COLORING t–DIMENSIONAL m–BOXES

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Abstract. Call the set $S_1 \times \cdots \times S_t$ t–dimensional m–box if $|S_i| = m$ for every $i = 1, \ldots, t$. Let $R_t(m, r)$ be the smallest integer R such that for every r–coloring of t–fold cartesian product of $[R]$ one can find a monochromatic t–dimensional m–box. We give a lower and an upper bound for $R_t(m, r)$. We also consider the discrepancy problem connected to this set-system. Among other bounds we prove that the discrepancy of the hypergraph of all 2–dimensional m–boxes in $[R] \times [R]$ is equal to $\Theta(R^{\frac{3}{2}})$ for m a constant fraction (less than $\frac{1}{2}$) of R.

1. Introduction and results

In this article we investigate the problem of coloring a t–dimensional grid. This raises two types of questions: Ramsey-theory asks for conditions that imply the existence of a monochromatic box, discrepancy theory asks for the maximal deviation that occurs in one box.

To be more precise: Let $R, m, t \in \boldsymbol{N}$. Denote by $[R]$ the set $\{1, 2, \ldots, R\}$ and by $[R]^t$ the t–fold cartesian product of $[R]$ with itself (note that this notation is different from the one in [5]). A t–dimensional m–box in $[R]^t$ is a set $B = S_1 \times \cdots \times S_t$ such that $S_i \subseteq [R]$ and $|S_i| = m$ for all $i \in [t]$.

Definition 1.1. *For nonnegative integers R, m, t we define $R_t(m, r)$ to be the least R such that for any function*

$$f : [R]^t \to [r],$$

*called r–*coloring*, or just* coloring*, there exist a monochromatic m–box in $[R]^t$.*

We show that for sufficiently large m

$$\tfrac{1}{4e^2} m r^{\frac{m^t - 1}{(m-1)t}} \leq R_t(m, r) \leq \left(m + \frac{m^2}{r^{m-1}}\right) r^{m^{t-1}}.$$

Referring to [7, p. 28], we note that $R_t(m, r)$ is the Product Ramsey Number $PRN(m, k, r; t)$ in the case $k = 1$. Here $PRN(m, k, r; t)$ is the general Product Ramsey Number $R(r, t; k_1, \ldots, k_t; m_1, \ldots, m_t)$ where $k_1 = \ldots = k_t = k$ and $m_1 = \ldots = m_t = m$.

For the second question let us shortly introduce the general setting combinatorial discrepancy theory deals with:

Definition 1.2. *Let $\mathcal{H} = (X, \mathcal{E})$ denote a finite* hypergraph. *Here X is a finite set, called* vertices, *and $\mathcal{E}$ is a family of subsets of X, called* hyperedges. *In this setting, a* coloring *of $\mathcal{H}$ is a mapping $f : X \to \{-1, +1\}$. For a hyperedge $E \in \mathcal{E}$ define $f(E) = \sum_{x \in E} f(x)$. The* discrepancy *of $\mathcal{H}$ is defined by*

$$\mathrm{disc}(\mathcal{H}) = \min_{f : X \to \{-1,+1\}} \max_{E \in \mathcal{E}} |f(E)|.$$

Discrepancy measures the maximal occurring imbalance with respect to an optimal coloring. For a deeper understanding we would like to recommend the invaluable survey of Beck and Sós [2]. Note that this framework only deals with two colors. A multi–color setting is investigated in [3].

In the second part of this article we estimate the discrepancy of the 2–dimensional m–boxes. We show that if m is a constant fraction (smaller than $\frac{1}{2}$) of R we have a discrepancy of $\Theta(m^{\frac{3}{2}})$. This exhibits a different phenomenon compared to the Ramsey-type results above: Already for m relatively large, we have a rather imbalanced m–box. We also investigate the behavior of the discrepancy in between this case and the monochromatic ("Ramsey"–) case and estimate the discrepancy of the hypergraph of all boxes in $[R]^2$. The precise results are collected in section 4.

2. A lower bound for $R_t(m, r)$

Using the probabilistic method we will give a lower bound for $R_t(m, r)$. This improves the bound given in [4, Cor. 12.5 p. 65] (by Erdős in 1965, also by probabilistic methods). Our argument is based on a standard application of the following "Local Lemma" of Lovász [1, Cor. 1.2 p. 55].

