Balanced Coloring: Equally Easy for all Numbers of Colors?

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Abstract. We investigate the problem to color the vertex set of a hypergraph $\mathcal{H} = (X, \mathcal{E})$ with a fixed number of colors in a balanced manner, i.e., in such a way that all hyperedges contain roughly the same number of vertices in each color (discrepancy problem). We show the following result:

Suppose that we are able to compute for each induced subhypergraph a coloring in c_1 colors having discrepancy at most D. Then there are colorings in arbitrary numbers c_2 of colors having discrepancy at most $\frac{11}{10}\,c_1^2 D$.
A c_2–coloring having discrepancy at most $\frac{11}{10}\,c_1^2 D + 3c_1^{-k}|X|$ can be computed from $(c_1 - 1)(c_2 - 1)k$ colorings in c_1 colors having discrepancy at most D with respect to a suitable subhypergraph of $\mathcal{H}$.
A central step in the proof is to show that a fairly general rounding problem (linear discrepancy problem in c_2 colors) can be solved by computing low-discrepancy c_1–colorings.

1 Introduction and Results

This paper deals with the problem of balanced hypergraph colorings (or equivalently, balanced partitions). A coloring in c colors is called balanced, if all hyperedges contain roughly the same number of vertices in each color. More precise, we define the discrepancy of a coloring to be the maximum deviation (taken over all hyperedges E and all colors d) of vertices in color d contained in E compared to the fair value $\frac{1}{c}|E|$.

Whereas the discrepancy problem was mostly studied in the context of 2 colors (see e. g. Beck and Sós [BS95], Matoušek [Mat99] or Chazelle [Cha00]), there has recently been work on the general problem (e. g. [DS99,BCC$^+$00]). In this paper, we are interested in the relation between the discrepancy problem in different numbers of colors. Since [Doe01a] gave a class of hypergraphs having very different discrepancies in different numbers of colors, one might be pessimistic. On the other hand, there are several classes of hypergraphs having similar discrepancies in all numbers of colors, cf. [DS01]. This paper tries to solve this dichotomy.

A first result of this type already appeared in the paper [DS99]. There a recursive algorithm was presented that computes c–colorings from low-discrepancy 2–colorings. In this paper we extend this result to arbitrary numbers of colors. Unfortunately, it is not possible to use similar methods. Roughly speaking, it is relatively easy to compute c_2–colorings from c_1–colorings for $c_2 > c_1$ by a recursive partitioning scheme. Imbalances inflicted in earlier rounds of the recursion are split up in the following ones in a balanced manner. Therefore the final discrepancy can be estimated by something similar to a geometric series.

For the case $c_2 < c_1$ things are different. Of course, the case that c_2 divides c_1 is relatively trivial, but the remaining situations need more effort. The problem becomes visible already if we try to compute 2–colorings from 3–colorings. A natural approach would be to find a low-discrepancy 3–coloring and then to recursively recolor the vertices in color 3 according to further 3-colorings having low discrepancy on these vertices. If we organize this in a suitable way, at most $O(1)$ vertices in color 3 are left after $O(\log n)$ iterations (assuming n to be the number of vertices), which may be colored arbitrarily.

The draw-back of this approach is that the imbalances of the 3–colorings might accumulate. Thus for the final 2–coloring we cannot obtain a better discrepancy guarantee than $O(\log |X|)$ times the maximum discrepancy of the 3–colorings (cf. Theorem 2 for a precise version of this statement).

Our objective in this paper though is to show that the size of the hypergraph does not matter: Suppose that one can color a hypergraph and all its subgraphs with a fixed number of colors such that the corresponding discrepancies are at most D. Then this hypergraph can be colored with any number of colors such that the discrepancy is at most a constant factor larger than D. More precisely, we show:

Theorem 1. *Let $\mathcal{H} = (X, \mathcal{E})$ be a hypergraph. Let $c_1, c_2 \in \mathbb{N}_{\geq 2}$ be arbitrary numbers of colors. Suppose that for each induced subhypergraph of $\mathcal{H}$ there is a c_1–coloring of the vertex set having discrepancy at most D. Then there exists a c_2–coloring of $\mathcal{H}$ having discrepancy at most $\frac{11}{10} c_1^2 D$.*

A c_2–coloring of $\mathcal{H}$ having discrepancy at most $\frac{11}{10} c_1^2 D + 3c_1^{-k}|X|$ can be computed from $(c_1 - 1)(c_2 - 1)k$ colorings in c_1 colors having discrepancy at most D with respect to a suitable induced subhypergraph of $\mathcal{H}$.

The key idea to prove Theorem 1 is to show an even more general result. Roughly speaking, in the setting of Theorem 1 it is also possible to round any floating coloring to an ordinary one with small discrepancy. This will be made precise in Section 3, where the necessary definitions introduced in Section 2 are available.

In a sense, Theorem 1 is best possible: Fix two numbers c_1, c_2 of colors. Since Theorem 1 works for arbitrary numbers of colors, we may apply it also with the roles of c_1 and c_2 interchanged. This shows that for all hypergraphs $\mathcal{H}$ the maximum discrepancy among the induced subhypergraphs is the same in c_1 and c_2 colors (apart from constant factors depending on c_1, c_2 only).

The reason why we have these strong results (compared to the negative example of [Doe01a]) is that we use the stronger assumption that all induced subhypergraphs admit a low-discrepancy coloring in c_1 colors. This excludes the class of hypergraphs exhibited in [Doe01a]. On the other hand, most results known in discrepancy theory are hereditary (refer to the maximum discrepancy among all subhypergraphs) since they relate the discrepancy to another hereditary property like maximum degree (Beck and Fiala [BF81], Srinivasan [Sri97]), VC-dimension and shatter functions (Matoušek, Welzl and Wernisch [MWW84], Matoušek [Mat95]) or total unimodularity (Ghouila-Houri [GH62], Doerr [Doe01b]).

