Matrix Rounding with Low Error in Small Submatrices

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Abstract

We show that any real valued matrix A can be rounded to an integer one B such that the error in all 2×2 (geometric) submatrices is less than 1.5, that is, we have $|a_{ij} - b_{ij}| < 1$ and $|\sum_{k=i}^{i+1} \sum_{\ell=j}^{j+1} (a_{k\ell} - b_{k\ell})| < 1.5$ for all i, j. More precisely, an error of less than $1.5 - 3^{-2mn} + 3^{-d+1}$ can be achieved in time $O(mnd)$.

1 Introduction and Results

Let m, n be non-negative integers. We write $[n] = \{i \in \mathbb{N} \mid i \le n\}$. Let A and B be real-valued $m \times n$ matrices. We call B a *rounding* of A if $b_{ij} \in \{\lfloor a_{ij} \rfloor, \lceil a_{ij} \rceil\}$ for all $i \in [m]$, $j \in [n]$. For $R \subseteq [m] \times [n]$ let $d(A, B, R) = \sum_{(i,j) \in R} (a_{ij} - b_{ij})$. A set $\{i, i+1\} \times \{j, j+1\}$ for some $i \in [m-1]$, $j \in [n-1]$ shall be called a 2×2 *box*. Denote by $\mathcal{R}$ the set of all 2×2 boxes and put

$$d(A, B) = \max_{R \in \mathcal{R}} |d(A, B, R)|.$$

In the context of an image processing application, Asano, Matsui and Tokuyama [AMT00] proved that for any A there is a rounding B such that $d(A, B) \le 1.75$. This was improved to a bound of 5/3 by Asano and Tokuyama [AT01]. Both proofs are highly complicated.

The difficulty of this problem seems to lie in the fact that the errors depend on very few (four) variables. Thus traditional approaches like randomized rounding and even the rounding algorithm of Karp et al. [KLR+87] for sparse problems are too coarse. The object of this paper is to give a short proof of an upper bound of 1.5, and a slightly longer one showing that this bound is never attained.

Theorem 1.1. *For any $A \in \mathbb{R}^{m \times n}$ there is a rounding $B \in \mathbb{Z}^{m \times n}$ such that $d(A, B) < 1.5$.*

The proof yields an algorithm that computes a rounding with error $d(A, B) < 1.5 - 3^{-2mn} + 3^{-d+1}$ in time $O(mnd)$. There is a lower bound of 1 stemming from an odd cycle argument ([AMT00]).

For reasons of space, most of the proofs are omitted in this abstract.

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2 Proof of the Bound $d(A, B) \le 1.5$

Lemma 2.1. *Let $m, n \in \mathbb{N}$. Let $\mathcal{E}$ be a set of subsets of $[m] \times [n]$ such that*

(i) $|E| = 3$ for all $E \in \mathcal{E}$,

(ii) for each $E \in \mathcal{E}$ there is an $i \in [m]$ (which shall be denoted by $i(E)$) such that $E \subseteq \{i\} \times [n]$,

(iii) for each two $E_1, E_2 \in \mathcal{E}$ such that $i := i(E_1) = i(E_2)$, we have $e_1 < e_2$ for all $(i, e_1) \in E_1, (i, e_2) \in E_2$, or $e_1 > e_2$ for all $(i, e_1) \in E_1, (i, e_2) \in E_2$.

Then there is a $T \subseteq [m] \times [n]$ such that $|T \cap E| = 1$ for all $E \in \mathcal{E}$ and such that for all $(s_1, s_2), (t_1, t_2) \in T$ we have $s_2 \ne t_2$ whenever $|s_1 - t_1| = 1$.

Proof. We use induction on $|\mathcal{E}|$. For $|\mathcal{E}| = 0$, there is nothing to show. Hence let us assume that $|\mathcal{E}| \ge 1$ and that the assertion of the lemma is true for smaller set systems. Let $E^* \in \mathcal{E}$ have least extension to the right, i.e., $\max\{j \mid (i(E^*), j) \in E^*\} \le \max\{j \mid (i(E), j) \in E\}$ for all $E \in \mathcal{E}$. Let $i^* = i(E^*)$.