Theorem 2.1. *Let $A_1, A_2, \ldots, A_n$ be events in some probability space. Suppose that each event A_i is mutually independent of the set of all but d events A_j, and that $P(A_i) \leq p$ for all $1 \leq i \leq n$. If $ep(d+1) \leq 1$, then $P(\bigcap_{i=1}^n A_i^c) > 0$.*

We show

Theorem 2.2. *For $m, r, t \geq 2$ we have*

$$R_t(m, r) \geq \frac{1}{4e^2} m r^{\frac{m^t - 1}{(m-1)t}}.$$

Proof. Let us consider a uniform random r-coloring f of the t–dimensional R–box:

$$P(f(\tilde{x}) = i) = 1/r \text{ for every } \tilde{x} \in R^t \text{ and } i \in [r].$$

For every t–dimensional m–box S, let A_S be the event that S is monochromatic. Then

$$P(A_S) = r^{-m^t + 1}.$$

Clearly every event A_S is mutually independent of all the events $A_{S'}$ but those with $S \cap S' \neq \emptyset$. Let $S = S_1 \times \cdots \times S_t$ and $S' = S'_1 \times \cdots \times S'_t$. For every

$i=1,\ldots,m$ we have exactly $\binom{R}{m}-\binom{R-m}{m}$ possibilities to choose the m–set S'_i with $S_i\cap S'_i\neq\emptyset$. Thus, the total number of t–dimensional m–boxes that intersect S is equal to

$$d=\left(\binom{R}{m}-\binom{R-m}{m}\right)^t.$$

It follows from theorem 2.1 that $ep(d+1)\leq 1$ implies the existence of a coloring of the t–dimensional R–box with no monochromatic t–dimensional m–box. Therefore, $R_t(m,r)$ is greater than the largest R, which satisfies

$$er^{-m^t+1}\left(\left(\binom{R}{m}-\binom{R-m}{m}\right)^t+1\right)\leq 1. \tag{1}$$

Since $\binom{a}{b}+\binom{a}{b-1}=\binom{a+1}{b}$ we get

$$\binom{R}{m}-\binom{R-m}{m}=\sum_{i=1}^{m}\left(\binom{R-i+1}{m}-\binom{R-i}{m}\right)=\sum_{i=1}^{m}\binom{R-i}{m-1},$$

hence, since $(m-1)!>\left(\frac{m-1}{e}\right)^{m-1}$, we get

$$\binom{R}{m}-\binom{R-m}{m}<m\binom{R-1}{m-1}<m\left(\frac{e(R-1)}{m-1}\right)^{m-1}.$$

If we let $T=m\left(\frac{e(R-1)}{m-1}\right)^{m-1}$, then the above inequality and (1) imply that $R_t(m,r)$ is greater than the largest R that satisfies $er^{-m^t+1}(T^t+1)\leq 1$, that is to say

$$R_t(m,r)>\left(\frac{1}{e}r^{m^t-1}-1\right)^{\frac{1}{t(m-1)}}\frac{m-1}{e}\frac{1}{\sqrt[m-1]{m}}+1.$$

Since $r,t\geq 2$ we have $\frac{1}{e}r^{m^t-1}-1>\frac{1}{2e}r^{m^t-1}$. Also since $\frac{m-1}{\sqrt[m-1]{m}}>\frac{m}{2\sqrt{2}}$ for $m\geq 2$ we get

$$\begin{aligned}R_t(m,r) &> \left(\frac{1}{2e}r^{m^t-1}\right)^{\frac{1}{t(m-1)}}\frac{m}{2e\sqrt{2}}\\ &= r^{\frac{m^t-1}{(m-1)t}}\left(\frac{1}{2}\right)^{\frac{1}{t(m-1)}}e^{-\frac{1}{t(m-1)}-1}\frac{m}{2\sqrt{2}}\\ &> \frac{1}{4e^2}mr^{\frac{m^t-1}{(m-1)t}}\end{aligned}$$

which yields required inequality. □

3. An upper bound for $R_t(m,r)$

Clearly for any m and r we have $R_1(m,r) = (m-1)r+1$ by usual Pigeon-Hole principle. Hence we will study $R_t(m,r)$ for $t \geq 2$.

In order to get an upper bound for $R_t(m,r)$ we will establish an upper bound for a function $P_t(m,r)$ that we know is larger than $R_t(m,r)$:

Definition 3.1. *For $m,r,t \geq 2$ let $P_t(m,r)$ be the least integer R such that every $\lceil R^t/r \rceil$-element subset of $[R]^t$ contains a t-dimensional m-box $S_1 \times \cdots \times S_t$ where each S_i has m elements.*

- Clearly $R_t(m,r) \leq P_t(m,r)$.
- We will bound $P_t(m,r)$ for large m, by induction on the dimension t.

Definition 3.2. *Define the following:*

1. *For a nonnegative $t \in \boldsymbol{N}$ let*

$$\pi_{1;t-1} : [R]^t \to [R]^{t-1}$$
$$\pi_{1;t-1}(i_1, \ldots, i_t) = (i_1, \ldots, i_{t-1}),$$

$$\pi_t : [R]^t \to [R]$$
$$\pi_t(i_1, \ldots, i_t) = i_t$$

   be the projection onto the first $t-1$ coordinates, and the very last coordinate, respectively.
2. *For a subset $M \subseteq [R]^t$ and $\tilde{\imath} = (i_1, \ldots, i_{t-1}) \in [R]^{t-1}$ let*

$$M_{\tilde{\imath}} = M \cap \pi_{1;t-1}^{-1}(i_1, \ldots, i_{t-1}).$$

   Hence we get a disjoint partition of M as $M = \bigcup_{\tilde{\imath} \in [R]^{t-1}} M_{\tilde{\imath}}$.

