THE HEREDITARY DISCREPANCY IS NEARLY INDEPENDENT OF THE NUMBER OF COLORS

BENJAMIN DOERR

ABSTRACT. We investigate the discrepancy (or balanced coloring) problem for hypergraphs and matrices in arbitrary numbers of colors. We show that the hereditary discrepancy in two different numbers $a, b \in \mathbb{N}_{\geq 2}$ of colors is the same apart from constant factors, i.e.,

$$\mathrm{herdisc}(\cdot, b) = \Theta(\mathrm{herdisc}(\cdot, a)).$$

This contrasts the ordinary discrepancy problem, where no correlation exists in many cases.

1. Introduction and Results

The discrepancy problem is to partition the vertex set of a hypergraph into two classes such that both contain roughly the same number of vertices in each hyperedge. Recently, cf. e.g. [BHK01, BČC$^+$02, Doe02, DS03], there has been interest in the general partitioning problem in arbitrary numbers of partition classes (or 'colors', as partitions are usually represented by colorings). These results show a strange dichotomy: On the one hand, several classical results can be extended to arbitrary numbers of colors, among them the theorem of Beck and Fiala [BF81], Spencer's 'six standard deviations' bound [Spe85] and the determination of the discrepancy of arithmetic progressions due to Roth [Rot64], Matoušek and Spencer [MS96]. On the other hand, there is no general correlation between the discrepancies of a hypergraph in different number of colors. There are classes of hypergraphs having discrepancy zero in some numbers of colors and high discrepancy in all others (cf. [Doe02]).

The main result of this paper is that the hereditary discrepancy, which is the maximum discrepancy among the induced subhypergraphs, behaves differently. We show that given two numbers of colors $a, b \in \mathbb{N}_{\geq 2}$, there are constants $C_1, C_2 > 0$ such that for *all* hypergraphs $\mathcal{H} = (X, \mathcal{E})$ we have

$$C_1 \,\mathrm{herdisc}(\mathcal{H}, a) \leq \mathrm{herdisc}(\mathcal{H}, b) \leq C_2 \,\mathrm{herdisc}(\mathcal{H}, a).$$

The interesting point is that these bounds are independent of the size of the hypergraph. A bound of type

$$\mathrm{herdisc}(\mathcal{H}, b) = O(\log(|X|)\,\mathrm{herdisc}(\mathcal{H}, a))$$

1991 *Mathematics Subject Classification.* Primary 11K38. Secondary 05C65.
Key words and phrases. discrepancy, hypergraphs.
partially supported (associate member) by the graduate school 'Effiziente Algorithmen und Multiskalenmethoden', Deutsche Forschungsgemeinschaft.

can be shown easily by a recursive approach (see the last section of this paper). We do not aim at optimizing the constants C_1, C_2, but are satisfied with the fact that they may be chosen polynomial in the numbers of colors.

The central step of our proof is to show that a fairly general rounding problem, the linear discrepancy problem in b colors, can be reduced to the hereditary discrepancy problem in a colors. We have

$$\operatorname{lindisc}(\cdot, b) = O(\operatorname{herdisc}(\cdot, a)).$$

This extends the well-known bound $\operatorname{lindisc}(\mathcal{H}, 2) \leq 2 \operatorname{herdisc}(\mathcal{H}, 2)$ due to Beck and Spencer [BS84] and Lovász, Spencer and Vesztergombi [LSV86]. All results stated above hold as well for discrepancies of matrices.

2. Notation and Preliminaries

Below we introduce the multi-color discrepancy notions used in this paper. Though no further prerequisites are needed, we would like to point the interested reader to the chapter Beck and Sós [BS95] or the books Matoušek [Mat99] and Chazelle [Cha00] for more information on combinatorial discrepancies.

