

L_p Linear Discrepancy of Totally Unimodular Matrices*

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Abstract

Let $p \in [1, \infty[$ and $c_p = \max_{a \in [0,1]} ((1-a)a^p + a(1-a)^p)^{1/p}$. We prove that the known upper bound $\text{lindisc}_p(A) \leq c_p$ for the L_p linear discrepancy of a totally unimodular matrix A is asymptotically sharp, i.e.,

$$\sup_A \text{lindisc}_p(A) = c_p.$$

We estimate $c_p = \frac{p}{p+1} \left(\frac{1}{p+1}\right)^{1/p} (1 + \varepsilon_p)$ for some $\varepsilon_p \in [0, 2^{-p+2}]$, hence $c_p = 1 - \frac{\ln p}{p}(1 + o(1))$. We also show that an improvement for smaller matrices as in the case of L_∞ linear discrepancy cannot be expected. For any $p \in \mathbb{N}$ we give a totally unimodular $(p+1) \times p$ matrix having L_p linear discrepancy greater than $\frac{p}{p+1} \left(\frac{1}{p+1}\right)^{1/p}$.

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1 Introduction and Results

Let $p \in [1, \infty[$ and let $A \in \mathbb{R}^{m \times n}$. Denote the rows of A by $a^{(1)}, \dots, a^{(m)} \in \mathbb{R}^n$. Let $x \in [0, 1]^n$. The L_p linear discrepancy of A with respect to x is

$$\begin{aligned} \text{lindisc}_p(A, x) &= \min_{y \in \{0,1\}^n} \frac{1}{m^{1/p}} \|A(x - y)\|_p \\ &= \min_{y \in \{0,1\}^n} \left(\frac{1}{m} \sum_{i=1}^m |a^{(i)} \cdot (x - y)|^p \right)^{1/p}. \end{aligned}$$

The L_p linear discrepancy of A is $\text{lindisc}_p(A) = \max_{x \in [0,1]^n} \text{lindisc}_p(A, x)$.

The matrix A is called *totally unimodular*, if each square submatrix has determinant $-1, 0$ or 1 . In particular, the entries of a totally unimodular matrix are from $\{-1, 0, 1\}$. Put $c_p = \max_{a \in [0,1]} ((1-a)a^p + a(1-a)^p)^{1/p}$. Motivated by an application in image processing, Asano, Katoh, Obokata, Tokuyama [2] (cf. also the survey Asano [1]) show and estimate

$$\text{lindisc}_p(A) \leq c_p \leq 1 - \frac{1}{p+1}. \quad (1)$$

They also note that the n -dimensional identity matrix I_n satisfies $\text{lindisc}_p(I_n) = \frac{1}{2}$ for all p and n . This shows that the first inequality in (1) is sharp for $p \leq 3$.

The objective of this note is to improve the lower bound for $p \geq 3$. We show that for all $n \in \mathbb{N}$, there is a totally unimodular matrix $A \in \{0, 1\}^{(n+1) \times n}$ such that

$$\text{lindisc}_p(A) \geq c_p(1 + o(1))$$

with $o(1)$ term depending on n only. Estimating c_p tighter than in [2] yields

$$\frac{p}{p+1} \left(\frac{1}{p+1} \right)^{1/p} \leq c_p \leq \frac{p}{p+1} \left(\frac{1}{p+1} \right)^{1/p} (1 + 2^{-p+2}).$$

Thus $c_p = 1 - \frac{1+p}{p}(1 + o(1))$. Finally, we give for any $p \in \mathbb{N}$ a totally unimodular $(p+1) \times p$ matrix A such that $\text{lindisc}_p(A) > \frac{p}{p+1} \left(\frac{1}{p+1} \right)^{1/p}$. This shows that an improvement of (1) for smaller matrices as was recently proven for the L_∞ linear discrepancy (see Section 3) cannot exist.

2 Proofs of the Main Results

Note first that $c_p = \max_{a \in [0, \frac{1}{2}]} ((1-a)a^p + a(1-a)^p)^{1/p}$ due to symmetry. Since the case $p \leq 3$ was already completely solved in [2], we assume $p \geq 3$ in the following. We prove the following lower bound.

Theorem 1. *For all $p \geq 3$ and all $n \in \mathbb{N}$, there is a totally unimodular matrix $A \in \{0, 1\}^{(n+1) \times n}$ such that*

$$\text{lindisc}_p(A) \geq c_p(1 + o(1))$$

with $o(1)$ term depending on n only.

Proof. Let $a \in]0, \frac{1}{2}]$ and $n \in \mathbb{N}$ be sufficiently large. Define $A \in \{0, 1\}^{(n+1) \times n}$ by $a_{ij} = 1$ if and only if $i = j$ or $i = n + 1$. Clearly, A is totally unimodular. Let $x = a\mathbf{1}_n \in \mathbb{R}^n$. Let $y \in \{0, 1\}^n$ such that $\|A(x - y)\|_p$ is minimal. Let k be the number of $i \in \{1, \dots, n\}$ such that $y_i = 1$. Then

$$\|A(x - y)\|_p^p = (n - k)a^p + k(1 - a)^p + |na - k|^p =: f(k).$$

Note that this value only depends on the number k , but not on the distribution of the ones in y . f viewed as function on the reals is convex and has a minimum at $k_0 = an - (\frac{(1-a)^p - a^p}{p})^{1/(p-1)}$. This yields

$$\begin{aligned} \|A(x - y)\|_p^p &\geq n((1-a)a^p + a(1-a)^p) - ((1-a)^p - a^p)^{p/(p-1)} \frac{p-1}{p^{p/(p-1)}} \\ &\geq n((1-a)a^p + a(1-a)^p) - 1. \end{aligned}$$

Thus

$$\text{lindisc}_p(A) \geq \max_{a \in [0, 1]} \sqrt[p]{\frac{n}{n+1}((1-a)a^p + a(1-a)^p) - \frac{1}{n+1}} = c_p(1 + o(1)). \quad (2)$$

□

Having shown that c_p is the supremum L_p linear discrepancy of a totally unimodular matrix, we now estimate the constant c_p .