Let $f : [R]^t \to [r]$ be a coloring function. By the pigeonhole principle, there is a subset $S \subseteq [R]^t$ such that $|S| \geq \lceil R^t/r \rceil$ that receives the same color by f. For each $\tilde{\imath} \in [R]^{t-1}$ let $s_{\tilde{\imath}} = |S_{\tilde{\imath}}|$. Trivially there are exactly $\binom{s_{\tilde{\imath}}}{m}$ m-sets of $S_{\tilde{\imath}}$, hence the total number of m-sets of $|S|$, that are mapped to a singleton under $\pi_{1;t-1}$, is

$$\sum_{\tilde{\imath} \in [R]^{t-1}} \binom{s_{\tilde{\imath}}}{m}.$$

Call the set of these m-sets $S^{(m)}$. Note that $\sum s_{\tilde{\imath}} = |S| \geq R^t/r$.

The idea for getting an upper bound for $P_t(m,r)$ is to introduce an extra integer variable x on $[R]$. To ensure that at least $\lceil R^{t-1}/x \rceil$ elements of $S^{(m)}$ are mapped to the same m-set of $[R]$ under π_t, we must have

$$\sum_{\tilde{\imath} \in [R]^{t-1}} \binom{s_{\tilde{\imath}}}{m} \geq \left(\left\lceil \frac{R^{t-1}}{x} \right\rceil - 1 \right) \binom{R}{m} + 1,$$

since the total number of m-sets in $[R]$ is $\binom{R}{m}$, and hence it suffices that

$$\sum_{\tilde{\imath}\in[R]^{t-1}} \binom{s_{\tilde{\imath}}}{m} > \frac{R^{t-1}}{x}\binom{R}{m}. \tag{2}$$

On the other hand, to ensure that the image of these $\lceil R^{t-1}/x\rceil$ elements of $S^{(m)}$ in $[R]^{t-1}$ under $\pi_{1;t-1}$ contains a $t-1$ dimensional m-box, we must by definition 3.1 have

$$R \geq P_{t-1}(m,x). \tag{3}$$

Hence, if both (2) and (3) are satisfied then, we can rest assured that there is a t-dimensional monochromatic m-box in $[R]^t$.

We will use conditions (2) and (3) to get an upper bound for $P_t(m,r)$. First we have two technical lemmas. For convenience, let us, for a fixed integer m, define the real function

$$\binom{x}{m}^* = \begin{cases} \frac{x(x-1)\cdots(x-m+1)}{m!} & \text{if } x \geq m-1 \\ 0 & \text{otherwise} \end{cases}$$

The first lemma and its proof can be found in [4, Lemma 12.2 p. 59]:

Lemma 3.3. *For fixed m and C let $x_1,\ldots,x_n$ be real positive variables subjected to the constraint $x_1+\cdots+x_n \geq C$. Then $\sum_{i=1}^n \binom{x_i}{m}^*$ reaches its minimum value when $x_1 = \ldots = x_n$.* □

The second lemma deals with polynomials in one variable:

Lemma 3.4. *For positive integers $m, r \geq 1$ and a real number $\alpha \in]0;1[$, the largest real solution of the polynomial equation*

$$X(X-r)\cdots(X-(m-1)r) = \alpha X(X-1)\cdots(X-m+1) \tag{4}$$

is asymptotically $R = \frac{m^2(1-r)}{2\log(\alpha)}$ when m tends to infinity. Here log *is the usual natural logarithm.*

NOTE: For fixed $r \geq 2$ and $\alpha \in]0;1[$, we see that there is a sufficiently large m such that the largest root of (4) is strictly less than $\frac{m^2 r}{2\log(\alpha^{-1})}$.

Proof. If we let

$$F_m(X) = \frac{X(X-r)(X-2r)\cdots(X-(m-1)r)}{X(X-1)(X-2)\cdots(X-m+1)}$$

then, by the use of Stirling's formula for estimating $n!$ for large n, together with the Taylor expansion of $\log(1+t)$ around $t=0$, we get

$$F_m(m^2rY) = \frac{r^m(m^2Y)!(m^2rY-m)!}{(m^2rY)!(m^2Y-m)!} \underset{m\to\infty}{\longrightarrow} e^{\frac{1}{2}\left(\frac{1-r}{rY}\right)}.$$

We note that this limit value is equal to α when $Y = \frac{1-r}{2r\log\alpha}$.