2.1. **Multi-Color Discrepancies.**

Let $\mathcal{H} = (X, \mathcal{E})$ be a finite hypergraph, that is, X is a finite set and $\mathcal{E} \subseteq 2^X$. Throughout this section let $c \in \mathbb{N}_{\geq 2}$ denote the number of classes we want to partition the vertices of $\mathcal{H}$ into. It is natural to represent the partition by a coloring. The partition classes then are formed by the sets of equally colored vertices. A c–coloring of $\mathcal{H}$ is a mapping $\chi : X \to M$, where M is some set of cardinality c. We often use $M = [c] := \{1, \ldots, c\}$.

The concept of measuring the deviation of a given partition from an ideal one motivates these definitions of the *discrepancy of* $\mathcal{H}$ *with respect to* χ and the *discrepancy of* $\mathcal{H}$ *in* c *colors*:

$$\begin{aligned} \operatorname{disc}(\mathcal{H}, \chi) &:= \max_{d \in M, E \in \mathcal{E}} \left| |\chi^{-1}(d) \cap E| - \tfrac{1}{c}|E| \right|, \\ \operatorname{disc}(\mathcal{H}, c) &:= \min_{\chi : X \to [c]} \operatorname{disc}(\mathcal{H}, \chi). \end{aligned}$$

For a subset $X_0 \subseteq X$ of vertices denote by $\mathcal{H}_{|X_0} = (X_0, \{E \cap X_0 | E \in \mathcal{E}\})$ the hypergraph induced by X_0. The *hereditary discrepancy in* c *colors* is defined by

$$\operatorname{herdisc}(\mathcal{H}, c) := \max_{X_0 \subseteq X} \operatorname{disc}(\mathcal{H}_{|X_0}, c).$$

2.2. **Discrepancies of Matrices.**

As in two colors, the multi-color notion of discrepancy has a natural extension to matrices. Let $A \in \mathbb{R}^{m \times n}$ be an arbitrary real matrix and $\chi : [n] \to [c]$. Then

$$\begin{aligned} \operatorname{disc}(A, \chi) &:= \max_{d \in [c], i \in [m]} \left| \sum_{j \in \chi^{-1}(d)} a_{ij} - \frac{1}{c} \sum_{j \in [n]} a_{ij} \right|, \\ \operatorname{disc}(A, c) &:= \min_{\chi : [n] \to [c]} \operatorname{disc}(A, \chi). \end{aligned}$$

Immediately we see that $\operatorname{disc}(A, c) = \operatorname{disc}(\mathcal{H}, c)$, if A is the incidence matrix of $\mathcal{H}$. For the problems we are concerned with in this article, it makes no difference whether we restrict ourselves to the special case of hypergraphs or the generalization to matrices. Hence from now on we will deal with the matrix case only. The notion of hereditary discrepancy translates to matrices in the obvious way: Write $A_{|J}$

to denote the submatrix of A containing the rows with index in J only. Then $\operatorname{herdisc}(A,c) := \max_{J\subseteq[n]} \operatorname{disc}(A_{|J},c)$.

2.3. **Linear Discrepancy.** The notion of linear discrepancy refers to the following rounding problem: For a given floating coloring assigning each vertex not a single color, but a weighted mixture of colors, we are looking for a 'pure' coloring (assigning each vertex a single color) such that each hyperedge in total receives every color roughly in the same amount by both colorings.

At this point it will be convenient to choose a different set of colors. Denote by $E^c = \{e^{(1)}, \dots, e^{(c)}\}$ the standard basis of $\mathbb{R}^c$, *i.e.*, $e^{(i)}$ is defined by $e^{(i)}_j = 1$ for $i = j$ and $e^{(i)}_j = 0$ for $i \neq j$. Denote by $\overline{E}^c$ the convex hull of E^c. In the hypergraph case our objective is to 'round' a floating coloring $p : X \to \overline{E}^c$ to a coloring $q : X \to E^c$ in such a way that the imbalances $\left\|\sum_{x\in E} p(x) - \sum_{x\in E} q(x)\right\|_\infty$ are small for all $E \in \mathcal{E}$.

