

Matrix Approximation and Tusnády's Problem

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Abstract

We consider the problem of approximating a given matrix by an integer one such that in all geometric submatrices the sum of the entries does not change by much. We show that for all integers $m, n \geq 2$ and real matrices $A \in \mathbb{R}^{m \times n}$ there is an integer matrix $B \in \mathbb{Z}^{m \times n}$ such that

$$\left| \sum_{i \in I} \sum_{j \in J} (a_{ij} - b_{ij}) \right| < 4 \log_2(\min\{m, n\})$$

holds for all intervals $I \subseteq [m]$, $J \subseteq [n]$. Such a matrix can be computed in time $O(mn \log(\min\{m, n\}))$. The result remains true, if we add the requirement $|a_{ij} - b_{ij}| < 2$ for all $i \in [m]$, $j \in [n]$. This is surprising, as the slightly stronger requirement $|a_{ij} - b_{ij}| < 1$ makes the problem equivalent to Tusnády's problem.

1 Introduction and Results

1.1 Matrix Rounding Problems

We consider the problem of approximating a given matrix by an integer one such that a particular error measure is small. Such matrix rounding and approximation problems have a rich history and found several applications.

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Baranyai [Bar75] showed that any matrix can be rounded to an integer one in such a way that the errors in all rows, all columns and the whole matrix are less than one. He used this “integer making lemma” to prove several deep partitioning theorems for complete uniform hypergraphs, among them the precise estimate of its chromatic index. Independently, Causey, Cox and Ernst [CCE85] proved the same rounding result and applied it to several problems in statistics. Further applications and results in this direction can be found in [SY93, BC04, HKL96, DFKO].

Raghavan [Rag88], Karp, Leighton, Rivest, Thompson, Vazirani and Vazirani [KLR⁺87] and other authors use matrix rounding algorithms to solve VLSI routing problems. Asano, Katoh, Obokata and Tokuyama [AKOT03] (see also [Asa03]) relate the digital halftoning problem to a matrix rounding problem. They suggest that if the average error in suitable regions is small, then the image represented by the rounded matrix is a high quality halftoning of the grey-scale image represented by the input matrix. In this context, the problem of rounding with small average error in all 2×2 geometric (contiguous) submatrices is investigated. A recent improvement of these rounding results can be found in [Doe04b].

In this paper, we are concerned with the errors in all geometric submatrices. Throughout the paper let $m, n \geq 2$ be integers. For $m \times n$ matrices A and B let

$$d(A, B) = \max_{I, J} \left| \sum_{i \in I} \sum_{j \in J} (a_{ij} - b_{ij}) \right|,$$

where $I \subseteq [m] := \{1, \dots, m\}$ and $J \subseteq [n]$ are intervals. We say that B is a *rounding* of A , if $b_{ij} \in \{\lfloor a_{ij} \rfloor, \lceil a_{ij} \rceil\}$ for all $i \in [m], j \in [n]$. In other words, B is integral and $|a_{ij} - b_{ij}| < 1$ holds for all $i \in [m], j \in [n]$.

The *matrix rounding problem* is, given $A \in \mathbb{R}^{m \times n}$, to find a rounding B of A that minimizes $d(A, B)$. Denote by $r_{m,n}$ the smallest $r \in \mathbb{R}$ such that for any $A \in \mathbb{R}^{m \times n}$ a rounding B exists with $d(A, B) \leq r$. This matrix rounding problem is equally hard as a famous problem from computational geometry known as Tusnády’s Problem. We briefly state the problem and the connection between the two.

1.2 Tusnády's Problem

Tusnády's Problem is to color a finite collection of points in the plane with two colors in such a way that in each rectangle the number of red points deviates not too much from the one of blue points. More precisely, let $P \subseteq \mathbb{R}^2$ be finite and $\chi : P \rightarrow \{-1, 1\}$. For a rectangle $R \subseteq \mathbb{R}^2$ let $\chi(R) = \sum_{p \in P \cap R} \chi(p)$ denote the difference of $(+1)$ -colored points to (-1) -colored points in R . The discrepancy of χ is $T(P, \chi) = \max_R |\chi(R)|$, where the maximum is taken over all rectangles R in the plane. The discrepancy $T(P)$ is the minimum $T(P, \chi)$ among all colorings $\chi : P \rightarrow \{-1, 1\}$. By t_n we denote the maximum discrepancy among all n -point sets in the plane. In other words, t_n is the smallest number such that any n -points can be two-colored in a way that in no rectangle the number of points in one color exceeds the one in the other by more than t_n . Note that this combinatorial discrepancy is to be distinguished from geometric discrepancies of point sets and their application in numerical integration (see e.g. Matoušek [Mat99] or Niederreiter [Nie92]).

