MULTICOLOR DISCREPANCY OF ARITHMETIC PROGRESSIONS –EXTENDED ABSTRACT–

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1. Introduction

We introduce combinatorial multi-color discrepancies and show for the hypergraph of arithmetic progressions in $\{1,\dots,n\}$ a lower bound for the discrepancy in c colors of $\frac{1}{25\sqrt{c}}\sqrt[4]{n}$ and an upper bound of $\mathcal{O}(c^{-0.16}\sqrt[4]{n})$. For $c=2$ this is the famous lower bound of Roth (1964) and the optimal upper bound of Matoušek and Spencer (1996).

2. Preliminaries

Combinatorial discrepancy theory deals with the problem of partitioning the vertices of a hypergraph (set-system) in such a way that all hyperedges are split into roughly equal-sized parts. Discrepancy measures the deviation of an optimal partition to an ideal one, that is one where all edges contain the same number of vertices in each class of the partition.

Usually one represents the partition by a coloring, that is a mapping from the vertices into some set such that the classes of equal images form the partition classes. In this language, most results known so far only deal with two colors. Recent results from communication complexity (e. g. [BHK98]) further motivate the study of multi-color discrepancies.

In this article we show how to construct multi-colorings with low discrepancy.

Up to now little is known about the discrepancy problem where we ask for a partition into more than two classes. The only one we found is

Theorem 2.1 ([BS95]). *Let $\mathcal{H}$ be any hypergraph such that the incidence matrix of $\mathcal{H}$ is unimodular. Then for any number $c \in \mathbb{N}$ there is a c–partition $X = X_1\dot\cup\dots\dot\cup X_c$ of X such that for any edge $E \in \mathcal{E}$ and $i \in [c]$*

$$\left| |E \cap X_i| - \frac{|E|}{c} \right| < 1.$$

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The first author is supported by the graduate school ‘Effiziente Algorithmen und Multiskalenmethoden’, Deutsche Forschungsgemeinschaft.

Let us introduce some notation concerning c–color discrepancies. A c–coloring of $\mathcal{H}$ is simply a mapping $\chi : X \to M$, where M is any set of cardinality c. For convenience, normally one has $M = [c] := \{1, \ldots, c\}$. Sometimes a different set M will be of advantage. Note that in applications to communication complexity M cab be a finite abelian group [BHK98]. The basic idea of measuring the deviation from the average motivates the definitions of the *discrepancy of an edge $E \in \mathcal{E}$ in color $i \in M$ with respect to χ* as

$$\operatorname{disc}_{\chi,i}(E) := \left| |\chi^{-1}(i) \cap E| - \frac{|E|}{c} \right|, \tag{1}$$

the *discrepancy of $\mathcal{H}$ with respect to χ* as

$$\operatorname{disc}(\mathcal{H}, \chi) := \max_{i \in M, E \in \mathcal{E}} \operatorname{disc}_{\chi,i}(E) \tag{2}$$

and the *discrepancy of $\mathcal{H}$ in c colors* as

$$\operatorname{disc}(\mathcal{H}, c) := \min_{\chi : X \to [c]} \operatorname{disc}(\mathcal{H}, \chi). \tag{3}$$

Immediately we see

Remark 2.2. $\operatorname{disc}(\mathcal{H}, 2) = \frac{1}{2} \operatorname{disc}(\mathcal{H})$.

To add some further motivation to what follows let us give an example which shows that a hypergraph may have very different discrepancies in different numbers of colors.

Example: Let $k \in \mathbb{N}$ and $n = 4k$. Set $\mathcal{H}_n = ([n], \{X \subseteq [n] \,|\, |X \cap [\frac{n}{2}]| = |X \setminus [\frac{n}{2}]|\})$. Obviously, $\mathcal{H}_n$ has discrepancy zero, but $\operatorname{disc}(\mathcal{H}_n, 4) = \frac{1}{8} n$.

Proof. Let $\chi : [n] \to [4]$ be any 4–coloring. Let $i \in [4]$ be a color such that $|\chi^{-1}(i)| \leq \frac{1}{4} n$. Then there are sets $E_1 \subseteq [\frac{n}{2}]$, $E_2 \subseteq [n] \setminus [\frac{n}{2}]$ such that $|E_j| = \frac{1}{4} n$ and $\chi^{-1}(i) \cap E_j = \emptyset$. Thus $E_1 \cup E_2$ is an edge in $\mathcal{H}$ and has discrepancy $\frac{1}{8} n$ in color i. On the other hand $\chi : x \mapsto \lceil \frac{4x}{n} \rceil$ is a coloring having discrepancy $\frac{1}{8} n$. □

In the above notion we can not express discrepancies simply by sums of colors. As this is very practical sometimes and a step towards the matrix concept, we describe the color $i \in [c]$ by a vector $m^{(i)} \in \mathbb{R}^c$ defined by

$$m_j^{(i)} := \begin{cases} \frac{c-1}{c} & \text{if } i = j \\ -\frac{1}{c} & \text{otherwise.} \end{cases}$$