For the matrix case let $\overline{\mathcal{C}^c} := \{p \,|\, p : [n] \to \overline{E}^c\}$ denote the set of all floating colorings and $\mathcal{C}^c := \{p \,|\, p : [n] \to E^c\}$ the set of all pure ones. For $p, q \in \overline{\mathcal{C}^c}$ put

$$d_A(p,q) := \max_{i\in[m]} \left\| \sum_{j=1}^{n} a_{ij}(p(j) - q(j)) \right\|_\infty .$$

The *linear discrepancy of A with respect to* $p \in \overline{\mathcal{C}^c}$ and the *linear discrepancy of A in c colors* now are

$$\begin{aligned} \operatorname{lindisc}(A,p) &:= \min_{q\in\mathcal{C}^c} d_A(p,q), \\ \operatorname{lindisc}(A,c) &:= \max_{p\in\overline{\mathcal{C}^c}} \operatorname{lindisc}(A,p). \end{aligned}$$

It is easy to see that the linear discrepancy problem contains the ordinary discrepancy problem: Let $p : [n] \to \overline{E}^c$ such that $p(i) = \frac{1}{c}\mathbf{1}_c$ for all $i \in [n]$. Then $\operatorname{disc}(A,c) = \operatorname{lindisc}(A,p)$.

Remark 1. *For all matrices A and all $c \in \mathbb{N}_{\geq 2}$, we have* $\operatorname{disc}(A,c) \leq \operatorname{lindisc}(A,c)$.

2.4. **Types.** Let $a, b \in \mathbb{N}_{\geq 2}$. A vector $t \in \{0, \dots, a-1\}^b$ shall be called (a,b)*–type* in this paper if $\|t\|_1 = \sum_{i\in[b]} t_i = a$. Denote by T_{ab} the set of all (a,b)–types. Put $n_{ab} := |T_{ab}|$ and $s_{ab} := \sum_{t\in T_{ab}} t$. The following two lemmata determine n_{ab} and s_{ab}.

Lemma 2. *The number of (a,b)–types is*

$$n_{ab} = \binom{a+b-1}{a} - b.$$

Proof. Put $\overline{T}_{ab} = \{t \in \mathbb{N}_0^b \,|\, \|t\|_1 = a\}$. Then $|\overline{T}_{ab}| = |T_{ab}| + b$. For all $k, n \in \mathbb{N}$, let $P_{nk} := \{t \in \mathbb{N}^k \,|\, \|t\|_1 = n\}$ denote the set of k–partitions of n. Then $|P_{nk}| = \binom{n-1}{k-1}$ can be found in any combinatorics book, e.g. [Aig97, 3.16 on p. 80]. The claim follows from the fact that $\overline{T}_{ab} \to P_{a+b,b}; t \mapsto (t_1+1, \dots, t_b+1)$ is a bijection. □

Lemma 3. *The sum of all (a,b)–types is $s_{ab} = \frac{a}{b} n_{ab} \mathbf{1}_b$.*

Proof. By symmetry, it is clear that $s_{ab} = \lambda \mathbf{1}_b$ for some $\lambda \in \mathbb{R}_{\geq 0}$. From

$$\lambda b = \|\lambda \mathbf{1}_b\|_1 = \|s_{ab}\|_1 = \sum_{t \in T_{ab}} \|t\|_1 = n_{ab} a$$

we get $\lambda = \frac{a}{b} n_{ab}$. □

Lemma 4. *Let $v \in \{0, \ldots, a-1\}^b$ such that a divides $\|v\|_1$. Then v is the sum of (a,b)–types each thereof occurring just once, i.e., there are $\varepsilon_t \in \{0,1\}, t \in T_{ab}$ such that $v = \sum_{t \in T_{ab}} \varepsilon_t t$. These $\varepsilon_t, t \in T_{ab}$, can be found efficiently by a Greedy-Algorithm.*