Let A and B be real numbers such that $A < \frac{1-r}{2r\log\alpha} < B$. Since the map $Y \mapsto \frac{1}{2}\left(\frac{1-r}{rY}\right)$ is increasing we get $\frac{1}{2}\left(\frac{1-r}{rA}\right) < \alpha < \frac{1}{2}\left(\frac{1-r}{rB}\right)$. Hence we can find m_0 such that

$$F_m(m^2rA) < \alpha < F_m(m^2rB) \tag{5}$$

for all $m \geq m_0$. For r, α given let $m_1 \geq m_0$ be such that $\frac{m^2(1-r)}{2\log\alpha} > (m-1)r$ for all $m \geq m_1$. Consider now a fixed $m \geq m_1$.

- Since the map $Y \mapsto F_m(m^2rY)$ is continuous we get from (5) that there is a real z_m in the open interval $]A; B[$ such that $F_m(m^2rz_m) = \alpha$.
- Since $m^2rB > \frac{m^2(1-r)}{2\log\alpha} > (m-1)r$ and $\frac{d}{dX}F_m(X) > 0$ for $X > (m-1)r$ we have that $F_m(X)$ is increasing for $X \geq m^2rB$ and hence $F_m(X) - \alpha$ has no real root greater than m^2rB.

We conclude that the largest real root, R_m, of the equation $F_m(X) = \alpha$ is contained in the interval $[m^2rz_m; m^2rB[\quad\subseteq\quad]m^2rA; m^2rB[$, or in other words, $\frac{R_m}{m^2r} \in]A; B[$. This holds for all $m \geq m_1$, hence

$$\lim_{m\to\infty} \frac{R_m}{m^2r} = \frac{1-r}{2r\log\alpha}.$$

□

In order for R to satisfy (2), it is sufficient by lemma 3.3, that R satisfies $R^{t-1}\binom{R/r}{m}^* > \frac{R^{t-1}}{x}\binom{R}{m}$, that is to say $\binom{R/r}{m}^* > \frac{1}{x}\binom{R}{m}$. Note that this condition does not depend on the dimension t at all. From condition (2) we therefore extract a new and more simple condition on R that, together with (3), ensures that the set S contains an m-box in $[R]^t$:

$$\binom{R/r}{m}^* > \frac{1}{x}\binom{R}{m} \tag{6}$$

where x is some integer variable $< R$. Since both sides of (6) are polynomials of degree m, we see that in order for (6) to hold for all sufficiently large R, we must have $x > r^m$. Since now (6) can be written in the form

$$R(R-r)\cdots(R-(m-1)r) > \frac{r^m}{x}R(R-1)\cdots(R-m+1)$$

we have by lemma 3.4, that for large enough m, it suffices for R to satisfy $R \geq \frac{m^2r}{2\log(x/r^m)}$ in order for (6) to hold.

Hence we see that from (3) and (6), that if $x > r^m$, and

$$R \geq \max\left(P_{t-1}(m,x), \frac{m^2r}{2\log(x/r^m)}\right)$$

then every $\lceil\frac{R^t}{r}\rceil$-subset of $[R]^t$ contains a t-dimensional m-box. Hence for large m, we get an inductive condition that bounds $P_t(m,r)$, and hence also

$R_t(m,r)$:

$$(7)\qquad \begin{aligned} P_1(m,r) &= (m-1)r+1 \\ P_t(m,r) &\le \min_{x>r^m}\left\{\max\left(P_{t-1}(m,x), \tfrac{m^2r}{2\log(x/r^m)}\right)\right\}. \end{aligned}$$

Since $R_t(m,r) \le P_t(m,r)$ we get from (7) the following theorem:

Theorem 3.5. *For $r,t \ge 1$ we have*

$$\limsup_{m\in\boldsymbol{N}}\left\{\frac{R_t(m,r)}{mr^{m^{t-1}}}\right\} \le 1.$$

Proof. We will prove that the statement of the theorem holds for $P_t(m,r)$. The theorem is clearly true when the dimension $t=1$. We assume that $r,t\ge 2$, that m is sufficiently large and proceed by induction on t:

Assume $P_{t-1}(m,r) \le a_{t-1}(m)r^{m^{t-2}}$ where $\lim_{m\to\infty}\frac{a_{t-1}(m)}{m}=1$. From (7) we get for $t\ge 2$

$$P_t(m,r) \le \min_{x>r^m}\left\{\max\left(a_{t-1}(m)x^{m^{t-2}}, \frac{m^2r}{2\log(x/r^m)}\right)\right\}.$$

We note that $f(x) = a_{t-1}(m)x^{m^{t-2}}$ is increasing with $\lim_{x\to\infty} f(x) = \infty$ and $g(x) = \frac{m^2r}{2\log(x/r^m)}$ is decreasing for $x > r^m$ with $\lim_{x\to\infty} g(x) = 0$ and $\lim_{x\to r^m+} g(x) = \infty$. Hence there is exactly one $x > r^m$ which would give the optimal bound for $P_t(m,r)$, namely the $x^* > r^m$ satisfying $f(x^*) = g(x^*)$. Let us examine this solution x^*:

If we let $y = x/r^m$ and hence $y \in]1;\infty[$, then the equation $f(x) = g(x)$ becomes

$$(8)\qquad y^{m^{t-2}}\log y = \frac{m^2r}{2a_{t-1}(m)r^{m^{t-1}}}.$$

We will show that the solution y^* is "close" to 1. Since $t\log t \ge t-1$ for all $t>0$ we have for all $y>1$ that $\log y \ge 1-\frac{1}{y} \ge 1-\frac{1}{y^{m^{t-2}}}$ Hence the solution y' to the equation

$$y^{m^{t-2}} - 1 = y^{m^{t-2}}\left(1-\frac{1}{y^{m^{t-2}}}\right) = \frac{m^2r}{2a_{t-1}(m)r^{m^{t-1}}}$$

will be larger than the solution y^* of (8). Therefore $1 < y^* < y'$ and hence $r^m < x^* < r^m y'$. Since now $f(x)$ is an increasing function we get that

$$P_t(m,r) \le f(x^*) < f(r^m y') = \left(a_{t-1}(m) + \frac{m^2r}{2r^{m^{t-1}}}\right)r^{m^{t-1}}.$$

Letting $a_t(m) = a_{t-1}(m) + \frac{m^2r}{2r^{m^{t-1}}}$ we get that $P_t(m,r) \le a_t(m)r^{m^{t-1}}$ where $\lim_{m\to\infty}\frac{a_t(m)}{m} = 1$, which completes the induction and hence proves our theorem. □

REMARKS: (i) $mr^{m^{t-1}}$ is the best asymptotic bound that (7) can yield for $P_t(m,r)$. (ii) Looking at the inductive definition for $a_t(m)$ in the last paragraph of the above proof, we get that

$$a_t(m) - m = a_t(m) - a_1(m) = \sum_{i=1}^{t-1} (a_{i+1}(m) - a_i(m)) \le \sum_{i=1}^{t-1} \frac{m^2 r}{2r^{m^i}}$$

and since $\sum_{i=1}^{t-1} \frac{1}{r^{m^i}} < \frac{1}{r^m - 1} < \frac{2}{r^m}$ we get in particular that

$$R_t(m,r) \le P_t(m,r) \le \left(m + \frac{m^2}{r^{m-1}}\right) r^{m^{t-1}}$$

for sufficiently large m.

4. Discrepancy

In this section we investigate a phenomenon related to the one of the preceding sections, namely the discrepancy of 2–dimensional m–boxes in $[R]^2$. Let $\mathcal{B}_{Rm} := \{A \times B : A, B \subseteq [R], |A| = |B| = m\}$ be the set of these boxes. Discrepancy measures how well $[R]^2$ can be colored with two colors such that all boxes are roughly split into equal parts by the two color–classes, that is, given m and R we try to determine

$$d_{Rm} := \operatorname{disc}(\mathcal{B}_{Rm}) = \min_{f:[R]^2 \to \{-1,+1\}} \max_{B \in \mathcal{B}_{Rm}} |f(B)|,$$

where $f(B) := \sum_{b \in B} f(b)$. In section 3 we showed that for $m2^m \lesssim R$ we have a monochromatic m-box, that is, $d_{Rm} = m^2$. This marks the extreme case occurring when m is very small compared to R. Here we are interested in the case when m is larger compared to R. It turns out that also for relatively large m there is always a "badly" colored m–box (of discrepancy $\Theta(m^{\frac{3}{2}})$, if $m < \frac{1}{2}R$ is a constant fraction of R.) The precise results are collected in the theorems 4.1 and 4.2 below. We start with an upper bound.

Theorem 4.1 (Upper bound). *$d_{Rm} \le 2m^{\frac{3}{2}}\sqrt{\log\left(\frac{eR}{m}\right)}$ for all $m \le R$.*

Proof. For any set–system consisting of M sets of size at most N we know that its discrepancy is less or equal $\sqrt{2N\log(2M)}$ [1, Thm. 1.1 p. 186]. Since $m! > (\frac{m}{e})^m$ we have

$$\begin{aligned} d_{Rm} &\le \sqrt{2m^2 \log\left(2\binom{R}{m}^2\right)} \\ &\le \sqrt{2m^2 \log\left(\left(\frac{eR}{m}\right)^{2m}\right)} \\ &= 2m^{\frac{3}{2}}\sqrt{\log\left(\frac{eR}{m}\right)}. \end{aligned}$$

□

Our lower bound is not far from the above result:

Theorem 4.2 (Lower bound). *For $m \leq \frac{1}{2}R$ we have*

$$d_{Rm} \geq m^{\frac{3}{2}}\left(1-\tfrac{2m}{R-1}\right)\sqrt{1-\tfrac{m}{R-1}}-3m.$$

More specifically, if $m \leq \frac{1}{2}(R-1)^{\frac{2}{3}}$ we have

$$d_{Rm} \geq \min\left\{m^{\frac{3}{2}}\sqrt{\tfrac{2}{5}\log_2\left(\tfrac{R}{2m^{\frac{3}{2}}}\right)}-4m,\ \tfrac{(m-1)^2}{3}-4m\right\}.$$

To prove the theorem, we need the following lemma.