Lemma 2. For $p \geq 3$,

$$\frac{p}{p+1} \left(\frac{1}{p+1} \right)^{1/p} \leq c_p \leq \frac{p}{p+1} \left(\frac{1}{p+1} \right)^{1/p} (1 + 2^{-p+2}).$$

In particular, $c_p = 1 - \frac{\ln p}{p}(1 + o(1))$.

Proof. We use the estimate $1 + x \leq e^x \leq 1 + \frac{x}{1-x}$ valid for all $x < 1$ several times. Putting $a = \frac{1}{p+1}$ in the definition of c_p , we obtain

$$c_p \geq \left(\frac{p^p + p}{(p+1)^{p+1}} \right)^{1/p} > \frac{p}{p+1} \left(\frac{1}{p+1} \right)^{1/p}.$$

For the upper bound, we have

$$\begin{aligned} c_p &= \left(\max_{a \in [0, \frac{1}{2}]} ((1-a)a^p + a(1-a)^p) \right)^{1/p} \\ &\leq \left(\max_{a \in [0, \frac{1}{2}]} (1-a)a^p + \max_{a \in [0, \frac{1}{2}]} a(1-a)^p \right)^{1/p} = \left(\frac{p^p}{(p+1)^{p+1}} + 2^{-p-1} \right)^{1/p}. \end{aligned}$$

Since $x \mapsto x^{1/p}$ is concave for $p \geq 1$, we conclude

$$\begin{aligned} c_p &\leq \left(\frac{p^p}{(p+1)^{p+1}} \right)^{1/p} + \frac{1}{p} \left(\frac{p^p}{(p+1)^{p+1}} \right)^{-1+(1/p)} 2^{-p} \\ &= \frac{p}{p+1} \left(\frac{1}{p+1} \right)^{1/p} \left(1 + \left(\frac{p+1}{p} \right)^{p+1} 2^{-p} \right). \end{aligned}$$

Finally, we estimate

$$\left(\frac{p+1}{p} \right)^{p+1} 2^{-p} \leq e^{(p+1)/p} 2^{-p} \leq e^{4/3} 2^{-p} \leq 4 \cdot 2^{-p}$$

using our assumption $p \geq 3$. From $\frac{p}{p+1} \left(\frac{1}{p+1} \right)^{1/p} = 1 - \frac{\ln p}{p}(1 + o(1))$ we derive the second claim. \square

3 Improving the $1 + o(1)$ term relative to n ?

We now turn to the dependency of n again. Recent results concerning the L_∞ linear discrepancy make this a natural problem. The L_∞ linear discrepancy

of A with respect to x is

$$\text{lindisc}_\infty(A, x) = \min_{y \in \{0,1\}^n} \|A(x - y)\|_\infty,$$

the L_∞ linear discrepancy of A is $\text{lindisc}_\infty(A) = \max_{x \in [0,1]^n} \text{lindisc}_\infty(A, x)$.

For the L_∞ linear discrepancy of totally unimodular matrices, the bound $\text{lindisc}_\infty(A) \leq 1$ is well known and follows easily from the theorem of Hoffman and Kruskal [6] (similar to the proof of the upper bound for the L_p linear discrepancy in [2]). However, this bound is not sharp. If an $m \times n$ matrix A is totally unimodular, then $\text{lindisc}_\infty(A) \leq \frac{n}{n+1}$ was shown in [4, 5] and independently in [3]. A similar improvement, as could be conjectured from equation (2), is not possible for the L_p linear discrepancy:

Theorem 3. *For any $p \in \mathbb{N}$, there is a totally unimodular matrix A having p columns only such that $\text{lindisc}_p(A) > \frac{p}{p+1} \left(\frac{1}{p+1}\right)^{1/p}$.*

Note that the bound above is less than a factor of $1 + 2^{-p+2}$ below our upper bound. We stated this result for integral p only to keep things simple. However, it is not difficult to see that similar statements can be made for arbitrary p .

Proof of Theorem 3. Let $p \in \mathbb{N}$. Let $\lambda \in \mathbb{N}_0$ and $n = \lambda(p+1) + p$. Let A be the $(n+1) \times n$ matrix defined in the proof of Theorem 1. Let $a = \frac{1}{p+1}$ and $y \in \{0,1\}^n$ such that $\|A(a\mathbf{1}_n - y)\|_p$ is minimal. Now $f(k)$ defined as in the proof of Theorem 1 but viewed as mapping from the integers is minimal for $k = \lfloor na \rfloor = na - \frac{p}{p+1}$ and $k = \lceil na \rceil = na + \frac{1}{p+1}$. In both cases we have $f(k) = (n+1) \frac{p^p + p}{(p+1)^{p+1}}$, i.e.,

$$\text{lindisc}_p(A, a\mathbf{1}_n) = \left(\frac{p^p + p}{(p+1)^{p+1}}\right)^{1/p} > \frac{p}{p+1} \left(\frac{1}{p+1}\right)^{1/p}.$$

□

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