Proof. Define a sequence of types by repeatedly choosing the lexicographically largest type $t \leq v$ and subtracting it from v. As a divides $\|v\|_1$ and $\|t\|_1$ for all types t, this terminates with $v = 0$. Since each iteration the number of non-zeroes of v increases, no type can be chosen more than once. □

3. Linear and Hereditary Discrepancy in Arbitrary Numbers of Colors

In this section we prove our main result by bounding the linear discrepancy in b colors in terms of the a–color hereditary discrepancy:

Lemma 5. *For any matrix A and arbitrary $a, b \in \mathbb{N}_{\geq 2}$ we have*

$$\operatorname{lindisc}(A, b) \quad \leq \quad \tfrac{a^2}{(a-1)b} n_{ab} \operatorname{herdisc}(A, a).$$

Note that for $a = b = 2$, Lemma 5 is just the result $\operatorname{lindisc}(A,2) \leq 2 \operatorname{herdisc}(A,2)$ of Beck and Spencer [BS84] and Lovász, Spencer and Vesztergombi [LSV86]. To give some intuition of the proof ideas, we start with a series of easy examples. Let $a = 7$ and $b = 5$.

(1) Let us assume that $p(j) = (\frac{2}{7}, \frac{3}{7}, \frac{1}{7}, \frac{1}{7}, 0)$ for all $j \in [n]$. By definition, there is a 7–coloring $\chi : [n] \to [7]$ such that $\operatorname{disc}(A, \chi) \leq \operatorname{herdisc}(A, 7)$. Thus for all $d \in [7]$ and $i \in [m]$ we have $\left|\sum_{\chi(j)=d} a_{ij} - \frac{1}{7} \sum_{j \in [n]} a_{ij}\right| \leq \operatorname{herdisc}(A, 7)$. Define a 5–coloring $q : [n] \to E^5$ by

$$q(j) := \begin{cases} e^{(1)} & \text{if } \chi(j) \in \{1,2\} \\ e^{(2)} & \text{if } \chi(j) \in \{3,4,5\} \\ e^{(3)} & \text{if } \chi(j) = 6 \\ e^{(4)} & \text{if } \chi(j) = 7. \end{cases}$$

For the first color and any $i \in [m]$ we compute

$$\begin{aligned} \left|\sum_{j \in [n]} a_{ij}(p_1(j) - q_1(j))\right| &= \left|\tfrac{2}{7} \sum_{j \in [n]} a_{ij} - \sum_{\chi(j) \in \{1,2\}} a_{ij}\right| \\ &\leq \left|\tfrac{1}{7} \sum_{j \in [n]} a_{ij} - \sum_{\chi(j)=1} a_{ij}\right| + \left|\tfrac{1}{7} \sum_{j \in [n]} a_{ij} - \sum_{\chi(j)=2} a_{ij}\right| \\ &\leq 2 \operatorname{herdisc}(A, 7). \end{aligned}$$

Similarly, we see that the discrepancies in color 2 to 5 are bounded by $3 \operatorname{herdisc}(A,7)$, $1 \operatorname{herdisc}(A,7)$, $1 \operatorname{herdisc}(A,7)$ and zero respectively. Thus $d_A(p,q) \leq 3 \operatorname{herdisc}(A,7)$.

(2) Now assume that $7p(j) \in T_{75}$ for all $j \in [n]$, but in addition to (1) these types may be different. Fix one type t and let $J := \{j \in [n] \,|\, 7p(j) = t\}$. As in (1), there is a 5–coloring $\tilde{p}^{(1)} : J \to E^5$ such that $d_{A_{|J}}(p_{|J}, \tilde{p}^{(1)}) \leq \|t\|_\infty \operatorname{herdisc}(A, 7)$. Extend $\tilde{p}^{(1)}$ to a floating 5–coloring $p^{(1)}$ of $[n]$ by putting $p^{(1)}(j) = p(j)$ for all $j \notin J$. Then $d_A(p, p^{(1)}) \leq \|t\|_\infty \operatorname{herdisc}(A, 7)$. Pick another type t' and do the same for this type to get $p^{(2)} : [n] \to \overline{E}^5$ such that $d_A(p^{(1)}, p^{(2)}) \leq \|t'\|_\infty \operatorname{herdisc}(A, 7)$. Note that $d_A(p, p^{(2)}) \leq \|t + t'\|_\infty \operatorname{herdisc}(A, 7)$. Repeating this procedure with all types, we end up with a true 5–coloring q such that $d_A(p, q) \leq s_{75} \operatorname{herdisc}(A, 7)$.