Lemma 4.3. *Let $d \leq m \leq R$ be given. Among all colorings $f:[R]\to\{-1,1\}$ the number of m–subsets of $[R]$ having discrepancy at most d is maximal if and only if the color classes of f deviate in size by at most one.*

Proof. Set $D := \{i \in [-d,d] | m+i \text{ even}\}$. Obviously the number of subsets of $[R]$ having discrepancy at most d with respect to a given coloring f depends only on the sizes of the color classes. Hence for $R_1, R_2 \leq R$ such that $R_1 + R_2 = R$

$$\delta(R_1,R_2) := \sum_{i\in D}\binom{R_1}{\frac{1}{2}(m-i)}\binom{R_2}{\frac{1}{2}(m+i)}$$

is the number of these sets. Assume $1 \leq R_1 \leq R_2$. Using induction is suffices to show $\delta(R_1,R_2) > \delta(R_1-1,R_2+1)$. Set $d_0 = \max D$. Then we have

$$\begin{aligned}
&\delta(R_1,R_2)-\delta(R_1-1,R_2+1)\\
&= \sum_{i\in D}\left[\binom{R_1-1}{\frac{1}{2}(m-i)-1}+\binom{R_1-1}{\frac{1}{2}(m-i)}\right]\binom{R_2}{\frac{1}{2}(m+i)}\\
&\quad -\sum_{i\in D}\binom{R_1-1}{\frac{1}{2}(m-i)}\left[\binom{R_2}{\frac{1}{2}(m+i)-1}+\binom{R_2}{\frac{1}{2}(m+i)}\right]\\
&= \sum_{i\in D}\binom{R_1-1}{\frac{1}{2}(m-i)-1}\binom{R_2}{\frac{1}{2}(m+i)}-\sum_{i\in D}\binom{R_1-1}{\frac{1}{2}(m-i)}\binom{R_2}{\frac{1}{2}(m+i)-1}\\
&= \binom{R_1-1}{\frac{1}{2}(m-d_0)-1}\binom{R_2}{\frac{1}{2}(m+d_0)}-\binom{R_1-1}{\frac{1}{2}(m-d_0)}\binom{R_2}{\frac{1}{2}(m+d_0)-1} > 0.
\end{aligned}$$

□

Now let us prove theorem 4.2.

Proof. Let R and $d \leq m \leq \frac{1}{2}R$ be given. Let $f:[R]^2 \to \{-1,+1\}$ be any coloring.

Our general approach is the following: For a row $\{i\}\times[R] \subseteq [R]^2$ estimate the number of sets $\{i\}\times B_0$ of size m that have discrepancy $|f(\{i\}\times B_0)|$

at least d. Call n_d the minimum possible number of these sets among all colorings $f : [R]^2 \to \{-1,+1\}$. Without loss of generality we may assume that for any f there are at least $\frac{1}{2}n_d$ such sets with $f(\{i\} \times B_0) \geq d$. We will find conditions that imply $\frac{1}{2}n_d R > (m-1)\binom{R}{m}$. We conclude from the pigeon–hole principle that in that case there are m different numbers $i_k, k \in [m]$ and an m–set $B_0 \subseteq [R]$, such that $f(\{i_k\} \times B_0) \geq d$ for all $k \in [m]$. Thus $\{i_1, \ldots, i_m\} \times B_0$ is an m–box having discrepancy at least md.

Let us first assume that R and m are both even. We will deal with the other values of m and R at the very end of this proof. Since a set M of even size always has an even discrepancy $d = |f(M)|$ with regard to all colorings f, we only need to consider even $d \geq 2$. By lemma 4.3 we may assume that both color classes of f have size $\frac{R}{2}$.
We will estimate n_d in two ways. Which one is better depends on whether $m \leq \frac{1}{2}R$, or more specifically $m \leq \frac{1}{2}R^{\frac{2}{3}}$.
First case: We start with the easier and more general case $m \leq \frac{1}{2}R$. Clearly we have

$$\begin{aligned} n_d &= \binom{R}{m} - \sum_{i=-\frac{d}{2}+1}^{\frac{d}{2}-1} \binom{\frac{R}{2}}{\frac{m}{2}-i}\binom{\frac{R}{2}}{\frac{m}{2}+i} \\ &\geq \binom{R}{m} - (d-1)\binom{\frac{R}{2}}{\frac{m}{2}}^2. \end{aligned}$$