Tusnády's problem has a rich history. With the focus on extending results of Komlós, Major and Tusnády [KMT75], Gábor Tusnády asked if t_n can be bounded by a constant independent of n . Beck [Bec81a] was the first to disprove this by showing a non-constant lower bound of $\Omega(\log n)$ for t_n via a reduction to geometric discrepancies and Schmidt's lower bound [Sch72]. In the same paper, he gave an $O(\log^4 n)$ upper bound, which he improved to $O(\log^{3.5+\varepsilon} n)$ in [Bec89]. Bohus [Boh90] derived an $O(\log^3 n)$ bound from his bound on the k permutation problem. Matoušek [Mat99] proved $O(\log^{2.5} n \sqrt{\log \log n})$ by a clever application of Beck's partial coloring method. The current best upper bound of $O(\log^{2.5} n)$ due to Srinivasan [Sri97] is obtained via the so-called entropy method (cf. Matousek [Mat99]). Note that only Beck's first-mentioned and Bohus' results are constructive.

The following connection between the matrix rounding problem and Tusnády's problem was used without proof in [AKOT03], so for reasons of completeness we give a short proof.

Theorem 1. *For all $m, n \in \mathbb{N}$, $m \geq n$, we have $\frac{1}{2}t_n \leq r_{m,n} \leq t_{mn}$.*

Proof. Let $P \subseteq \mathbb{R}^2$, $|P| = n$. Note that $T(P)$ is invariant under strictly monotone transformations in the coordinates: Let $\sigma, \tau : \mathbb{R} \rightarrow \mathbb{R}$ be strictly

monotone mappings and $\pi : \mathbb{R}^2 \rightarrow \mathbb{R}^2; (x, y) \mapsto (\sigma(x), \tau(y))$. Then $T(P) = T(\pi(P))$, as $T(P, \chi) = T(\pi(P), \chi \circ \pi^{-1})$ for all $\chi : P \rightarrow \{-1, 1\}$. Thus we may assume that $P \subseteq [n] \times [n]$.

Let $A \in \{0, \frac{1}{2}\}^{m \times n}$ such that $a_{ij} = \frac{1}{2}$ if and only if $(i, j) \in P$. Let B be an optimal rounding of A . Let $\chi : P \rightarrow \{-1, 1\}$ such that $\chi((i, j)) = -1$, if $b_{ij} = 0$, and $\chi((i, j)) = 1$, if $b_{ij} = 1$, for all $(i, j) \in P$. Now for any rectangle R , we have

$$\sum_{p \in P \cap R} \chi(p) = 2 \sum_{(i,j) \in P \cap R} (b_{ij} - a_{ij}) = 2 \sum_{(i,j) \in R \cap [m] \times [n]} (b_{ij} - a_{ij}).$$

Hence $T(P) \leq d(A, B) \leq r_{m,n}$.

To prove the second inequality, we use the result of Lovász, Spencer and Vesztergombi [LSV86] stating that the hereditary linear discrepancy of a hypergraph is at most twice its hereditary discrepancy (the factor of 2 can be replaced by $2(1 - \frac{1}{2m})$, cf. [Doe00], but we shall not need this). Since r_{mn} is nothing else than the hereditary linear discrepancy of the hypergraph of rectangles in $[m] \times [n]$, we have

$$r_{mn} \leq 2 \max_{A \in \{0, \frac{1}{2}\}^{m \times n}} \min_B d(A, B),$$

where B runs over all roundings of A . Let $A \in \{0, \frac{1}{2}\}^{m \times n}$. Put $P = \{(i, j) \in [m] \times [n] \mid a_{ij} = \frac{1}{2}\}$. Let $\chi : P \rightarrow \{-1, 1\}$ such that $T(P, \chi) \leq t_{|P|} \leq t_{mn}$. Define $B \in \{0, 1\}^{m \times n}$ through $b_{ij} = 1$ if and only if $(i, j) \in P$ and $\chi((i, j)) = -1$. Then for all intervals $I \subseteq [m]$ and $J \subseteq [n]$, we compute

$$\left| \sum_{i \in I} \sum_{j \in J} (a_{ij} - b_{ij}) \right| = \left| \frac{1}{2} \sum_{(i,j) \in P \cap (I \times J)} \chi((i, j)) \right| \leq \frac{1}{2} T(P, \chi).$$

Hence $d(A, B) \leq \frac{1}{2} t_{mn}$. We conclude $r_{m,n} \leq t_{mn}$. □

Corollary 2. For $m \geq n$,

$$\begin{aligned} r_{m,n} &= \Omega(\log n), \\ r_{m,n} &= O(\log^{2.5} m). \end{aligned}$$

1.3 Our Results

The result presented in this paper is that we do get an upper bound of $O(\log n)$ if we do not require B to be a rounding of A .