(3) For the general case we use the 7–ary expansion of the $p(j)$. Then example (2) roughly sketches how to round a given p to a q having 7–ary length one less than p. There is a difference in one aspect: If $p(j) = \sum_{k=0}^{l} z_k 7^{-k}$ for some $z_i \in \{0, \ldots, 6\}^5$ and $l \geq 2$, then z_l is not necessarily a type. All we know is that 7 divides $\|z_l\|_1$. This is where Lemma 4 comes into play. We refer to the following proof of Lemma 6 for the details.

We recall the fact that for every number $x \in [0, 1]$ that has a finite c–ary expansion $x = \sum_{k=0}^{l} z_k c^{-k}$ for some $z_k \in \{0, \ldots, c-1\}$, $z_l \neq 0$, this expansion is unique (among all finite expansions). Denote by $l_c(x) := l$ the length of this c–ary expansion of x. Put $M_{c,l} := \{x \in [0, 1] \,|\, l_c(x) \leq l\}$ and $\mathcal{C}^b_{a,l} := \{p \,|\, p : [n] \to \overline{E}^b \cap M^b_{a,l}\}$ for all $l \in \mathbb{N}_0$.

Lemma 6. *Let $p \in \mathcal{C}^b_{a,l}$ for some $l \in \mathbb{N}$. Then there is a $q \in \mathcal{C}^b_{a,l-1}$ such that*

$$d_A(p, q) \leq \tfrac{1}{b} a^{-l+2} n_{ab} \operatorname{herdisc}(A, a).$$

Proof. We give an algorithmic solution. Set $q^{(0)} = p$. Let $t^{(1)}, \ldots, t^{(n_{ab})}$ be an enumeration of T_{ab} in descending lexicographic order. For all $r = 1, \ldots, n_{ab}$ do the following:

> *Iteration r:* For every $j \in [n]$ let $q^{(r-1)}(j) = \sum_{k=0}^{l} z_k^{(r-1)}(j) a^{-k}$ for some $z_k^{(r-1)}(j) \in \{0, \ldots, a-1\}^b$ denote the a–ary expansion of the vector $q^{(r-1)}(j)$. Set $J^{(r)} := \{j \in [n] \,|\, z_l^{(r-1)}(j) \geq t^{(r)}\}$. Choose a coloring $\chi^{(r)} : J^{(r)} \to [a]$ such that $\operatorname{disc}(A_{|J^{(r)}}, \chi^{(r)}) \leq \operatorname{herdisc}(A, a)$. Choose a function $f^{(r)} : [a] \to [b]$ such that $|(f^{(r)})^{-1}(d)| = t_d^{(r)}$ for all $d \in [b]$. For all $j \in [n]$ put

$$q^{(r)}(j) := \begin{cases} q^{(r-1)}(j) & \text{if } j \notin J^{(r)} \\ q^{(r-1)}(j) - a^{-l} t^{(r)} + a^{-l+1} e^{(f^{(r)} \circ \chi^{(r)}(j))} & \text{if } j \in J^{(r)}. \end{cases}$$

Finally put $q := q^{(n_{ab})}$.