It suffices therefore to find the largest d such that

$$\tfrac{1}{2}R\left(\binom{R}{m} - (d-1)\binom{\frac{R}{2}}{\frac{m}{2}}^2\right) > (m-1)\binom{R}{m},$$

or equivalently $d - 1 < \left(1 - \frac{2(m-1)}{R}\right)\binom{R}{m}\binom{\frac{R}{2}}{\frac{m}{2}}^{-2}$. Define

$$g(n,m) := \frac{\sqrt{n}}{\sqrt{2\pi m(n-m)}}\left(\frac{n}{m}\right)^m \left(\frac{n}{n-m}\right)^{n-m}. \tag{9}$$

From a sharp version of the Stirling formula $\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n+1}} < n! < \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n}}$ due to Robbins [6] we see that for all $1 \leq m \leq \frac{1}{2}n$ we have $e^{-\frac{1}{6m}}g(n,m) \leq \binom{n}{m} \leq g(n,m)$. Using this we get

$$\begin{aligned} \frac{\left(1 - \frac{2(m-1)}{R}\right)\binom{R}{m}}{\binom{\frac{R}{2}}{\frac{m}{2}}^2} &\geq \frac{\left(1 - \frac{2(m-1)}{R}\right)e^{-\frac{1}{6m}}g(R,m)}{g(\frac{R}{2},\frac{m}{2})^2} \\ &= \sqrt{\frac{\pi m}{2}}\, e^{-\frac{1}{6m}}\left(1 - \frac{2(m-1)}{R}\right)\sqrt{1 - \frac{m}{R}}. \end{aligned}$$

We may assume $m \geq 10$ as otherwise our bound is negative and there is nothing to prove. Hence a sufficient condition for the existence of an m–box of discrepancy md is that $d-1 \leq \frac{6}{5}\sqrt{m}(1-\frac{2m}{R})\sqrt{1-\frac{m}{R}}$. Choosing d even and maximal we get $d \geq \frac{6}{5}\sqrt{m}(1-\frac{2m}{R})\sqrt{1-\frac{m}{R}}-1$ and hence $d_{Rm} \geq \frac{6}{5}m^{\frac{3}{2}}(1-\frac{2m}{R})\sqrt{1-\frac{m}{R}}-m$.

Second case: We will now consider the more specific case when $m \leq \frac{1}{2}R^{\frac{2}{3}}$. In this situation it is advantageous to estimate

$$n_d \geq 2\binom{\frac{R}{2}}{\frac{m-d}{2}}\binom{\frac{R}{2}}{\frac{m+d}{2}}.$$

We use the elementary fact that for $t > 1$ the function $t \mapsto (1-\frac{1}{t})^t$ is monotone increasing and tends to $\frac{1}{e}$. In particular we have $\frac{1}{4} \leq (1-\frac{1}{t})^t < \frac{1}{2}$ for all $t \geq 2$. We have therefore

$$\mathbf{a}(x) := \left(\frac{x^2}{x^2-d^2}\right)^{\frac{x}{2}} \geq 2^{\frac{d^2}{2x}} \text{ for } x > d > 0, \tag{10}$$

$$\mathbf{b}(x) := \left(\frac{x-d}{x+d}\right)^{\frac{d-1}{2}} \geq 4^{-\frac{d(d-1)}{x+d}} \text{ for } x > 3d > 0. \tag{11}$$

Assume that $m > 3d$. We now find a condition on d that will imply $\frac{1}{2}n_d > (m-1)\binom{R}{m}$. Using the Stirling formula we get

$$\begin{aligned}
\frac{\frac{1}{2}n_d}{(m-1)\binom{R}{m}} &> \frac{R\binom{\frac{R}{2}}{\frac{m-d}{2}}\binom{\frac{R}{2}}{\frac{m+d}{2}}}{(m-1)\binom{R}{m}} \\
&\geq \frac{Re^{-\frac{1}{6\frac{m-d}{2}}}g(\frac{R}{2},\frac{m-d}{2})e^{-\frac{1}{6\frac{m+d}{2}}}g(\frac{R}{2},\frac{m+d}{2})}{(m-1)g(R,m)} \\
&= \frac{\sqrt{R(R-m)}\cdot cR\sqrt{m}\cdot \mathbf{a}(m)\mathbf{b}(m)\mathbf{a}(R-m)\mathbf{b}(R-m)}{(R-m+d)(m-1)(m+d)},
\end{aligned}$$

where $c = \sqrt{\frac{2}{\pi}}\, e^{\frac{-2m}{3(m^2-d^2)}}$.