Our main result admits several corollaries. We present two in Section 4, one concerning hypergraph having more vertices than hyperedges, the other one showing a connection between multi-color discrepancies and the problem of integral approximate solutions of linear equations.

2 Notation and Preliminaries

2.1 Multi-Color Discrepancies

Let $\mathcal{H} = (X, \mathcal{E})$ be a finite hypergraph, that is, X is a finite set and $\mathcal{E} \subseteq 2^X$. Throughout this section let $c \in \mathbb{N}_{\geq 2}$ denote the number of classes we want to partition the vertices of $\mathcal{H}$ into. It is natural to represent the partition by a coloring. The partition classes then are formed by the sets of equally colored vertices. A c–coloring of $\mathcal{H}$ is a mapping $\chi : X \to M$, where M is any set of cardinality c. For convenience, normally one has $M = [c] := \{1, \dots, c\}$. Sometimes a different set M will be advantageous.

The basic idea of measuring the deviation of a given partition from the ideal one motivates these definitions: The *discrepancy of a hyperedge $E \in \mathcal{E}$ in color $d \in M$ with respect to χ* is

$$\mathrm{disc}_{\chi,d}(E) := \left| |\chi^{-1}(d) \cap E| - \frac{|E|}{c} \right|,$$

the *discrepancy of $\mathcal{H}$ with respect to χ* is

$$\mathrm{disc}(\mathcal{H}, \chi) := \max_{d \in M, E \in \mathcal{E}} \mathrm{disc}_{\chi,d}(E)$$

and the *discrepancy of $\mathcal{H}$ in c colors* is

$$\mathrm{disc}(\mathcal{H}, c) := \min_{\chi : X \to [c]} \mathrm{disc}(\mathcal{H}, \chi).$$

For a subset $X_0 \subseteq X$ of vertices denote by $\mathcal{H}_{|X_0} = (X_0, \mathcal{E}_{|X_0})$ the hypergraph induced by X_0, i. e., $\mathcal{E}_{|X_0} := \{E \cap X_0 | E \in \mathcal{E}\}$. The *hereditary discrepancy* in c colors is defined by

$$\mathrm{herdisc}(\mathcal{H}, c) := \max_{X_0 \subseteq X} \mathrm{disc}(\mathcal{H}_{|X_0}, c).$$

As in 2 colors, the notion of multi-color discrepancy has a natural extension to matrices. Let $A \in \mathbb{R}^{m \times n}$ be any real matrix and $\chi : [n] \to M$, where M is again an arbitrary set of cardinality c. Then the *discrepancy of A with respect to χ* is defined by

$$\mathrm{disc}(A, \chi) := \max_{d \in M, i \in [m]} \left| \sum_{j \in \chi^{-1}(d)} a_{ij} - \frac{1}{c} \sum_{j \in [n]} a_{ij} \right|.$$

The *discrepancy of A in c colors* is

$$\operatorname{disc}(A,c) := \min_{\chi:[n]\to[c]} \operatorname{disc}(A,\chi).$$

Immediately we see that $\operatorname{disc}(A,c) = \operatorname{disc}(\mathcal{H},c)$ if A is the incidence matrix of $\mathcal{H}$. The extension to matrices is justified by two reasons. Firstly, it is an interesting optimization problem on its own right to partition the column vectors $a^{(j)}, j \in [n]$ of A into balanced classes, i. e., in a way that for all partition classes $\chi^{-1}(d)$ the sums $\sum_{j\in\chi^{-1}(d)} a^{(j)}$ are roughly equal. Secondly, even for some hypergraph problems the matrix notion is more convenient, e. g., to prove the Beck–Fiala theorem.

For the problems we are concerned with in this article it makes no difference whether we restrict ourselves to the special case of hypergraphs or the generalization to matrices. Hence from now on we will deal with the matrix case only. The notion of hereditary discrepancy translates to matrices in the obvious way: Write $A_{|J}$ to denote the submatrix of A containing the columns with index in J only. Then

$$\operatorname{herdisc}(A,c) := \max_{J\subseteq[n]} \operatorname{disc}(A_{|J},c).$$

If one allows a logarithmic dependence on the size of the hypergraph, an elementary proof shows some relation between the discrepancy problem in different numbers of colors:

Theorem 2. *Let $A \in \mathbb{R}^{m\times n}$ and $c_2 < c_1$. If c_2 divides c_1, then any c_1–coloring $\chi_1 : [n] \to [c_1]$ yields a c_2–coloring χ_2 for A such that*

$$\operatorname{disc}(A,\chi_2) \le \tfrac{c_1}{c_2} \operatorname{disc}(A,\chi_1).$$

Otherwise, a c_2–coloring χ such that

$$\begin{aligned}\operatorname{disc}(A,\chi_2) &\le \left\lfloor \tfrac{c_1}{c_2} \right\rfloor \log_{1/(1-\frac{c_2}{c_1}\lfloor\frac{c_1}{c_2}\rfloor)}(n) D \\ &\le \left\lfloor \tfrac{c_1}{c_2} \right\rfloor \log_2(n) D\end{aligned}$$

can be computed from $\log_2 n$ colorings in c_1 colors having discrepancy at most D with respect to suitable submatrices of A.