Let T be as assured by the lemma with respect to the set system $\mathcal{E} \setminus \{E^*\}$. By construction, there is at most one set $E \in \mathcal{E}$ in each of the rows $i^* - 1$ and $i^* + 1$ such that $\{j \mid (i(E), j) \in E\}$ intersects $\{j \mid (i^*, j) \in E^*\}$ non-trivially. Since T has exactly one vertex in these (at most) two sets and $|E^*| = 3$, there is a $j \in [n]$ such that $(i^* - 1, j) \notin T$, $(i^* + 1, j) \notin T$ and $(i^*, j) \in E^*$. Thus $T \cup \{(i^*, j)\}$ satisfies the claim.

Lemma 2.2. *Let $n \in \mathbb{N}$. Let $a_1, a_n \in \{0, 1\}$ and $a_2, \ldots, a_{n-1} \in \{\frac{1}{3}, \frac{2}{3}\}$. Then*

(i) there are $b_1, \ldots, b_n \in \{0, 1\}$ such that $b_1 = a_1$, $b_n = a_n$ and $|b_i - a_i + b_{i+1} - a_{i+1}| \le \frac{1}{3}$ for all $i \in [n-1]$,

(ii) or there are three distinct numbers $x^{(1)}, x^{(2)}, x^{(3)} \in [n-1]$ and for each $k \in [3]$ there are $b_1^{(k)}, \ldots, b_n^{(k)} \in \{0, 1\}$ such that $b_1^{(k)} = a_1$, $b_n^{(k)} = a_n$ and for all $i \in [n-1] \setminus \{x^{(k)}\}$ we have $|b_i^{(k)} - a_i + b_{i+1}^{(k)} - a_{i+1}| \le \frac{1}{3}$, whereas $|b_{x^{(k)}}^{(k)} - a_{x^{(k)}} + b_{x^{(k)}+1}^{(k)} - a_{x^{(k)}+1}| = \frac{2}{3}$.

The heart of the main proof is the following fact concerning roundings of matrices with entries in $\{0, \frac{1}{3}, \frac{2}{3}, 1\}$.

LEMMA 2.3. *For any* $A \in \{0, \frac{1}{3}, \frac{2}{3}, 1\}^{m\times n}$ *there is a rounding* $B \in \{0,1\}^{m\times n}$ *such that* $d(A,B) \le 1$.

Proof. [Sketch] Let $A \in \{0, \frac{1}{3}, \frac{2}{3}, 1\}^{m\times n}$. Each row consists of strings of $\frac{1}{3}$s and $\frac{2}{3}$s, separated by zeroes and ones. All zeroes and ones shall be unchanged in B. For the intermediate strings of $\frac{1}{3}$ and $\frac{2}{3}$, Lemma 2.2 yields a rounding with error at most $\frac{1}{3}$ in each two consecutive entries except for possibly one location, where an error of $\frac{2}{3}$ cannot be avoided. Should this occur, however, we may choose this location out of at least three different possibilities. Invoking Lemma 2.1, we shall round the rows in such a way that this large error never occurs at the same location of two adjacent rows. Thus each 2×2 box contains at most one error of $\frac{2}{3}$ in one of its (two) rows, leading to a total error of at most 1 in the box.

The proof of the weaker bound now follows easily from regarding the ternary expansion of A and rounding "digit by digit" in a similar manner as in Beck and Spencer [BS84] (there done with binary expansions). We say that a matrix $A \in [0,1]^{m\times n}$ has ternary length ℓ for some $\ell \in \mathbb{N}$, if there are $A_1, \dots, A_\ell \in \{0, \frac{1}{3}, \frac{2}{3}, 1\}^{m\times n}$ such that $A = \sum_{i=1}^{\ell} 3^{-i+1} A_i$ and $A_\ell \neq 0$. It has ternary length zero, if it is binary.

Proof. Ignoring the integral parts of A, we may assume that $A \in [0,1]^{m\times n}$ and thus have to show the existence of a binary B such that $d(A,B) \le 1.5$ and $b_{ij} = a_{ij}$ whenever a_{ij} is integral. Assume that A has a finite ternary expansion of length ℓ. Let $A_\ell \in \{0, \frac{1}{3}, \frac{2}{3}, 1\}^{m\times n}$ such that $A - 3^{-\ell+1} A_\ell$ has ternary length less than ℓ. Let B_ℓ be a rounding of A_ℓ as in Lemma 2.3. Then $\tilde{A} = A - 3^{-\ell+1}(A_\ell - B_\ell)$ has ternary length less than ℓ and $d(A, \tilde{A}) = 3^{-\ell+1} d(A_\ell, B_\ell) \le 3^{-\ell+1}$. Hence an easy induction yields a binary B such that $d(A,B) \le \sum_{i=0}^{\ell-1} 3^{-i} \le 1.5$. Note that by Lemma 2.3 we also have $b_{ij} = a_{ij}$ whenever a_{ij} is integral. Density of the set of all matrices having finite ternary expansion and the fact that $A' \mapsto \min\{d(A', B) \,|\, B \text{ is a rounding of } A\}$ is continuous in A, yield the claim for arbitrary A.