By (10) and (11) we have $\mathbf{a}(x)\mathbf{b}(x) \geq 2^{\frac{d^2}{2x}}4^{-\frac{d(d-1)}{x+d}} > 2^{\frac{-3d^2}{2x}}$ for $x > 3d$. Since $m > 3d$ we have $\frac{\sqrt{R(R-m)}}{R-m+d} > 1$ and $\frac{\sqrt{m}}{(m-1)(m+d)} > \frac{3}{4}m^{-\frac{3}{2}}$. Hence we get

$$\frac{\frac{1}{2}n_d}{(m-1)\binom{R}{m}} > \frac{3}{4}c\frac{R}{m^{\frac{3}{2}}}2^{\frac{-3d^2}{2}(\frac{1}{m}+\frac{1}{R-m})}.$$

We may assume $m \geq 14$. Hence $c = \sqrt{\frac{2}{\pi}}\ e^{\frac{-2m}{3(m^2-d^2)}} \geq \sqrt{\frac{2}{\pi}}\ e^{-\frac{3}{4m}} \geq \frac{3}{4}$ and $\frac{1}{m} + \frac{1}{R-m} \leq \frac{9}{10}\frac{1}{m}$. Therefore we get

$$\frac{\frac{1}{2}n_d}{(m-1)\binom{R}{m}} > \frac{9}{16}\frac{R}{m^{\frac{3}{2}}}2^{-\frac{5d^2}{3m}}.$$

Thus our condition $\frac{1}{2}n_d R > (m-1)\binom{R}{m}$ is fulfilled when we have $m > 3d$ and $\frac{9}{16}\frac{R}{m^{\frac{3}{2}}}2^{-\frac{5d^2}{3m}} \geq 1$,

that is to say, that for all even d such that $d < \frac{m}{3}$ and $d \leq \sqrt{\frac{3}{5}m\log_2\left(\frac{9}{16}\frac{R}{m^{\frac{3}{2}}}\right)}$, there exists an m–box of discrepancy md. Hence there is an even d such that $d \geq \sqrt{\frac{3}{5}m\log_2\left(\frac{9}{16}\frac{R}{m^{\frac{3}{2}}}\right)} - 2$ or $d \geq \frac{m}{3} - 2$. Therefore we have $d_{Rm} \geq \min\left\{m^{\frac{3}{2}}\sqrt{\frac{3}{5}\log_2\left(\frac{9}{16}\frac{R}{m^{\frac{3}{2}}}\right)} - 2m, \frac{m^2}{3} - 2m\right\}$.

Until now we have only considered R and m to be even. We will now consider general values of R and m. If $f : [R]^2 \to \{-1,+1\}$ is a coloring, then by restriction, we get a coloring $[R-1]^2 \to \{-1,+1\}$, and hence we have $d_{Rm} \geq d_{R-1\,m}$. Also note that an $(m-1)$–box of discrepancy $d_{R\,m-1}$ in $[R]^2$ will yield an m–box of discrepancy at least $d_{R\,m-1} - (2m-1)$, simply by adding a vertical and horizontal line to our $(m-1)$–box. Hence $d_{Rm} \geq d_{R\,m-1} - 2m + 1$. Therefore we have

$$d_{Rm} \geq \tfrac{6}{5}(2\lfloor\tfrac{m}{2}\rfloor)^{\frac{3}{2}}\left(1 - \tfrac{2m}{R-1}\right)\sqrt{1 - \tfrac{m}{R-1}} - 3m, \tag{12}$$

for all $m \leq \frac{1}{2}R$ and

$$d_{Rm} \geq \min\left\{(m-1)^{\frac{3}{2}}\sqrt{\tfrac{3}{5}\log_2\left(\tfrac{9}{16}\tfrac{R-1}{m^{\frac{3}{2}}}\right)} - 4m, \tfrac{(m-1)^2}{3} - 4m\right\}.$$

for all $m \leq \frac{1}{2}(R-1)^{\frac{2}{3}}$. From $m \geq 10$ in the first case and $m \geq 14$ in the second we get the theorem. □

Theorem 4.4. *For all $R \in \mathbf{N}$ the discrepancy d_R of the hypergraph of all m-boxes in $[R]^2$ is bounded by*

$$\tfrac{1}{15}R^{\frac{3}{2}} - \tfrac{4}{5}R \leq d_R \leq 2R^{\frac{3}{2}}.$$

Proof. Note that the lower bound is trivial for $R \leq 144$. For the remaining values taking $m = \frac{1}{4}(R-1)$ in equation (12) of the proof of theorem 4.2 is enough.

For the upper we proceed as in the proof of theorem 4.1, and get

$$\begin{aligned} d_R &\leq \sqrt{2R^2 \log\left(2\sum_{m=1}^{R}\binom{R}{m}^2\right)} \\ &\leq \sqrt{2R^2 \log\left(2^{2R+1}\right)} \\ &\leq \sqrt{2R^2 (2R+1) \log(2)} \\ &< 2R^{\frac{3}{2}}. \end{aligned}$$

Note that we actually computed the upper bound of the hypergraph of all rectangles in $[R]^2$, but the loss is marginal. □

We may remark that these constants could be improved a little. In particular by using another probabilistic argument (not published yet), the upper bounds can be lowered by a factor of about $\sqrt{2}$.

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