For small hypergraphs, this result is even superior to our main result. In general, of course, it leads into the wrong direction (suggesting that the size of the hypergraph might have an influence on how different hypergraphs behave in the discrepancy problem in different numbers of colors).

2.2 Linear Discrepancies

A rather general concept is the one of *linear discrepancy*. Here every vertex has an individual weight which describes in which ratio the vertex in average should contribute to each color class of each hyperedge it belongs to. A less obscure way of viewing this problem is to recognize it as rounding problem: For a given floating coloring assigning each vertex not a single color, but a weighted mixture of colors, we are looking for a 'pure' coloring (assigning each vertex a single color) such that each hyperedge in total receives every color roughly in the same amount by both colorings. This rounding aspect will be a central theme of the main proof.

At this point it will be convenient to choose a different set of colors. Denote by $E^c = \{e^{(1)}, \dots, e^{(c)}\}$ the standard basis of $\mathbb{R}^c$. Denote by $\overline{E}^c$ the convex hull of E^c, which is nothing more than the set of all $p \in [0,1]^c$ such that $\|p\|_1 = 1$. We call these vectors *c–color weights*. In the hypergraph case our objective hence is to 'round' a floating coloring $p : X \to \overline{E}^c$ to a coloring $q : X \to E^c$ in such a way that the imbalances $\left\|\sum_{x \in E} p(x) - \sum_{x \in E} q(x)\right\|_\infty$ are small for all $E \in \mathcal{E}$.

For matrices we define: A mapping $p : [n] \to \overline{E}^c$ is called a *floating coloring*. Denote by $\overline{\mathcal{C}^c}$ the set of all floating colorings and by $\mathcal{C}^c := \{p \,|\, p : [n] \to E^c\}$ the set of all pure colorings. For $p, q \in \overline{\mathcal{C}^c}$ put

$$d_A(p,q) := \max_{i \in [m]} \left\| \sum_{j=1}^{n} a_{ij}(p(j) - q(j)) \right\|_\infty .$$

It is clear that d_A is pseudo-metric on $\overline{\mathcal{C}^c}$, in particular it satisfies the triangle inequality. The *linear discrepancy of A with respect to $p \in \overline{\mathcal{C}^c}$* now is

$$\operatorname{lindisc}(A,p) := d_A(p, \mathcal{C}^c) = \min_{q \in \mathcal{C}^c} d_A(p,q).$$

Let $\overline{p} : [n] \to \overline{E}^c; j \mapsto \frac{1}{c}\mathbf{1}_c$. Then $d_A(\overline{p}, q) \le D$ just means that q is a c–coloring such that $\mathrm{disc}(A, q) \le D$. Thus the linear discrepancy problem is a direct generalization of the discrepancy problem.

2.3 Types

Let $c_1, c_2 \in \mathbb{N}_{\ge 2}$. A vector $t \in \{0, \ldots, c_1 - 1\}^{c_2}$ shall be called (c_1, c_2)–*type* in this paper if $\|t\|_1 = \sum_{i \in [c_2]} t_i = c_1$. Denote by T_{c_1,c_2} the set of all (c_1, c_2)–types. Put $n_{c_1,c_2} := |T_{c_1,c_2}|$ and $s_{c_1,c_2} := \sum_{t \in T_{c_1,c_2}} t$. The following three lemmata (proofs omitted) give some properties of these types.

Lemma 1. *The number of (c_1, c_2)–types is*

$$n_{c_1,c_2} = \binom{c_1 + c_2 - 1}{c_1} - c_2.$$

Lemma 2. *The sum of all (c_1, c_2)–types is $s_{c_1,c_2} = \frac{c_1}{c_2} n_{c_1,c_2} \mathbf{1}_{c_2}$.*

Lemma 3. *Let $v \in \{0, \ldots, c_1 - 1\}^{c_2}$ such that c_1 divides $\|v\|_1$. Then v is the sum of (c_1, c_2)–types each thereof occurring just once, i.e., there are $\varepsilon_t \in \{0, 1\}, t \in T_{c_1,c_2}$ such that $v = \sum_{t \in T_{c_1,c_2}} \varepsilon_t t$. These $\varepsilon_t, t \in T_{c_1,c_2}$, can be found efficiently by a Greedy-Algorithm.*

3 Linear and Hereditary Discrepancy in Arbitrary Numbers of Colors

In this section we show how linear discrepancies in c_2 colors can be bounded in terms of the c_1–color hereditary discrepancy. Recall from Section 2 that the linear discrepancy problem in particular solves the ordinary discrepancy problem. For a matrix A we write $\|A\|_\infty := \max_{i \in [m]} \sum_{j \in [n]} |a_{ij}|$ to denote the operator norm induced by the maximum norm.

Theorem 3. *Let A be any real matrix, $c_1, c_2 \in \mathbb{N}_{\ge 2}$ and $p \in \overline{\mathcal{C}^{c_2}}$. Then there is a $q \in \mathcal{C}^{c_2}$ such that*

$$d_A(p, q) \le \tfrac{c_1^2}{(c_1 - 1)c_2} n_{c_1,c_2} \,\mathrm{herdisc}(A, c_1).$$

A $q \in \mathcal{C}^{c_2}$ satisfying

$$d_A(p,q) \le \tfrac{c_1^2}{(c_1-1)c_2} n_{c_1,c_2} D + c_1^{-k}\|A\|_\infty$$

can be computed from kn_{c_1,c_2} colorings in c_1 colors having discrepancy at most D with respect to some submatrix of A.