3 Improvement to Strictly Less Than 1.5

In this section, we show how to extend the proof of the previous section to obtain a bound of strictly less than 1.5. The key idea is to make sure that for each box R not all $\{0, \frac{1}{3}, \frac{2}{3}, 1\}$ intermediate roundings have the same error $d(A, B, R)$ of -1 or 1.

LEMMA 3.1. *In the situation of Lemma 2.1, for all* $x \in [m] \times [n]$, *there is a* $T \subseteq ([m] \times [n]) \setminus \{x\}$ *such that* $|T \cap E| = 1$ *for all* $E \in \mathcal{E}$ *and such that for all* $(s_1, s_2), (t_1, t_2) \in T$ *we have* $s_2 \neq t_2$ *whenever* $|s_1 - t_1| = 1$.

LEMMA 3.2. *Let* $A \in \{0, \frac{1}{3}, \frac{2}{3}, 1\}^{m\times n}$, $R \in \mathcal{R}$ *and* $\varepsilon \in \{-1, 1\}$. *Then there is a rounding* B *of* A *such that* $d(A,B) \le 1$ *and* $d(A,B,R) \neq \varepsilon$.

Now it is easy to prove the stronger version of the theorem. Note that we did not try to optimize the 3^{-M} term in the $1.5 - 3^{-M}$ bound.

Proof. Let $M = 2(m-1)(n-1)$. Let $(R_1, \varepsilon_1), \dots, (R_M, \varepsilon_M)$ be an enumeration of $\mathcal{R} \times \{-1, 1\}$. For sake of notational convenience, let $(R_i, \varepsilon_i) \in \mathcal{R} \times \{-1, 1\}$ be arbitrary for $i > M$. Let $A^{(\ell)} := A \in [0,1]^{m\times n}$ have finite ternary expansion of length ℓ. Inductively, we define a sequence $A^{(i)} \in [0,1]^{m\times n}$, $i = \ell - 1, \dots, 0$ such that $A^{(i)}$ has ternary length at most i, $d(A^{(i+1)}, A^{(i)}) \le 3^{-i}$ and $d(A^{(i+1)}, A^{(i)}, R_{i+1}) \neq \varepsilon_{i+1} 3^{-i}$. Assume that $A^{(i+1)}$ is already defined. Let A_{i+1} be a $\{0, \frac{1}{3}, \frac{2}{3}, 1\}$ matrix such that $A^{(i+1)} - 3^{-i} A_{i+1}$ has ternary length at most i. Apply Lemma 3.2 on A_{i+1}, R_{i+1}, ε_{i+1} to obtain a rounding B_{i+1}. Put $A^{(i)} = A^{(i+1)} - 3^{-i}(A_{i+1} - B_{i+1})$. Now $A^{(i)}$ has the desired properties. In particular, $A^{(0)} \in \{0,1\}^{m\times n}$. It remains to compute that $d(A^{(\ell)}, A^{(0)}) \le 1.5 - 3^{-M}$.

4 Algorithmics

Let $\tilde{A}$ be obtained from rounding A to a matrix of ternary length d, i.e., we have $|\tilde{a}_{ij} - a_{ij}| \le \frac{1}{2} 3^{-d}$ and $3^d \tilde{a}_{ij} \in \mathbb{Z}$ for all i, j. Now if B is a rounding of $\tilde{A}$ as above, then $d(A,B) \le d(A, \tilde{A}) + d(\tilde{A}, B) \le 2 \cdot 3^{-d} + 1.5 - 3^{-M}$. Computing B means solving d rounding problems as in Lemma 3.2. Since both the one-dimensional roundings in Lemma 2.2 and the transversal in Lemma 3.1 can be computed in time $O(mn)$, this yields an overall time complexity of $O(mnd)$.

References

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