Theorem 3. *Let $m, n \geq 2$ be integers. Then for all real matrices $A \in \mathbb{R}^{m \times n}$ there is an integer matrix $B \in \mathbb{Z}^{m \times n}$ such that*

$$d(A, B) < 4 \log_2(\min\{m, n\}).$$

Such a matrix can be computed in time $O(mn \log(\min\{m, n\}))$. The result remains true, if we add the requirement $|a_{ij} - b_{ij}| < 2$ for all $i \in [m], j \in [n]$.

Our result has an interesting implication: Either, the rounding restriction $|a_{ij} - b_{ij}| < 1$ makes the matrix approximation problem significantly harder, or both Tusnády's problem and the matrix rounding problem can be solved with error $O(\log n)$. We believe the latter to be more likely.

Since we do not even have a non-constant lower bound, we did not try to optimize the constant in Theorem 3. Indeed, there is room for improvements, e.g., by writing arbitrary intervals not only as disjoint unions of canonical intervals, but also allowing differences in Lemma 5.

Note that the one-dimensional version of our problem is elementary. A simple greedy algorithm, sometimes called 1D error diffusion, yields the following result.

Lemma 4. *Let $a_1, \dots, a_m \in \mathbb{R}$. Then there are $b_1, \dots, b_m \in \mathbb{Z}$ such that $|\sum_{i=i_1}^{i_2} (a_i - b_i)| < 1$ holds for all $1 \leq i_1 \leq i_2 \leq m$. They can be computed in time $O(m)$.*

In fact, much more about this problem is known, e.g., the number of such roundings and how to compute an optimal one in time $O(m \log m)$, see [STT01, Doe04a]. The novel contribution of this paper is to combine Lemma 4 successfully with classical ideas like the decomposition into canonical intervals. In fact, our approach works for any set of regions of type $I \times J$, where J is an interval and I is a hyperedge of a totally unimodular hypergraph (note that Lemma 4 apart from the computational statement holds for arbitrary totally unimodular hypergraphs instead of the hypergraph of intervals).

2 Proof of the Main Result

We denote by \mathbb{N} the positive integers, by \mathbb{N}_0 the nonnegative ones. For arbitrary numbers $p, q \in \mathbb{R}$ we set $[p] = \{i \in \mathbb{N} \mid i \leq p\}$ and $[p..q] = \{i \in \mathbb{N}_0 \mid p \leq i \leq q\}$.

Let $n, m \in \mathbb{N}$. By $\text{Int}([n]) = \{[i..j] \mid 1 \leq i \leq j \leq n\}$ we denote the set of intervals in $[n]$. A set of form $[\lambda 2^k + 1..(\lambda + 1)2^k]$ for some $\lambda \in \mathbb{Z}$ is called a *canonical interval* of length 2^k . We write $\text{CInt}_k([n])$ for the set of all canonical interval of length 2^k that are contained in $[n]$. Similar as in Beck [Bec81b], we have the following decomposition lemma.

Lemma 5. *For $n \geq 2$, any interval in $[n]$ is the disjoint union of at most $2 \log_2 n$ canonical intervals.*

Let A and B be $m \times n$ matrices. For any $S \subseteq [m] \times [n]$ we use $A(S)$ to denote the sum $\sum_{(i,j) \in S} a_{ij}$ of the entries of A in S . The distance of A and B with respect to S is $d(A, B, S) = |A(S) - B(S)|$.

Lemma 6. *Let $\ell \in \mathbb{N}_0$, $n = 2^\ell$ and $A \in \mathbb{R}^{m \times n}$. Then there is an $\tilde{A} \in \mathbb{R}^{m \times n}$ that has integral row sums $\sum_{j \in [n]} \tilde{a}_{ij}$, $i \in [m]$, and $d(A, \tilde{A}, I \times J) < 2^{-\ell+k}$ for all $I \in \text{Int}([m])$, $k \in [0..\ell]$ and $J \in \text{CInt}_k([n])$. It can be computed in time $O(mn)$.*

Proof. Let $s_i = \sum_{j=1}^n a_{ij}$ for all $i \in [m]$. By Lemma 4, there are $t_i \in \mathbb{Z}$, $i \in [m]$, such that $D(i_1, i_2) := \sum_{i=i_1}^{i_2} (t_i - s_i)$ has absolute value less than one for all $i_1, i_2 \in [m]$. Put $\tilde{a}_{ij} = a_{ij} + \frac{1}{n}(t_i - s_i)$ for all $i \in [m], j \in [n]$. Since $\sum_{j \in [n]} \tilde{a}_{ij} = t_i$ for all $i \in [m]$, \tilde{A} has integral row sums. Let $I \in \text{Int}([m])$, $k \in [0..\ell]$ and $J \in \text{CInt}_k([n])$. Then

$$d(A, \tilde{A}, I \times J) = \left| \frac{|J|}{n} \sum_{i=\min I}^{\max I} (t_i - s_i) \right| = \frac{1}{n} 2^k |D(\min I, \max I)| < 2^{-\ell+k}.$$

□

The proof of Theorem 3 depends heavily on the following lemma.