Note that for $c_1 = c_2 = 2$, Theorem 3 is just the result $\operatorname{lindisc}(A,2) \le 2\operatorname{herdisc}(A,2)$ of Beck and Spencer [BS84] and Lovász, Spencer and Vesztergombi [LSV86].[1] We recall the fact that for every real number $x \in [0,1]$ that has a finite c–ary expansion $x = \sum_{k=0}^{l} b_k c^{-k}$ for some $b_k \in \{0,\dots,c-1\}, b_l \neq 0$, this expansion is unique (among all finite expansions). Denote by $l_c(x)$ the length of this c–ary expansion of x. Put $M_{c,l} := \{x \in [0,1] \,|\, l_c(x) \le l\}$ and $\mathcal{C}^{c_2}_{c_1,l} := \{p \,|\, p : [n] \to \overline{E}^{c_2} \cap M^{c_2}_{c_1,l}\}$ for all $l \in \mathbb{N}_0$.

The following lemma is the heart of our proof. It analyzes how well (with respect to $d_A(\cdot,\cdot)$) a floating coloring having a c_1–ary expansion of length l can be rounded to one of length $l-1$.

Lemma 4. *Let $p \in \mathcal{C}^{c_2}_{c_1,l}$ for some $l \in \mathbb{N}$. Then a $q \in \mathcal{C}^{c_2}_{c_1,l-1}$ such that*

$$d_A(p,q) \le \tfrac{1}{c_2} c_1^{-l+2} n_{c_1,c_2} D$$

can be computed from n_{c_1,c_2} colorings in c_1 colors having discrepancy at most D with respect to some submatrix of A.

Proof. This algorithm solves the problem. Set $q^{(0)} = p$. Let $t^{(1)},\dots,t^{(n_{c_1,c_2})}$ be an enumeration of T_{c_1,c_2} such that $\|t^{(r)}\|_\infty \ge \|t^{(r+1)}\|_\infty$ for all $r \in [n_{c_1,c_2}-1]$. For all $r = 1,\dots,n_{c_1,c_2}$ do:

> *Iteration r:* For every $j \in [n]$ let $q^{(r-1)}(j) = \sum_{k=0}^{l} b_k^{(r-1)}(j) c_1^{-k}$ for some $b_k^{(r-1)}(j) \in \{0,\dots,c_1-1\}^{c_2}$ denote the c_1–ary expansion of the vector $q^{(r-1)}(j)$. Set $J^{(r)} := \{j \in [n] | b_l^{(r-1)}(j) \ge t^{(r)}\}$. Choose a coloring $\chi^{(r)} : J^{(r)} \to [c_1]$ such that $\operatorname{disc}(A_{|J^{(r)}}, \chi^{(r)}) \le D$. Choose a function $f^{(r)} : [c_1] \to [c_2]$ such

[1] By $\operatorname{lindisc}(A,c)$ we denote the maximum value $\operatorname{lindisc}(A,p)$ among all floating c–colorings $p \in \overline{\mathcal{C}^c}$. For two colors, this is equivalent to $\operatorname{lindisc}(A,2) := \max_{p\in[0,1]^n} \min_{q\in\{0,1\}^n} \|A(p-q)\|_\infty$.

that $|(f^{(r)})^{-1}(d)| = t_d^{(r)}$ for all $d \in [c_2]$. For all $j \in [n], d \in [c_2]$ put

$$q^{(r)}(j)_d := \begin{cases} q^{(r-1)}(j)_d & \text{if } j \notin J^{(r)} \\ q^{(r-1)}(j)_d + (c_1 - t_d^{(r)})c_1^{-l} & \text{if } j \in J^{(r)}, d = f^{(r)}(\chi^{(r)}(j)) \\ q^{(r-1)}(j)_d - t_d^{(r)} c_1^{-l} & \text{if } j \in J^{(r)}, d \neq f^{(r)}(\chi^{(r)}(j)). \end{cases}$$

Finally set $q := q^{(n_{c_1,c_2})}$.

We defer the correctness proof to the full version of this paper.

□

The proof of Theorem 3 (also omitted) mainly consist of a repeated application of Lemma 4. As linear discrepancies are a generalization of the discrepancy problem, this already shows $\operatorname{disc}(A, c_2) = O_{c_1,c_2}(\operatorname{herdisc}(A, c_1))$. Unfortunately, the implicit constants are exponential in the numbers of colors.

4 Proof of the Main Results

In this section, we replace the exponential dependency on the number of colors by a quadratic dependency on c_1 only.

Theorem 4. *Let A be an m by n matrix. Let $c_1, c_2 \in \mathbb{N}_{\geq 2}$ be arbitrary numbers of colors. Suppose that for each submatrix of A there is a c_1–coloring having discrepancy at most D. Then there exists a c_2–coloring for A having discrepancy at most $\frac{11}{10} c_1^2 D$.*

A c_2–coloring for A having discrepancy at most $\frac{11}{10} c_1^2 D + 3c_1^{-k}\|A\|_\infty$ can be computed from $(c_1 - 1)(c_2 - 1)k$ colorings in c_1 colors having discrepancy at most D with respect to a suitable submatrix of A.

This improvement is made possible by a detour through 2 colors. Already by applying Theorem 3 first on the numbers of colors c_1 and 2 and then a second time on 2 and c_2, we can lower the constant to $O(c_1^2 c_2)$. We do slightly better (completely removing the dependence on c_2) by invoking the result of Srivastav and the author [DS99]. Using a recursive approach, they show that c–color discrepancies can be bounded in terms of 2–color discrepancies: For any hypergraph $\mathcal{H}$ and any number c of colors, a c–coloring χ satisfying

$$\operatorname{disc}(\mathcal{H}, \chi) \leq 2.0005\, D$$

can be computed from at most $(c-1)$ 2–colorings for induced subhypergraphs of $\mathcal{H}$ having discrepancy at most D with respect to a suitable weight. From the proof it is clear that an analogous result holds as well for discrepancies of matrices. We will use this fact without further proof.

