

Partial Colorings of Unimodular Hypergraphs

Benjamin Doerr

*Max-Planck-Institut für Informatik
Saarbrücken, Germany*

Abstract

Combinatorial discrepancy theory asks for vertex-colorings of hypergraphs such that all hyperedges contain all colors in (as good as possible) equal quantity. Unimodular hypergraphs are good in this sense: They (and all their induced subhypergraphs as well) can be two-colored in a way that in each hyperedge the number of vertices in one color deviates from that in the other color by at most one. Note that this means that even cardinality hyperedges are perfectly balanced, whereas odd ones have a deviation of exactly one. This observation raises the question whether one can spare these deviations of one by leaving some vertices uncolored. In this work, we give a complete characterization of when this is possible.

Keywords: Discrepancy of hypergraphs, unimodular hypergraph, totally unimodular matrix.

1 Introduction and Results

A *hypergraph* $\mathcal{H} = (V, \mathcal{E})$ consists of a (here) finite set V of *vertices* and a set $\mathcal{E} \subseteq 2^V$ of (*hyper-*)*edges*. A hypergraph is called *unimodular* if its incidence matrix is *totally unimodular*, that is, it has the property that each square submatrix has determinant -1 , 0 or 1 . An alternative characterization is given by the theorem of Ghouila-Houri [6].

Theorem 1.1 *A hypergraph $\mathcal{H} = (V, \mathcal{E})$ is unimodular if and only if for each $V_0 \subseteq V$ there is a two-coloring $\chi : V_0 \rightarrow \{1, 2\}$ such that for all $E \in \mathcal{E}$,*

$$||E \cap \chi^{-1}(1)| - |E \cap \chi^{-1}(2)|| \leq 1.$$

Hence unimodular hypergraphs (and all their induced subhypergraphs) can be 2-colored as balanced as possible: There is a 2-coloring of the vertex set such that in each hyperedge the number of vertices in one color deviates from that in the other by at most one. In the language of discrepancy theory (see e.g. the survey by Beck and Sós [3]), this means that unimodular hypergraphs are exactly the ones that have hereditary discrepancy (at most) one.

Note that this deviation of one occurs if and only if there is an odd cardinality hyperedge (where such an imbalance naturally cannot be avoided). An interesting question therefore is whether we can avoid such an imbalance by not coloring all the vertices. We call a mapping $\chi : V \rightarrow \{0, 1, 2\}$ *partial 2-coloring*. The vertices in $\chi^{-1}(0)$ are called *uncolored*. We say that χ is *perfectly balanced*, if for each hyperedge $E \in \mathcal{E}$, we have $|\chi^{-1}(1) \cap E| = |\chi^{-1}(2) \cap E|$.

Clearly, mapping each vertex to 0 defines a perfectly balanced partial coloring, so usually we are looking for partial colorings that do leave not all, or better, only leave a certain fraction of the vertices uncolored. We shall call a mapping not constant zero *non-trivial* in the following. Repeatedly coloring the uncolored vertices with a non-trivial partial coloring is the heart of Beck's partial coloring method [2], which proved many famous results.

Since, by the theorem of Ghoula-Houri, unimodular hypergraphs admit very balanced full colorings, one might expect that they also have non-trivial perfectly balanced partial colorings. Surprisingly, this is not the case, as hypergraphs containing each vertex as singleton hyperedge demonstrate. A less trivial counter-example is the hypergraph of all intervals of length 3 and 5 in $[n] := \{1, \dots, n\}$ for $n \in \mathbb{N}_{\geq 6}$.

On the other hand, for $k \geq 2$ all k -uniform (all hyperedges contain exactly k vertices) unimodular hypergraphs do have this property. Proof sketch: Use the theorem of Hoffman and Kruskal [7] (or the later mentioned one of de Werra [4]) to obtain a non-empty subset of vertices that intersects each hyperedge in an even number of vertices. Now use the theorem of Ghoula-Houri to color these vertices perfectly balanced.

In this paper, we characterize the unimodular hypergraphs that have a perfectly balanced non-trivial partial coloring. For convenience, let us abbreviate $Z_p := \{0, \frac{1}{p}, \frac{2}{p}, \dots, \frac{p-1}{p}\}$ and $\bar{Z}_p = Z_p \cup \{1\}$ for all positive integers p . Then our main result is as follows.

