# Global Roundings of Sequences

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**Abstract**

For a given sequence $a = (a_1, \dots, a_n)$ of numbers, a global rounding is an integer sequence $b = (b_1, \dots, b_n)$ such that the rounding error $|\sum_{i \in I}(a_i - b_i)|$ is less than one in all intervals $I \subseteq \{1, \dots, n\}$. We give a simple characterization of the set of global roundings of $a$. This allows to compute optimal roundings in time $O(n \log n)$ and generate a global rounding uniformly at random in linear time under a non-degeneracy assumption and in time $O(n \log n)$ in the general case.
**Key words:** Combinatorial problems, rounding, integral approximation, discrepancy.

## 1 Introduction and Results

In connection with an application in image processing, Sadakane, Takki-Chehibi and Tokuyama [2] study the problem to round a sequence $a = (a_1, \dots, a_n)$ of numbers in such a way that the errors in all intervals are less than one. By rounding we mean finding numbers $b_i \in \mathbb{Z}$ such that $|a_i - b_i| < 1$. Clearly, we may ignore the integral parts and hence assume that $a_i \in [0, 1)$ and $b_i \in \{0, 1\}$ for all $i \in [n] := \{1, \dots, n\}$. We

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put $\operatorname{disc}(a,b) = \max_I |\sum_{i\in I}(a_i - b_i)|$, where the maximum is taken over all intervals $I \subseteq [n]$. The sequence $b$ is called *global rounding*, if $\operatorname{disc}(a,b) < 1$, and *optimal rounding*, if $\operatorname{disc}(a,b)$ is minimal. Denote by $\operatorname{Rd}(a)$ the set of all global roundings of $a$. It is not difficult to see that any sequence has global roundings, i.e, that $\operatorname{Rd}(a) \neq \emptyset$.

Sadakane, Takki-Chehibi and Tokuyama [2] show that any sequence has at most $n+1$ different global roundings. It has exactly $n+1$ different global roundings, if it is *non-degenerate*, that is, if $\sum_{i\in I} a_i$ is non-integral for all intervals $\emptyset \neq I \subseteq [n]$. They also provide an algorithm computing all global roundings respectively an optimal rounding in time $O(n^2)$. More precisely, their algorithm builds up an $O(n^2)$–space data structure in time $O(n^2)$ which allows to generate any global rounding in linear time. The ability to access several global roundings or a random global rounding is important in applications, see also Sadakane, Takki-Chehibi and Tokuyama [3, 4].

In this note, we give an easy characterization of the set $\operatorname{Rd}(a)$ of all global roundings of $a$. This makes the costly datastructure used in [2] obsolete. In consequence, we may

- compute an optimal rounding in time $O(n \log n)$;
- compute $k$ different global roundings in time $O(kn)$;
- compute a global rounding uniformly at random in linear time in the non-degenerate case and in time $O(n \log n)$ in the general case; more precisely, in the general case one can either do an $O(n \log n)$ preprocessing and then access global roundings uniformly at random in time $O(n)$, or one skips the preprocessing (which is merely sorting out duplicates) and obtains random global roundings non-uniformly, but each with probability at least $\frac{1}{n+1}$.
- We also obtain the fact that a sequence has less than $n+1$ global roundings in the degenerate case.

All algorithms are simple and require $O(n)$ space. Our characterization also yields shorter proofs for other results in this area like the fact that an optimal global rounding has discrepancy at most $\frac{n}{n+1}$ and a characterization of this worst-case.

The elementary, but crucial observation is that any global rounding $b$ of $a$ satisfies

$$\gamma - 1 < \sum_{i \in [k]} (a_i - b_i) \leq \gamma$$

for some $\gamma \in [0, 1)$ and all $k \in [n]$. On the other hand, for any such $\gamma$ there is exactly one global rounding of $a$ satisfying the above condition. It turns out that only those $\gamma$ have to be regarded that are the fractional parts of the partial sums of $a$, that is, the numbers $\gamma_k = \{\sum_{i \in [k]} a_i\}$ for $k \in [n]_0 := \{0, \ldots, n\}$ (where we write $\{x\} := x - \lfloor x \rfloor$ to denote the fractional part of a number $x$). The following proofs make these ideas precise.

# 2 Proofs

Throughout this note let $a = (a_1, \ldots, a_n)$ denote a finite sequences of numbers in $[0, 1)$.