Let $d \in [b]$ and $i \in [m]$. We compute

$$\begin{aligned}
&\left| \sum_{j \in [n]} a_{ij} \left(q^{(r-1)}(j)_d - q^{(r)}(j)_d \right) \right| \\
&= a^{-l+1} \left| \sum_{\substack{j \in J^{(r)} \\ f^{(r)} \circ \chi^{(r)}(j) = d}} a_{ij} - \frac{t_d^{(r)}}{a} \sum_{j \in J^{(r)}} a_{ij} \right| \\
&\leq a^{-l+1} \sum_{e \in (f^{(r)})^{-1}(d)} \left| \sum_{j \in (\chi^{(r)})^{-1}(e)} a_{ij} - \frac{1}{a} \sum_{j \in J^{(r)}} a_{ij} \right| \\
&\leq a^{-l+1} t_d^{(r)} \operatorname{disc}(A_{|J^{(r)}}, \chi^{(r)}) \\
&\leq a^{-l+1} t_d^{(r)} \operatorname{herdisc}(A, a).
\end{aligned}$$

By Lemma 3 we have

$$\begin{aligned}
d_A(p,q) &= \max_{i \in [m]} \left\| \sum_{r=1}^{n_{ab}} \sum_{j \in [n]} a_{ij} \left(q^{(r-1)}(j) - q^{(r)}(j) \right) \right\|_\infty \\
&\leq a^{-l+1} \left\| \sum_{r=1}^{n_{ab}} t^{(r)} \right\|_\infty \operatorname{herdisc}(A, a) \\
&\leq \tfrac{1}{b} a^{-l+2} n_{ab} \operatorname{herdisc}(A, a).
\end{aligned}$$

Next we claim that $q^{(r)}(j) \in \overline{E}^b \cap M_{a,l}^b$ for all $r \in \{0, \dots, n_{ab}\}, j \in [n]$. This is clear for $r = 0$ as $q^{(0)} = p$. We proceed by induction. Let $r \in [n_{ab}]$ and $j \in [n]$ such that $q^{(r-1)}(j) \in \overline{E}^b \cap M_{a,l}^b$. If $j \notin J^{(r)}$, then $q^{(r)}(j) = q^{(r-1)}(j)$ and there is nothing to show. Assume $j \in J^{(r)}$. Since $z_l^{(r-1)}(j) \geq t^{(r)}$, we have $q^{(r-1)}(j) \geq t^{(r)} a^{-l}$ and hence $q^{(r)}(j)$ is not negative by definition. We compute

$$\sum_{d \in [b]} q^{(r)}(j)_d = \sum_{d \in [b]} q^{(r-1)}(j)_d - a^{-l} \sum_{d \in [b]} t_d^{(r)} + a^{-l+1} = \sum_{d \in [b]} q^{(r-1)}(j)_d = 1.$$

Since $q^{(r)}(j)$ is nonnegative, we conclude $q^{(r)}(j)_d \leq 1$ for all $d \in [b]$. Finally, from the definition of $q^{(r)}(j)$ it is clear that $q^{(r)}(j) \in M_{a,l}^b$ if and only if $q^{(r-1)}(j) \in M_{a,l}^b$. This proves the claim.

The definitions of $q^{(\cdot)}(j)$ and $z_l^{(\cdot)}(j)$ also yield $z_l^{(r)}(j) = z_l^{(r-1)}(j) - t^{(r)}$ for all $j \in J^{(r)}, r \in [n_{ab}]$. Since a divides $\|z_l^{(0)}(j)\|_1$ for all $j \in [n]$, Lemma 4 shows that the range of q is actually contained in $M_{a,l-1}^b$. □

Proof of Lemma 5. We show that for any $p : [n] \to \overline{E}^b$ there is a $q : [n] \to E^b$ such that $d_A(p,q) \leq \frac{a^2}{(a-1)b} n_{ab} \operatorname{herdisc}(A, c)$. Let us first assume that $p : [n] \to \overline{E}^b \cap M_{a,l}^b$ for some $l \in \mathbb{N}$.