Proof (of Theorem 4, sketched). By Theorem 3, we can compute good 2–colorings with respect to weights, since the weighted discrepancy problem is just the linear discrepancy one restricted to constant floating colorings (in the language of hypergraphs: All vertices have the same weight). Note that Theorem 3 works for induced submatrices as well. Therefore, combining Theorem 3 with the result cited above, we have Theorem 4 and its hypergraph version Theorem 1. □

We end this section with two corollaries. If A has more columns than rows, a reduction by linear algebra due to Spencer [Spe87] can be applied: In two colors, the linear discrepancy is at most the maximum linear discrepancy among all submatrices containing at most m columns of A. This yields:

Corollary 1. *For any $m \times n$ matrix A and any number of colors $c \in \mathbb{N}_{\geq 2}$, we have*

$$\operatorname{disc}(A, c) \leq \tfrac{11}{10} c^2 \max_{\substack{J \subseteq [n] \\ |J| \leq m}} \operatorname{disc}(A_{|J}, c).$$

A c–coloring χ for A having discrepancy $\operatorname{disc}(A,\chi) \leq \frac{11}{10}c^2 D + 3c^{-k}\|A\|_\infty$ can be computed from $(c-1)^2 k$ colorings in c colors having discrepancy at most D for a suitable submatrix of A having at most m columns.

Proof (sketched). We use Theorem 3 to step down to 2 colors, apply Spencer's reduction and return to c colors again (using [DS99]). □

The linear discrepancy in two colors also describes how well a solution of a linear system can be rounded to an approximate integer one (this is a folklore result easily being deduced from the definition). Combined with Theorem 3 we derive:

Corollary 2. *Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ such that the linear system $Ax = b$ has a solution x. Then there is a $z \in \mathbb{Z}^n$ such that $\|x - z\|_\infty \leq 1$ and $\|Az - b\|_\infty \leq \frac{11}{10}c^2 \operatorname{herdisc}(A, c)$ for all $c \in \mathbb{N}_{\geq 2}$.*

5 Concluding Remarks

In this paper we showed that the hereditary discrepancy in nearly independent of the numbers of colors. This strongly contrasts the ordinary discrepancy. Our result suggests that the hereditary discrepancy is a very general measure of how well a hypergraph behaves in partitioning problems.

To prove the main result $\mathrm{herdisc}(A, c_2) = \Theta_{c_1}(\mathrm{herdisc}(A, c_1))$ we needed a detour through linear discrepancies. It seems to be an interesting question whether this is necessary, or if a bound of type $\mathrm{herdisc}(A, c_2) = \Theta_{c_1,c_2}(\mathrm{herdisc}(A, c_1))$ can be proven more directly. The best result avoiding the detour we have contains a logarithmic dependence of the number of columns, cf. Theorem 2.

A second open problem is the precise influence of the numbers of colors. Our bound contains a factor of $O(c_1^2)$, whereas we only know examples justifying a factor of $\Omega(c_1)$.

References

[BCC+00] T. Biedl, E. Cenek, T. M. Chan, E. D. Demaine, M. L. Demaine, R. Fleischer, and Ming-Wei Wang. Balanced k-colorings. In *Proceedings of the 25th International Symposium on Mathematical Foundations of Computer Science (MFCS 2000)*, volume 1893 of *Lecture Notes in Computer Science*, pages 202–211, Berlin–Heidelberg, 2000. Springer Verlag.

[BF81] J. Beck and T. Fiala. "Integer making" theorems. *Discrete Applied Mathematics*, 3:1–8, 1981.

[BS84] J. Beck and J. Spencer. Integral approximation sequences. *Math. Programming*, 30:88–98, 1984.

[BS95] J. Beck and V. T. Sós. Discrepancy theory. In R. Graham, M. Grötschel, and L. Lovász, editors, *Handbook of Combinatorics*. 1995.

[Cha00] B. Chazelle. *The Discrepancy Method*. Princeton University, 2000.

[Doe01a] B. Doerr. Discrepancy in different numbers of colors. *Discrete Mathematics*, 2001. To appear.

[Doe01b] B. Doerr. Lattice approximation and linear discrepancy of totally unimodular matrices. In *Proceedings of the 12th Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 119–125, 2001. To appear in *Combinatorica*.

[DS99] B. Doerr and A. Srivastav. Approximation of multi-color discrepancy. In D. Hochbaum, K. Jansen, J. D. P. Rolim, and A. Sinclair, editors, *Randomization, Approximation and Combinatorial Optimization (Proceedings of APPROX-RANDOM 1999)*, volume 1671 of *Lecture Notes in Computer Science*, pages 39–50, Berlin–Heidelberg, 1999. Springer Verlag.

[DS01] B. Doerr and A. Srivastav. Recursive randomized coloring beats fair dice random colorings. In A. Ferreira and H. Reichel, editors, *Proceedings of the 18th Annual Symposium on Theoretical Aspects of Computer Science*

(STACS) 2001, volume 2010 of *Lecture Notes in Computer Science*, pages 183–194, Berlin–Heidelberg, 2001. Springer Verlag.

[GH62] A. Ghouila-Houri. Caractérisation des matrices totalement unimodulaires. *C. R. Acad. Sci. Paris*, 254:1192–1194, 1962.