Theorem 1.2 \mathcal{H} has a perfectly balanced non-trivial partial coloring if and only if there are $p \in \mathbb{N}_{\geq 2}$ and a non-trivial mapping $w : V \rightarrow \mathbb{Z}_p$ such that for all $E \in \mathcal{E}$, $w(E) := \sum_{v \in E} w(v)$ is an integer.

We may choose the partial coloring in a way that a least half of the vertices in $\{v \in V \mid w(v) \neq 0\}$ become truly colored.

From the theorem, we immediately see that for $k \geq 2$ all k -uniform hypergraphs have perfectly balanced non-trivial partial colorings: Clearly, $p = k$ and $w : V \rightarrow \mathbb{Z}_k; v \mapsto \frac{1}{k}$ satisfy the assumptions of the theorem.

Besides a natural interest of the problem considered, the theorem above also is the heart of a recent result [5] of the author on how to generate randomized roundings with cardinality constraints.

2 Proofs

To prove the theorem, we need a couple of intermediate results. The first one extends an old result of de Werra [4], namely that unimodular hypergraphs have p -color discrepancy less than one (that is, there is a p -coloring $\chi : V \rightarrow [p]$ such that in each hyperedge E the number of vertices in each color deviates from the fair value $\frac{1}{p}|E|$ by less than one).

Lemma 2.1 Let $p \in \mathbb{N}$. Let B be a totally unimodular $m \times n$ matrix and $x \in \overline{\mathbb{Z}}_p^n$ such that $Bx \in \mathbb{Z}^m$. Then there are $x^{(1)}, \dots, x^{(p)} \in \{0, \frac{1}{p}\}^n$ such that

- (i) $x = \sum_{k=1}^p x^{(k)}$;
- (ii) $\forall k \in [p] : Bx^{(k)} = \frac{1}{p}Bx$;
- (iii) $\forall i \in [n] : x_i = 0 \Rightarrow \forall k \in [p] : x_i^{(k)} = 0$.

De Werra's result transferred to the above setting would be the special case that $x = (\frac{1}{p}, \dots, \frac{1}{p})$.

Proof. [Lemma 2.1] We use induction on p . For $p = 1$, there is little to show. Let $p \geq 2$. Since B is totally unimodular and Bx is integral, by the theorem of Hoffman and Kruskal [7] (integrality of polyhedra defined via totally unimodular matrixes) there is a $y \in \{0, 1\}^n$ such that $B y = B x$ and $\|y - x\|_\infty < 1$. In particular, $y_i = 0$ if $x_i = 0$. Let $x' = x - \frac{1}{p}y$. Note that $x' \in \mathbb{Z}_p$ and $\frac{p}{p-1}Bx' = Bx \in \mathbb{Z}^{m_B}$. Hence we may apply induction to $\tilde{x} := \frac{p}{p-1}x'$, which is in $\overline{\mathbb{Z}}_{p-1}$, and obtain $\tilde{x}^{(1)}, \dots, \tilde{x}^{(p-1)} \in \{0, \frac{1}{p-1}\}^n$ satisfying (i) to (iii) with p replaced by $p - 1$. For $k \in [p - 1]$ let $x^{(k)} := \frac{p-1}{p}\tilde{x}^{(k)}$. Let $x^{(p)} = \frac{1}{p}y$. Then $x^{(1)}, \dots, x^{(p)}$ are as desired. \square

Without proof, we state the following elementary fact.

Lemma 2.2 *Let $p \in \mathbb{N}_{\geq 2}$ and $\emptyset \neq J \subset [p]$. Let I be an even cardinality subset of $[p]$ drawn uniformly at random. Then $\Pr(|I \cap J| \text{ odd}) = \frac{1}{2}$.*

The final ingredient for the proof of Theorem 1.2 is the following.

Lemma 2.3 *Let B be a totally unimodular $m \times n$ matrix and $x \in Z_p^n$ such that $Bx \in \mathbb{Z}^m$. Then there is an $x' \in Z_2^n$ such that*

- (i) for all $i \in [n]$, $x'_i = \frac{1}{2} \Rightarrow x_i \neq 0$;
- (ii) $Bx' \in \mathbb{Z}^m$;
- (iii) $|\{i \in [n] \mid x'_i = \frac{1}{2}\}| \geq \frac{1}{2}|\{i \in [n] \mid x_i \neq 0\}|$.

Proof. To ease the presentation, we may ignore components of x that are 0 (and the corresponding columns of B) and later set x' to be zero in such components. By Lemma 2.1, we find $x^{(1)}, \dots, x^{(p)} \in \{0, \frac{1}{p}\}^n$ such that $x = \sum_{k=1}^p x^{(k)}$ and $Bx^{(k)} = \frac{1}{p}Bx$ for all $k \in [p]$.