Lemma 7. Let $\ell \in \mathbb{N}_0$, $n = 2^\ell$ and $A \in \mathbb{R}^{m \times n}$ such that all row sums $\sum_{j \in [n]} a_{ij}$, $i \in [m]$, are integral. Then there is a $B \in \mathbb{Z}^{m \times n}$ such that for all $I \in \text{Int}([m])$, $k \in [0.. \ell]$ and $J \in \text{CInt}_k([n])$ we have

$$d(A, B, I \times J) \leq \begin{cases} \sum_{i=0}^{\ell-1-k} 2^{-i} & \text{if } k < \ell \\ 0 & \text{if } k = \ell. \end{cases}$$

It can be computed in time $O(mn \log n)$.

Proof. We use induction on ℓ . For $\ell = 0$, we have $n = 1$ and thus $A \in \mathbb{Z}^{m \times n}$ by assumption. Hence $B = A$ proves the claim.

Let $\ell > 0$ and assume the claim to be true for $\ell - 1$. Let $A^L, A^R \in \mathbb{R}^{m \times (n/2)}$ such that $A = (A^L \ A^R)$. Obtain \tilde{A}^L from A^L as in Lemma 6. Set $\tilde{A}^R = A^R - (\tilde{A}^L - A^L)$. Then both \tilde{A}^L and \tilde{A}^R have integral row sums. By induction, there are $B^L, B^R \in \mathbb{Z}^{m \times (n/2)}$ such that the pairs (\tilde{A}^L, B^L) and (\tilde{A}^R, B^R) in the roles of (A, B) both fulfill the assertion of this lemma. Put $B = (B^L \ B^R)$.

Let $I \in \text{Int}([m])$, $k \in [0.. \ell]$ and $J \in \text{CInt}_k([n])$. If $J = [n]$, then

$$\begin{aligned} B(I \times J) &= B^L(I \times [\frac{n}{2}]) + B^R(I \times [\frac{n}{2} + 1 .. n]) \\ &= \tilde{A}^L(I \times [\frac{n}{2}]) + \tilde{A}^R(I \times [\frac{n}{2} + 1 .. n]) \\ &= A(I \times J). \end{aligned}$$

Hence $d(A, B, I \times J) = 0$ as claimed. If $J \neq [n]$, i.e., $k < \ell$, then either $J \subseteq [\frac{n}{2}]$ or $J \subseteq [\frac{n}{2} + 1 .. n]$. In the first case, we compute

$$\begin{aligned} d(A, B, I \times J) &= d(A^L, B^L, I \times J) \\ &\leq d(A^L, \tilde{A}^L, I \times J) + d(\tilde{A}^L, B^L, I \times J) \\ &< 2^{-\ell+1+k} + \sum_{i=0}^{\ell-2-k} 2^{-i} \\ &= \sum_{i=0}^{\ell-1-k} 2^{-i}. \end{aligned}$$

An analogous argument shows the claim for $J \subseteq [\frac{n}{2} + 1 .. n]$. \square

Proof of Theorem 3. Let $A \in \mathbb{R}^{m \times n}$. Without loss of generality, we may assume $m \geq n$. Let $n_0 = 2^\ell$ be the smallest power of two not less than n . By

appending columns of zeroes, we obtain $A' \in \mathbb{R}^{m \times n_0}$ from A . By Lemma 6, we compute a matrix \tilde{A} having integral row sums such that for all $I \in \text{Int}([m])$, $k \in [0..\ell]$ and $J \in \text{CInt}_k([n_0])$ we have $d(A', \tilde{A}, I \times J) < 2^{-\ell+k}$. Applying Lemma 7 to \tilde{A} , we obtain a matrix B that fulfills $d(A', B, I \times J) < \sum_{i=0}^{\ell-k} 2^{-i} < 2$ for all $I \in \text{Int}([m])$, $k \in [0..\ell]$ and $J \in \text{CInt}_k([n_0])$. By Lemma 5, we infer $d(A, B, I \times J) = d(A', B, I \times J) < 4 \log_2 n$ for all $I \in \text{Int}([m])$, $J \in \text{Int}([n])$.

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