[LSV86] L. Lovász, J. Spencer, and K. Vesztergombi. Discrepancies of set-systems and matrices. *Europ. J. Combin.*, 7:151–160, 1986.

[Mat95] J. Matoušek. Tight upper bound for the discrepancy of half-spaces. *Discr. & Comput. Geom.*, 13:593–601, 1995.

[Mat99] J. Matoušek. *Geometric Discrepancy.* Springer-Verlag, Berlin, 1999.

[MWW84] J. Matoušek, E. Welzl, and L. Wernisch. Discrepancy and approximations for bounded VC–dimension. *Combinatorica*, 13:455–466, 1984.

[Spe87] J. Spencer. *Ten Lectures on the Probabilistic Method.* SIAM, 1987.

[Sri97] A. Srinivasan. Improving the discrepancy bound for sparse matrices: better approximations for sparse lattice approximation problems. In *Proceedings of the Eighth Annual ACM-SIAM Symposium on Discrete Algorithms (New Orleans, LA, 1997)*, pages 692–701, New York, 1997. ACM.

6 Appendix

This appendix contains proofs omitted for reasons of space.

Proof (of Theorem 2). Let $\chi : [n] \to [c_1]$ be a c_1–coloring such that $\mathrm{disc}(A, \chi) \le \mathrm{herdisc}(A, c_1)$. Let $f : [c_1] \to [c_2] \cup \{0\}$ be any mapping such that $|f^{-1}(i)| = \left\lfloor \frac{c_1}{c_2} \right\rfloor$ for all $i \in [c_2]$. Set

$$\eta : [n] \to [c_2] \cup \{0\}; x \mapsto f(\chi(x)).$$

Then $\mathrm{disc}(A_{0\,|\eta^{-1}([c_2])}, \eta) \le \left\lfloor \frac{c_1}{c_2} \right\rfloor \mathrm{herdisc}(A, c_1)$ since each color class of η is the union of $\left\lfloor \frac{c_1}{c_2} \right\rfloor$ color classes of χ. If c_2 divides c_1, $\eta^{-1}([c_2]) = [n]$ and we are done. Let us therefore assume that c_2 does not divide c_1. We may choose f in such a way that it maps the $c_1 - c_2 \left\lfloor \frac{c_1}{c_2} \right\rfloor$ smallest color classes of χ to 0. Hence at least $c_2 \frac{n}{c_1} \left\lfloor \frac{c_1}{c_2} \right\rfloor$ points are colored by η, i. e., are not mapped to 0. We repeat this procedure on $\eta^{-1}(0)$ until all points are colored. This takes at most $\log_{1/(1-\frac{c_2}{c_1}\left\lfloor \frac{c_1}{c_2} \right\rfloor)}(n) \le \log_2(n)$ iterations. □

Proof (of Lemma 1). Each type $t \in T_{c_1,c_2}$ can be uniquely described by the set N_t of indices i such that $t_i \ne 0$ and a vector $\bar{t} \in \mathbb{N}^{|N_t|}$ which results from deleting the zero entries from t.[2] Conversely, for any $N \subseteq [c_2]$, $|N| \ge 2$, and any $t_0 \in \mathbb{N}^{|N|}$, $\|t_0\|_1 = c_1$ there is a type $t \in T_{c_1,c_2}$ such that $N_t = N$ and $\bar{t} = t_0$. Denote by r_i the number of $t_0 \in \mathbb{N}^i$ such that $\|t_0\|_1 = c_1$. Then we just showed

$$n_{c_1,c_2} = \sum_{i=2}^{c_2} \binom{c_2}{i} r_i.$$

The numbers r_i are easy to compute: There is a one-one mapping ϕ from the set of i–dimensional vectors in $\mathbb{N}$ having sum c_1 onto the $i-1$ element subsets of $[c_1 - 1]$. Each vector t is mapped on the set $\phi(t) = \{\sum_{j=1}^{k} t_j | k \in [i-1]\}$ of all but the last of its partial sums.

[2] We denote by $\mathbb{N}$ the set of positive integers, by $\mathbb{N}_0$ the nonnegative ones.

Hence $r_i = \binom{c_1-1}{i-1}$. As $\binom{c_1-1}{i-1} = \binom{c_1-1}{c_1-i}$, we conclude

$$\begin{aligned} n_{c_1,c_2} &= \sum_{i=2}^{c_2} \binom{c_2}{i} r_i = \sum_{i=2}^{c_2} \binom{c_2}{i}\binom{c_1-1}{c_1-i} \\ &= \sum_{i=1}^{c_2} \binom{c_2}{i}\binom{c_1-1}{c_1-i} - c_2 = \binom{c_1+c_2-1}{c_1} - c_2. \end{aligned}$$

□

Proof (of Lemma 2). For every permutation $\pi \in S_{c_2}$ denote by $\overline{\pi} : \mathbb{R}^{c_2} \to \mathbb{R}^{c_2}$ the (linear) mapping such that $\overline{\pi}(v)_i = v_{\pi(i)}$ for all $i \in [c_2], v \in \mathbb{R}^{c_2}$. Set $\overline{S}_{c_2} := \{\overline{\pi} | \pi \in S_{c_2}\}$. From the definitions we see that $\overline{S}_{c_2}$ leaves T_{c_1,c_2} invariant, i. e., $\overline{\pi}(T_{c_1,c_2}) = T_{c_1,c_2}$ for all $\overline{\pi} \in \overline{S}_{c_2}$. Hence s_{c_1,c_2} is a fixed point of $\overline{S}_{c_2}$, i. e., $\overline{\pi}(s_{c_1,c_2}) = s_{c_1,c_2}$ for all $\overline{\pi} \in \overline{S}_{c_2}$. The only fixed points of $\overline{S}_{c_2}$ are multiples of $\mathbf{1}_{c_2}$, hence there is a $\lambda \in \mathbb{R}_{\geq 0}$ such that $s_{c_1,c_2} = \lambda \mathbf{1}_{c_2}$. From