**Lemma 1.** *For each $\gamma \in [0, 1)$ there is exactly one sequence $b := \mathrm{rd}(a, \gamma)$ in $\{0, 1\}^n$ such that*

$$\gamma - 1 < \sum_{i \in [k]} (a_i - b_i) \leq \gamma$$

*holds for all $k \in [n]$.*

*Proof.* Define the $b_i$ recursively. Assume that for some $k \in [n]$ the $b_i$, $i < k$, are already defined and satisfy $\gamma - 1 < \sum_{i \in [k']} (a_i - b_i) \leq \gamma$ for all $k' < k$. If $\sum_{i \in [k-1]} (a_i - b_i) + a_k \leq \gamma$, put $b_k = 0$, if $\sum_{i \in [k-1]} (a_i - b_i) + a_k > \gamma$, put $b_k = 1$. Then $\gamma - 1 < \sum_{i \in [k]} (a_i - b_i) \leq \gamma$ holds in both cases.

Assume that there are two different sequences $b, b' \in \{0, 1\}^n$ as above. Let $k \in [n]$ be minimal such that $b_k \neq b'_k$. Then $|\sum_{i \in [k]} b_i - \sum_{i \in [k]} b'_i| = 1$ and thus $|\sum_{i \in [k]} (a_i - b_i) - \sum_{i \in [k]} (a_i - b'_i)| = 1$, contradicting our assumption that $\sum_{i \in [k]} (a_i - b_i)$ and $\sum_{i \in [k]} (a_i - b'_i)$ are in $(\gamma - 1, \gamma]$. □

The rounding procedure above can be interpreted as a one-dimensional error diffusion algorithm [1] with treshold $\gamma$. Let $\gamma_k = \gamma_k(a) = \{\sum_{i\in[k]} a_i\}$ for all $k \in [n]_0$.

**Lemma 2.** *Any global rounding $b$ of $a$ equals some* $\mathrm{rd}(a, \gamma_k)$*, $k \in [n]_0$.*

*Proof.* Let $b$ be a global rounding of $a$ and $k^* \in [n]_0$ such that $\gamma := \sum_{i\in[k^*]}(a_i - b_i) = \max\{\sum_{i\in[k]}(a_i - b_i) \,|\, k \in [n]_0\}$. Since $b$ is a global rounding, $\gamma \in [0, 1)$ and $\gamma = \gamma_{k^*}$. We show that $\gamma - 1 < \sum_{i\in[k]}(a_i - b_i) \le \gamma$ for all $k \in [n]$, which implies $b = \mathrm{rd}(a, \gamma_{k^*})$ by Lemma 1. Assume that $\sum_{i\in[k]}(a_i - b_i) \notin (\gamma - 1, \gamma]$ for some $k \in [n]$. Then $\sum_{i\in[k]}(a_i - b_i) \le \gamma - 1$ by definition of $\gamma$. If $k < k^*$, then

$$\sum_{i=k+1}^{k^*} (a_i - b_i) = \sum_{i=1}^{k^*}(a_i - b_i) - \sum_{i=1}^{k}(a_i - b_i) \ge \gamma - (\gamma - 1) = 1,$$

if $k > k^*$, then

$$\sum_{i=k^*+1}^{k} (a_i - b_i) = \sum_{i=1}^{k}(a_i - b_i) - \sum_{i=1}^{k^*}(a_i - b_i) \le (\gamma - 1) - \gamma = -1.$$

In both cases we have the contradiction that $b$ is no global rounding of $a$. □

Put $\Gamma(a) = \{\gamma_i \,|\, i \in [n]_0\}$. Let $0 = \gamma^{(1)} < \ldots < \gamma^{(\ell)}$ be an increasing enumeration of $\Gamma(a)$. Write $\gamma^{(\ell+1)} = 1$. The following lemma determines the discrepancy of the roundings $\mathrm{rd}(a, \gamma^{(j)})$, $j \in [\ell]$. In particular, it shows that all are global roundings.

