2, \dots, k+1$, if y_1 can be determined then every y_i is also determined. When n odd, since $1 = \sum_{i=n+1}^{2n+1} x_i = 2 \sum_{j=1}^{k+1} y_j = 2 \sum_{j=1}^{k+1} (\frac{R_n}{2})^{j-1}$, so we have $y_1 = \frac{1}{2 \sum_{j=1}^{k+1} (\frac{R_n}{2})^{j-1}}$. Similarly, when n even, since $1 = \sum_{i=n+1}^{2n+1} x_i = y_1 + 2 \sum_{j=2}^{k+1} y_j$, we have $y_1 = \frac{1}{1 + 2 \sum_{j=2}^{k+1} (\frac{R_n}{2})^{j-1}}$.

Observation 2. From $y_i = \frac{2}{R_n} y_{i+1}$ and $\frac{2}{R_n}$ is greater than 1, $y_1 \geq y_2 \geq \dots \geq y_{k+1}$.

In order to show the correctness of this strategy and the optimality, we have to prove that the interval sequence $(x_{i+1}, \dots, x_{2i}, x_{2i+1})$ generated by p_i is sorted in non-increasing order with respect to their length for every $1 \leq i \leq n$.

Lemma 4. Whenever our strategy partitions the interval x_i for every $1 \leq i \leq n$, the resulting interval sequence $(x_{i+1}, x_{i+2}, \dots, x_{2i})$ is sorted in non-increasing order, that is, inequalities $x_{i+1} \geq x_{i+2} \geq \dots \geq x_{2i} \geq x_{2i+1}$ are satisfied.

Proof. It can be proved by induction on level of the tree configuration with size $2n+1$. The level of a node v is defined as $\lceil \log(2n+1) \rceil - \text{height of } v$. So, all leaf nodes may be in the level 0 or 1, and the level of root x_1 is $\lceil \log(2n+1) \rceil$, (see Figure 2).

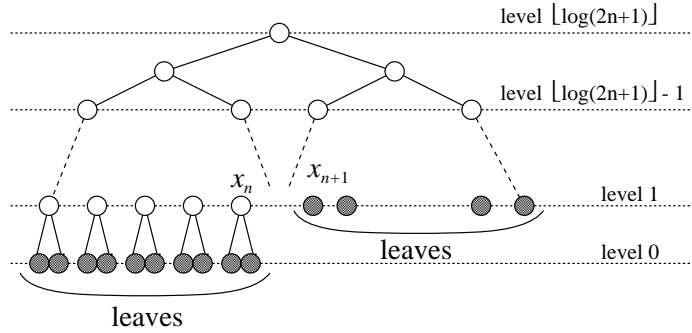


Fig. 2. The levels of tree corresponding the behavior of our strategy.

When $2^h = n+1$, where $h = \lceil \log(2n+1) \rceil$, all leaf nodes are in level 0. In this case, from Observation 2, the statement $x_{n+1} \geq \dots \geq x_{2n+1}$ is hold. When $n+1 \neq 2^h$, the intervals x_n and x_{n+1} are in same level 1. Then,

$$x_n = 2y_{k+1} = 2 \left(\frac{R_n}{2} \right)^k y_1 = \frac{R_n^{k+1}}{R_n 2^{k-1}} y_1 = \frac{2^k}{R_n 2^{k-1}} y_1 = \frac{2}{R_n} y_1 > y_1 = x_{n+1}.$$

On the remaining nodes in level 1, the both children of a node are the intervals which has a same length so as $2y_j$. For the consequence two intervals x_i and x_{i+1} , $x_i \geq x_{i+1}$ by Observation 2.

Let $I^i = (z_1^i, \dots, z_{2^i}^i)$ be the intervals in level i , where z_1^i and $z_{2^i}^i$ are the leftmost and rightmost intervals in level i , respectively. Now, we assume that the statement holds up to level i , that is, $z_1^i \geq z_2^i \geq \dots \geq z_{2^i}^i \geq \dots \geq x_{2^{n+1}}$. By this induction hypothesis, $z_{2^{i-1}}^{i-1} = z_{2^{i-1}}^i + z_{2^i}^i \geq z_1^{i+1} + z_2^{i+1} = z_1$. For the others, $z_j^{i-1} \geq z_{j+1}^{i-1}$ are hold by similar arguments in the base step. \square

Thus, we conclude on 1-dimensional dispersion problem.

Theorem 1. *For given an integer n , our strategy gives an optimal solution with the maximum gap ratio is $2^{\frac{\lfloor n/2 \rfloor}{\lfloor n/2 \rfloor + 1}}$ on 1-dimensional dispersion problem in $O(n)$ time.*

4 Conclusions

In this paper we presented a preliminary result on dispersion. One of the most important future works is to extend the result to higher dimensions. We had some results on lower and upper bounds of the maximum gap ratio for the planar case, but none in the higher dimensions.

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References

1. B. Chazelle: "The Discrepancy Method: Randomness and Complexity," Cambridge University Press, 2000.
2. J. Matoušek: "Geometric Discrepancy," Springer, 1991.