$$\lambda c_2 = \|\lambda \mathbf{1}_{c_2}\|_1 = \|s_{c_1,c_2}\|_1 = \sum_{t \in T_{c_1,c_2}} \|t\|_1 = n_{c_1,c_2} c_1$$

we get $\lambda = \frac{c_1}{c_2} n_{c_1,c_2}$. □

Proof (of Lemma 3). Let $t^{(1)}, \ldots, t^{(n_{c_1,c_2})}$ be an enumeration of T_{c_1,c_2} such that $\|t^{(r)}\|_\infty \geq \|t^{(r+1)}\|_\infty$ for all $r \in [n_{c_1,c_2} - 1]$. We claim that the following greedy algorithm is correct:

```
v_0 := v;
for r=1 to n_{c_1,c_2} do  e[r] := 0;
for r=1 to n_{c_1,c_2} do
  if t[r] <= v then {v := v - t[r]; e[r] := 1}
```

It is easy to see that $\sum_{r=1}^{n_{c_1,c_2}} e[r]t[r] + v = v_0$ is satisfied after the execution of each line. Furthermore v is nonnegative all the time. Therefore is suffices to show that $v = 0$ holds after the termination of our algorithm.

Assume that $v \neq 0$ after the termination of the algorithm. By induction we see that c_1 divides $\|v\|_1$, in particular $\|v\|_1 \geq c_1$. Hence there is an $r \in [n_{c_1,c_2}]$ such that $t^{(r)} \leq v$. At first we note that this forces $e[r] = 1$. Denote by $v^{(r-1)}$ the value of v after termination of iteration $r-1$. As $e[r]=1$ we have $2t^{(r)} \le v^{(r-1)}$. We show that this can never happen:

Let s be minimal such that $2t^{(s)} \le v^{(s-1)}$. Then $\|t^{(s)}\|_\infty \le \frac{c_1-1}{2}$. Hence there is a type $t^{(u)}$ such that $\|t^{(u)}\|_\infty = 2\|t^{(s)}\|_\infty$ and $t^{(u)} \le 2t^{(s)}$. We conclude $u < s$. By minimality of s we have $e[u]=0$. This is a contradiction as $t^{(u)} \le 2t^{(s)} \le v^{(s-1)} \le v^{(u-1)}$. □

Proof (of Lemma 4). Let us analyze a single iteration r first: Let $d \in [c_2]$ and $i \in [m]$. We compute

$$
\begin{aligned}
&\left|\sum_{j\in[n]} a_{ij}\left(q^{(r-1)}(j)_d - q^{(r)}(j)_d\right)\right| = \left|\sum_{j\in J^{(r)}} a_{ij}\left(q^{(r-1)}(j)_d - q^{(r)}(j)_d\right)\right| \\
&= c_1^{-l+1}\left|\sum_{\substack{j\in J^{(r)}\\ f^{(r)}(\chi^{(r)}(j))=d}} \frac{c_1 - t_d^{(r)}}{c_1} a_{ij} - \sum_{\substack{j\in J^{(r)}\\ f^{(r)}(\chi^{(r)}(j))\neq d}} \frac{t_d^{(r)}}{c_1} a_{ij}\right| \\
&= c_1^{-l+1}\left|\sum_{\substack{j\in J^{(r)}\\ f^{(r)}(\chi^{(r)}(j))=d}} a_{ij} - \frac{t_d^{(r)}}{c_1}\sum_{j\in J^{(r)}} a_{ij}\right| \\
&= c_1^{-l+1}\left|\sum_{e\in(f^{(r)})^{-1}(d)}\left(\sum_{j\in(\chi^{(r)})^{-1}(e)} a_{ij} - \frac{1}{c_1}\sum_{j\in J^{(r)}} a_{ij}\right)\right| \\
&\le c_1^{-l+1}\sum_{e\in(f^{(r)})^{-1}(d)}\left|\sum_{j\in(\chi^{(r)})^{-1}(e)} a_{ij} - \frac{1}{c_1}\sum_{j\in J^{(r)}} a_{ij}\right| \\
&\le c_1^{-l+1} t_d^{(r)}\, \mathrm{disc}(A_{|J^{(r)}}, \chi^{(r)}) \\
&\le c_1^{-l+1} t_d^{(r)} D.
\end{aligned}
$$

Thus we have

$$\begin{aligned}\left|\sum_{j\in[n]} a_{ij}\left(p(j)_d - q(j)_d\right)\right| &= \left|\sum_{r=1}^{n_{c_1,c_2}} \sum_{j\in[n]} a_{ij}\left(q^{(r-1)}(j)_d - q^{(r)}(j)_d\right)\right| \\ &\le c_1^{-l+1} D \sum_{r=1}^{n_{c_1,c_2}} t_d^{(r)} \\ &= c_1^{-l+1}(s_{c_1,c_2})_d D.\end{aligned}$$

Lemma 2 gives

$$\begin{aligned}d_A(p,q) &\le \|s_{c_1,c_2}\|_\infty c_1^{-l+1} D \\ &\le \tfrac{1}{c_2} c_1^{-l+2} n_{c_1,c_2} D.\end{aligned}$$