Then

$$\operatorname{disc}(\mathcal{H}, \chi) = \max_{E \in \mathcal{E}} \left\| \sum_{x \in E} m^{(\chi(x))} \right\|_\infty . \tag{4}$$

Set $M_c := \{m^{(i)} | i \in [c]\}$. Apparently, we have

$$\operatorname{disc}(\mathcal{H}, c) = \min_{\chi : X \to M_c} \max_{E \in \mathcal{E}} \left\| \sum_{x \in E} \chi(x) \right\|_\infty . \tag{5}$$

As for 2 colors, the notion of multi-color discrepancy has a natural extension to matrices via the tensor product. Let $A \in \mathbb{R}^{m \times n}$ be any matrix. Let $\overline{A}$ be the matrix which results from replacing every a_{ij} in A by $a_{ij} I_c$, where I_c shall denote the unit matrix of dimension c. $\overline{A}$ is the tensor product of A and I_c. Identifying a $\chi : [n] \to M_c$ by a cn–dimensional vector in the natural way, we get

Theorem 2.3.

$$\operatorname{disc}(A, c) = \min_{\chi : [n] \to M_c} \|\overline{A}\chi\|_\infty . \tag{6}$$

3. The Main Result

For $a, d, l \in \mathbb{N}$ denote by $A_{adl} := \{a + id | 0 \leq i \leq l - 1\}$ the arithmetic progression with starting point a, difference d and length l. Denote by $\mathcal{E}_n$ the set of all arithmetic progressions in $[n]$, that is $\mathcal{E}_n = \{A_{adl} \cap [n] | a, d, l \in [n]\}$. Set $\mathcal{H}_n = ([n], \mathcal{E}_n)$. Roth [Rot64] proved the celebrated lower bound $\operatorname{disc}(\mathcal{H}_n) \geq \frac{1}{20}\sqrt[4]{n}$. It took 20 years until Matoušek and Spencer [MS96] solved the discrepancy problem for $\mathcal{H}_n$ by proving the upper bound $\mathcal{O}(\sqrt[4]{n})$.

We have

Theorem 3.1. *The hypergraph of arithmetic progressions fulfills*

$$0.04 \frac{1}{\sqrt{c}} \sqrt[4]{n} \leq \operatorname{disc}(\mathcal{H}_n, c) \leq \alpha c^{-0.16} \sqrt[4]{n},$$

where α is an absolute constant.

Proof idea for the upper bound. For some 2–color discrepancy results the proofs seem to rely heavily on the fact that only two colors are used. A prominent example is Spencer's $\mathcal{O}(\sqrt{n})$ bound for hypergraphs having n vertices and edges. One key step in the proof is to construct a low discrepancy partial coloring $\chi := \frac{1}{2}(\chi_1 - \chi_2)$ from two colorings χ_1, χ_2 with $\chi_1(E) \approx \chi_2(E)$ for all $E \in \mathcal{E}$. It is not clear to us how this idea can be generalized to c colors. As the partial coloring method has been a major break-through in 2–color discrepancy theory we somehow need a similar method for c colors too. What we do in this section is not partial coloring, i. e. enlarging the partition classes successively coloring points, but recursive 2–coloring, i. e. successively enlarging the number of partition classes. The basic idea is to find a suitable 2–coloring of X with color classes X_1, X_2 and then to iterate this process on the subhypergraphs induced by X_1 and X_2. If the weighted 2–color discrepancy of all induced subhypergraphs is uniformly bounded, such a recursive method can be analyzed, even for c not a power of 2.

The proof is quite technical as we have to keep track of decreasing discrepancies in subhypergraphs along a suitable partition tree.

Proof idea for the lower bound. Using the tensor product calculus we have the following c–color version of a result attributed to Lovász and Sós in [BS95].

Theorem 3.2. *Let* $A \in \mathbb{R}^{m \times n}$. *Then* $\operatorname{disc}(A, c) \geq \sqrt{\frac{n(c-1)}{mc^2} \lambda_{\min}(A^\top A)}$.

For the proof of the lower bound we use the above theorem in the following way. Set $k = \lfloor \sqrt{\frac{1}{6} n} \rfloor$. Recall that a matrix is called circulant if the i-th row can be obtained from the first by shifting it $i-1$ times to the right. The incidence matrix $A = (a_{ij}) \in \{0,1\}^{6kn \times n}$ defined by $a_{ij} = 1$ if and only if $j \in E_i$ consists of $6k$ circulant sub-matrices. As sum and product of two circulant matrices is circulant again, $A^\top A$ is circulant. The eigenvectors of circulant matrices are known to be of the form $(1, \varepsilon, \varepsilon^2, \ldots, \varepsilon^{n-1})^\top$, where ε is an nth root of unity. Using this one gets that the minimum eigenvalue $\lambda_{\min}(A^\top A)$ of $A^\top A$ is greater than $\frac{1}{4}k^2$. Using Theorem 3.2 we have $\operatorname{disc}(([n], \mathcal{E}_n), c)^2 \geq \frac{n(c-1)}{6knc^2} \frac{1}{4} k^2 = \frac{(c-1)k}{24c^2}$ and an easy calculation finishes the proof.

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