Now let I be an even cardinality subset drawn uniformly at random from $[p]$. Let us use the notation $\{r\} := r - \lfloor r \rfloor$ for the fractional part of a number $r \in \mathbb{R}$. Define x' via $x'_i = \{\sum_{k \in I} \frac{p}{2}x_i^{(k)}\}$ for all $i \in [n]$. In other words, x'_i is $\frac{1}{2}$ if and only if an odd number of the $(x_i^{(k)})_{k \in I}$ are $\frac{1}{p}$.

Let $i \in [m]$. From $\{(Bx')_i\} = \{\sum_{k \in I} \frac{p}{2}(Bx^{(k)})_i\} = \{|I| \frac{p}{2} (Bx)_i\} = 0$, we see that $(Bx')_i$ is integral. Finally, let $i \in [n]$ and recall that we assumed $x_i \notin \{0, 1\}$. Then by construction and Lemma 2.2, with probability a half an odd number of the $(x_i^{(k)})_{k \in I}$ are non-zero and thus $x'_i = \frac{1}{2}$. Hence the expected number of components of x' that are $\frac{1}{2}$, is $\frac{1}{2}|\{i \in [n] \mid x_i \neq 0\}|$, and there exists an x as claimed by applying the elementary probabilistic method (see, e.g., Alon and Spencer [1]). \square

We are now ready to prove the theorem.

Proof. Let $\mathcal{H} = (V, \mathcal{E})$ be unimodular, $p \in \mathbb{N}_{\geq 2}$ and $x : V \rightarrow Z_p$ be such that for all $E \in \mathcal{E}$, $x(E) := \sum_{v \in E} x(v)$ is an integer. Let $B \in \{0, 1\}^{\mathcal{E} \times V}$ be the incidence matrix of \mathcal{H} . Then B is totally unimodular and $Bx \in \mathbb{Z}^{\mathcal{E}}$. Let x' be as in Lemma 2.3. Let $V_0 := \{v \in V \mid x(v) = \frac{1}{2}\}$. Since $Bx' \in \mathbb{Z}^{\mathcal{E}}$, $V_0 \cap E$ has even cardinality for each hyperedge $E \in \mathcal{E}$. Since the hypergraph induced by V_0 is unimodular again, by the theorem of Ghouila-Houri, there is a 2-coloring of V_0 such that in each hyperedge, there is the same number of vertices in one color as in the other. Viewing this coloring as a partial coloring of V , we obtain the desired partial coloring.

The converse statement is trivial. Assume that there is a perfectly balanced non-trivial partial 2-coloring of \mathcal{H} . Let $p \in \mathbb{N}_{\geq 2}$. Define $w(v)$ by $w(v) := \frac{1}{p}$, if v was colored 1, $w(v) := \frac{p-1}{p}$, if v was colored 2 and $w(v) := 0$ if v was uncolored. As we startet with a perfectly balanced coloring, we $w(E)$ is integral for all $E \in \mathcal{E}$. \square

References

- [1] Alon, N. and J. H. Spencer, “The Probabilistic Method,” Wiley, 2000, 2nd edition.
- [2] Beck, J., *Roth’s estimate of the discrepancy of integer sequences is nearly sharp*, *Combinatorica* **1** (1981), pp. 319–325.
- [3] Beck, J. and V. T. Sós, *Discrepancy theory*, in: R. Graham, M. Grötschel and L. Lovász, editors, *Handbook of Combinatorics*, Elsevier, 1995 pp. 1405–1446.
- [4] de Werra, D., *Equitable colorations of graphs*, *Rev. Française Informat. Recherche Opérationnelle* **5** (1971), pp. 3–8.
- [5] Doerr, B., *Randomly rounding rationals with cardinality constraints and derandomizations*, in: W. Thomas and P. Weil, editors, *STACS 2007*, Lecture Notes in Computer Science **4393** (2007), pp. 441–452.
- [6] Ghouila-Houri, A., *Caractérisation des matrices totalement unimodulaires*, *C. R. Acad. Sci. Paris* **254** (1962), pp. 1192–1194.
- [7] Hoffman, A. J. and J. B. Kruskal, *Integral boundary points of convex polyhedra*, in: H. W. Kuhn and A. W. Tucker, editors, *Linear Inequalities and Related Systems*, Princeton University Press, Princeton, NJ, 1956 pp. 223–246.