Inductively, we define a sequence $(p^{(i)})_{i \in [l]}$ such that $p^{(i)} : [n] \to \overline{E}^b \cap M_{a,l-i}^b$. Set $p^{(0)} := p$. Having defined $p^{(i)}$ for some $i < l$ we apply Lemma 6 on $p^{(i)}$ and get $p^{(i+1)} : [n] \to \overline{E}^b \cap M_{a,l-i-1}^b$ such that $d_A(p^{(i)}, p^{(i+1)}) \leq \frac{a^{-l+i+2}}{b} n_{ab} \operatorname{herdisc}(A, a)$.

Let $q := p^{(l)}$. Note that $p^{(l)} : [n] \to E^b$. We have

$$\begin{aligned} d_A(p,q) &\leq \sum_{i=0}^{l-1} d_A(p^{(i)}, p^{(i+1)}) \\ &\leq \tfrac{a^2}{(a-1)b} n_{ab} \operatorname{herdisc}(A,a). \end{aligned}$$

Since $\bigcup_{l\in\mathbb{N}_0} \mathcal{C}^b_{a,l}$ is dense in $\overline{\mathcal{C}^b}$ and $\operatorname{lindisc}(A,\cdot) : \overline{\mathcal{C}^b} \to \mathbb{R}; p \mapsto \operatorname{lindisc}(A,p)$ is continuous, we have $\operatorname{lindisc}(A,p) \leq \frac{a^2}{(a-1)b} n_{ab} \operatorname{herdisc}(A,a)$ for all $p \in \overline{\mathcal{C}^b}$. □

We finally derive our main result from Lemma 5. Since $\operatorname{disc}(A,c) \leq \operatorname{lindisc}(A,c)$ for all A and c, Lemma 5 implies $\operatorname{disc}(A,b) \leq \frac{a^2}{(a-1)b} n_{ab} \operatorname{herdisc}(A,a)$ for *all* A and all $a,b \in \mathbb{N}_{\geq 2}$. Hence we have

$$\begin{aligned} \operatorname{disc}(A_{|J}, b) &\leq \tfrac{a^2}{(a-1)b} n_{ab} \operatorname{herdisc}(A_{|J}, a) \\ &\leq \tfrac{a^2}{(a-1)b} n_{ab} \operatorname{herdisc}(A,a) \end{aligned}$$

for all $J \subseteq [n]$, and thus $\operatorname{herdisc}(A,b) \leq \frac{a^2}{(a-1)b} n_{ab} \operatorname{herdisc}(A,a)$. Applying this twice, we obtain

$$\begin{aligned} \operatorname{herdisc}(A,b) &\leq 2(b-1) \operatorname{herdisc}(A,2) \\ &\leq a^2(b-1) \operatorname{herdisc}(A,a). \end{aligned}$$

Theorem 7. *For all matrices A and all $a,b \in \mathbb{N}_{\geq 2}$,*

$$\operatorname{herdisc}(A,b) \leq a^2(b-1) \operatorname{herdisc}(A,a).$$

4. Discussion

To prove the main result $\operatorname{herdisc}(\cdot,b) = \Theta(\operatorname{herdisc}(\cdot,a))$ we needed a detour through linear discrepancies. It seems to be an interesting question whether this is necessary. The best result avoiding the detour we have contains a logarithmic dependence of the number of columns:

Theorem 8. *Let $b < a$. If b divides a, then* $\operatorname{herdisc}(A,b) \leq \frac{a}{b} \operatorname{herdisc}(A,a)$. *Otherwise we have*

$$\begin{aligned} \operatorname{herdisc}(A,b) &\leq \left\lfloor \tfrac{a}{b} \right\rfloor \log_{1/(1-\frac{b}{a}\lfloor \frac{a}{b} \rfloor)}(n) \operatorname{herdisc}(A,a) \\ &\leq \left\lfloor \tfrac{a}{b} \right\rfloor \log_2(n) \operatorname{herdisc}(A,a). \end{aligned}$$