**Lemma 3.** *For all $j \in [\ell]$,* $\mathrm{disc}(a, \mathrm{rd}(a, \gamma^{(j)})) = 1 - (\gamma^{(j+1)} - \gamma^{(j)})$*.*

*Proof.* Let $b = \mathrm{rd}(a, \gamma^{(j)})$. Then

$$\sum_{i\in[k]}(a_i - b_i) \in (\gamma^{(j)} - 1, \gamma^{(j)}] \cap \{\gamma^{(i)}, \gamma^{(i)} - 1 \,|\, i \in [n]_0\} \subseteq [\gamma^{(j+1)} - 1, \gamma^{(j)}]$$

for all $k \in [n]$. Hence for all $1 \le k_1 \le k_2 \le n$ we compute

$$\sum_{i=k_1}^{k_2}(a_i - b_i) = \sum_{i\in[k_2]}(a_i - b_i) - \sum_{i\in[k_1-1]}(a_i - b_i) \begin{cases} \le \gamma^{(j)} - (\gamma^{(j+1)} - 1) \\ \ge (\gamma^{(j+1)} - 1) - \gamma^{(j)}. \end{cases}$$

For $k_1, k_2$ such that $\{\gamma_{k_1-1}, \gamma_{k_2}\} \equiv \{\gamma^{(j)}, \gamma^{(j+1)}\} \pmod 1$, one of the inequalities becomes an equality. Hence $\mathrm{disc}(a, \mathrm{rd}(a, \gamma^{(j)})) = 1 - (\gamma^{(j+1)} - \gamma^{(j)})$. □

**Theorem 4.** *The mapping $\Gamma(a) \to \mathrm{Rd}(a); \gamma \mapsto \mathrm{rd}(a, \gamma)$ is a bijection. In particular,*

- *a random global rounding of $a$ can be computed in linear time assuming non-degeneracy, and in time $O(n \log n)$ in the general case,*
- *for $k \leq |\Gamma(a)|$, $k$ distinct global roundings of $a$ can be computed in time $O(kn)$,*
- *an optimal rounding of $a$ can be computed in time $O(n \log n)$,*
- *$a$ has $n + 1$ different global roundings if and only if $\sum_{i \in I} a_i$ is non-integral for all non-empty intervals $I \subseteq [n]$.*

*Proof.* The main statement just combines Lemma 1, 2 and 2. To compute a global rounding uniformly at random in the non-degenerate case, simple choose a $k \in [n]_0$ uniformly at random and compute $\gamma_k(a)$ and $\mathrm{rd}(a, \gamma_k(a))$ each in linear time. This does the job, since all $\mathrm{rd}(a, \gamma_k(a))$ are distinct (see below). In the general case, compute $\Gamma(a)$ in time $O(n \log n)$ by sorting the sequence of $\gamma_k(a)$ and removing duplicates, pick a $\gamma \in \Gamma(a)$ uniformly at random and compute $\mathrm{rd}(a, \gamma)$ in linear time.

Let $k \leq |\Gamma(a)|$. To compute $k$ distinct global roundings, compute the sequence of $\gamma_j(a)$, $j \in [n]_0$, and mark them all undone. For $i = 1, \ldots, k$ find an undone $\gamma_j(a)$, compute $\mathrm{rd}(a, \gamma_j(a))$ and mark all $\gamma_{j'}(a)$ such that $\gamma_{j'}(a) = \gamma_j(a)$ as done.

To compute an optimal global rounding, compute and sort $\Gamma(a)$ in time $O(n \log n)$ to obtain an increasing enumeration $\gamma^{(1)} < \ldots < \gamma^{(\ell)}$ of $\Gamma(a)$, put $\gamma^{(\ell+1)} = 1$, find in linear time a $j \in [\ell]$ such that $\gamma^{(j+1)} - \gamma^{(j)}$ is maximal and compute $\mathrm{rd}(a, \gamma^{(j)})$ in linear time. This has minimal discrepancy according to Lemma 3.

If $\sum_{i \in I} a_i$ is integral for some interval $\emptyset \neq I = \{k_1, \ldots, k_2\} \subseteq [n]$, then $\gamma_{k_1-1}(a) = \gamma_{k_2}(a)$. Hence the number $|\Gamma(a)|$ of global roundings is at most $n$. Conversely, if $\gamma_{k_1} = \gamma_{k_2}$ for some $0 \leq k_1 < k_2 \leq n$, then $\sum_{i=k_1+1}^{k_2} a_i$ is integral. □

Note that if we just want to compute a random global rounding without caring too much about the probabilities, we may simple follow the approach for the non-degenerate case also in the general one. This yields each global rounding with probability at least $\frac{1}{n+1}$. In applications like the one cited in the introduction, this is probably the preferred alternative.

The theorem in conjunction with Lemma 3 also describes the extremal situations: An optimal global rounding has discrepancy at most $\frac{n}{n+1}$. We have equality if and only if $\Gamma(a) = \{\frac{k}{n+1} \mid k \in [n]_0\}$, which is equivalent to saying that $a$ is non-degenerate and all $a_i$ are multiples of $\frac{1}{n+1}$.

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# References

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