Next we claim that $q^{(r)}(j) \in \overline{E}^{c_2} \cap M_{c_1,l}^{c_2}$ for all $r \in \{0,\dots,n_{c_1,c_2}\}, j \in [n]$. This is clear for $r = 0$ as $q^{(0)} = p$. We proceed by induction. Let $r \in [n_{c_1,c_2}]$ and $j \in [n]$ such that $q^{(r-1)}(j) \in \overline{E}^{c_2} \cap M_{c_1,l}^{c_2}$. If $j \notin J^{(r)}$, then $q^{(r)}(j) = q^{(r-1)}(j)$ and there is nothing to show. Assume $j \in J^{(r)}$. Since $b_l^{(r-1)}(j) \ge t^{(r)}$, we have $q^{(r-1)}(j) \ge t^{(r)} c_1^{-l}$ and hence $q^{(r)}(j)$ is not negative by definition. We compute

$$\begin{aligned}\sum_{d\in[c_2]} q^{(r)}(j)_d &= q^{(r-1)}(j)_{f^{(r)}(\chi^{(r)}(j))} + (c_1 - t^{(r)}_{f^{(r)}(\chi^{(r)}(j))}) c_1^{-l} \\ &\quad + \sum_{\substack{d\in[c_2] \\ d\ne f^{(r)}(\chi^{(r)}(j))}} (q^{(r-1)}(j)_d - t_d^{(r)} c_1^{-l}) \\ &= \sum_{d\in[c_2]} q^{(r-1)}(j)_d + (c_1 - \sum_{d\in[c_2]} t_d^{(r)}) c_1^{-l} \\ &= \sum_{d\in[c_2]} q^{(r-1)}(j)_d = 1.\end{aligned}$$

Since $q^{(r)}(j)$ is nonnegative, we conclude $q^{(r)}(j)_d \le 1$ for all $d \in [c_2]$. Finally, from the definition of $q^{(r)}(j)$ it is clear that $q^{(r)}(j) \in M_{c_1,l}^{c_2}$ if and only if $q^{(r-1)}(j) \in M_{c_1,l}^{c_2}$. This proves the claim.

The definitions of $q^{(\cdot)}(j)$ and $b_l^{(\cdot)}(j)$ also yield $b_l^{(r)}(j) = b_l^{(r-1)}(j) - t^{(r)}$ for all $j \in J^{(r)}, r \in [n_{c_1,c_2}]$. Since c_1 divides $\|b_l^{(0)}(j)\|_1$ for all

$j \in [n]$, Lemma 3 shows that the range of q is actually contained in $M^{c_2}_{c_1,l-1}$. □

Proof (of Theorem 3). Let $p : [n] \to \overline{E}^{c_2}$. We show that there is a $q : [n] \to E^{c_2}$ such that $d_A(p,q) \leq \frac{c_1^2}{(c_1-1)c_2} n_{c_1,c_2} \operatorname{herdisc}(A,c_1)$ and how to approximate it. Let us first assume that $p : [n] \to \overline{E}^{c_2} \cap M^{c_2}_{c_1,l}$ for some $l \in \mathbb{N}$.

Set $p^{(0)} := p$. Inductively, we define a sequence $(p^{(i)})_{i\in[l]}$ such that $p^{(i)} : [n] \to \overline{E}^{c_2} \cap M^{c_2}_{c_1,l-i}$. In particular, $p^{(l)} : [n] \to E^{c_2}$. Having defined $p^{(i)}$ for some $i < l$ we apply Lemma 4 on $p^{(i)}$ and get $p^{(i+1)} : [n] \to \overline{E}^{c_2} \cap M^{c_2}_{c_1,l-i-1}$ such that $d_A(p^{(i)}, p^{(i+1)}) \leq \frac{c_1^{-l+i+2}}{c_2} n_{c_1,c_2} D$. Set $q := p^{(l)}$. Then

$$\begin{aligned} d_A(p,q) &\leq \sum_{i=0}^{l-1} d_A(p^{(i)}, p^{(i+1)}) \\ &\leq \sum_{i=0}^{l-1} \tfrac{1}{c_2} c_1^{-l+i+2} n_{c_1,c_2} D \\ &\leq \frac{c_1^2}{(c_1-1)c_2} n_{c_1,c_2} D. \end{aligned}$$

For arbitrary p and $l \in \mathbb{N}$ do the following: Choose a $p^{(0)} : [n] \to \overline{E}^{c_2} \cap M^{c_2}_{c_1,l}$ such that $\|p(j) - p^{(0)}(j)\|_\infty \leq c_1^{-l}$. Then $d_A(p, p^{(0)}) \leq c_1^{-l} \|A\|_\infty$. With $p^{(0)}$ proceed as above and get a $q : [n] \to E^{c_2}$ such that $d_A(p^{(0)}, q) \leq \frac{c_1^2}{(c_1-1)c_2} n_{c_1,c_2} D$. Since $d_A(\cdot,\cdot)$ obeys the triangle inequality, we are done.

For the existence result choose $D = \operatorname{herdisc}(A, c_1)$. Since $\bigcup_{l\in\mathbb{N}_0} \mathcal{C}^{c_2}_{c_1,l}$ is dense in $\overline{\mathcal{C}^{c_2}}$ and $\operatorname{lindisc}(A,\cdot) : \overline{\mathcal{C}^{c_2}} \to \mathbb{R}; p \mapsto \operatorname{lindisc}(A,p)$ is continuous, we have $\operatorname{lindisc}(A,p) \leq \frac{c_1^2}{(c_1-1)c_2} n_{c_1,c_2} \operatorname{herdisc}(A,c_1)$ for all $p \in \overline{\mathcal{C}^{c_2}}$.

□