Proof. Let $A_0 \in \mathbb{R}^{m\times n_0}$ be a submatrix of A. Let $\chi : [n_0] \to [a]$ be a a–coloring such that $\operatorname{disc}(A_0,\chi) \leq \operatorname{herdisc}(A,a)$. Let $f : [a] \to [b] \cup \{0\}$ be any mapping such that $|f^{-1}(i)| = \left\lfloor \frac{a}{b} \right\rfloor$ for all $i \in [b]$. Set

$$\eta : [n_0] \to [b] \cup \{0\}; x \mapsto f(\chi(x)).$$

Then $\operatorname{disc}(A_{0\,|\eta^{-1}([b])}, \eta) \leq \left\lfloor \frac{a}{b} \right\rfloor \operatorname{herdisc}(A,a)$ since each color class of η is the union of $\left\lfloor \frac{a}{b} \right\rfloor$ color classes of χ. If b divides a, $\eta^{-1}([b]) = [n_0]$ and we are done. Let us therefore assume that b does not divide a. We may choose f in such a way that it maps the $a - b\left\lfloor \frac{a}{b} \right\rfloor$ smallest color classes of χ to 0. Hence at least $b\frac{n_0}{a}\left\lfloor \frac{a}{b} \right\rfloor$ points are colored by η, i. e., are not mapped to 0. We repeat this procedure on $\eta^{-1}(0)$ until all points are colored. This takes at most $\log_{1/(1-\frac{b}{a}\lfloor \frac{a}{b} \rfloor)}(n_0) \leq \log_2(n_0)$ iterations. □

Acknowledgments

I was very happy to receive fruitful comments from an anonymous referee, which significantly improved the presentation.

References

[Aig97] M. Aigner. *Combinatorial Theory.* Classics in Mathematics. Springer-Verlag, Berlin, 1997.

[BČC+02] T. Biedl, E. Čenek, T. Chan, E. Demaine, M. Demaine, R. Fleischer, and M. Wang. Balanced k-colorings. *Discrete Mathematics*, 254:19–32, 2002.

[BF81] J. Beck and T. Fiala. "Integer making" theorems. *Discrete Applied Mathematics*, 3:1–8, 1981.

[BHK01] L. Babai, T. P. Hayes, and P. G. Kimmel. The cost of the missing bit: communication complexity with help. *Combinatorica*, 21:455–488, 2001.

[BS84] J. Beck and J. Spencer. Integral approximation sequences. *Math. Programming*, 30:88–98, 1984.

[BS95] J. Beck and V. T. Sós. Discrepancy theory. In R. Graham, M. Grötschel, and L. Lovász, editors, *Handbook of Combinatorics*, pages 1405–1446. Elsevier, 1995.

[Cha00] B. Chazelle. *The Discrepancy Method.* Cambridge University Press, 2000.

[Doe02] B. Doerr. Discrepancy in different numbers of colors. *Discrete Mathematics*, 250:63–70, 2002.

[DS03] B. Doerr and A. Srivastav. Multi-color discrepancies. *Combinatorics, Probability and Computing*, 2003. To appear.

[LSV86] L. Lovász, J. Spencer, and K. Vesztergombi. Discrepancies of set-systems and matrices. *Europ. J. Combin.*, 7:151–160, 1986.

[Mat99] J. Matoušek. *Geometric Discrepancy.* Springer-Verlag, Berlin, 1999.

[MS96] J. Matoušek and J. Spencer. Discrepancy in arithmetic progressions. *J. Amer. Math. Soc.*, 9:195–204, 1996.

[Rot64] K. F. Roth. Remark concerning integer sequences. *Acta Arithmetica*, 9:257–260, 1964.

[Spe85] J. Spencer. Six standard deviations suffice. *Trans. Amer. Math. Soc.*, 289:679–706, 1985.

Benjamin Doerr, Mathematisches Seminar, Christian–Albrechts–Universität zu Kiel, Christian–Albrechts–Platz. 4, D–24098 Kiel, Germany

E-mail address: bed@numerik.uni-kiel.de