Structured Randomized Rounding and Coloring

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Abstract. In this paper we propose an advanced randomized coloring algorithm for the problem of balanced colorings of hypergraphs (discrepancy problem). It allows to use structural information about the hypergraph in the design of the random experiment. This yields colorings having smaller discrepancy than those independently coloring the vertices. We also obtain more information about the coloring, or, conversely, we may enforce the random coloring to have special properties. Due to the dependencies, these random colorings need fewer random bits to be constructed, and computing their discrepancy can be done faster. We apply our method to hypergraphs of d–dimensional boxes. Among others, we observe a factor $2^{d/2}$ decrease in discrepancy and a reduction of the number of random bits needed by a factor of 2^d.
Since the discrepancy problem is a particular rounding problem, our approach is a randomized rounding strategy for the corresponding ILP-relaxation that beats the usual randomized rounding.
Key words: randomized algorithms, hypergraph coloring, discrepancy, randomized rounding, integer linear programming.

1 Introduction and Results

1.1 The Discrepancy Problem and Integer Linear Programs

In this paper we deal with a special kind of integer linear programs, namely those that model discrepancy problems. Roughly speaking, the *combinatorial discrepancy problem* is to partition the vertex set of a given hypergraph into two classes in a balanced manner, i.e., such that each hyperedge contains the same number of vertices in each of the two partition classes. To be precise:

We call a pair $\mathcal{H} = (X, \mathcal{E})$, where X is finite set and $\mathcal{E} \subseteq 2^X$, a *hypergraph*. The elements of X are called *vertices*, those of $\mathcal{E}$ *hyperedges*. A partition of X into

* supported by the graduate school 'Effiziente Algorithmen und Multiskalenmethoden', Deutsche Forschungsgemeinschaft

two classes is usually represented by a coloring $\chi : X \to C$ for some two-element set C. The partition then is formed by the color classes $\chi^{-1}(i), i \in C$. It turns out to be useful to select -1 and $+1$ as colors. For a coloring $\chi : X \to \{-1, 1\}$ and a hyperedge $E \in \mathcal{E}$ then the expression

$$\chi(E) := \sum_{x \in E} \chi(x)$$

counts how many of the $+1$–vertices of E cannot be matched by -1–vertices. $|\chi(E)|$ is therefore a measure of how balanced the hyperedge E is colored by χ. As it is our aim to color all hyperedges simultaneously in a balanced manner, we define the discrepancy of χ with respect to $\mathcal{H}$ by

$$\mathrm{disc}(\mathcal{H}, \chi) := \max_{E \in \mathcal{E}} |\chi(E)|.$$

The discrepancy problem originated from number theoretical problems (e. g. van der Waerden [vdW27] or Roth [Rot64]), but due to a wide range of applications and connections it has received an increased attention by computer scientists and applied mathematicians during the last twenty years.

Most notably is the connection to uniformly distributed sets and sequences which play a crucial role in *numerical integration* in higher dimensions (quasi-Monte Carlo methods). This area is also called geometric discrepancy theory. Via the so-called "transference principle", geometric and combinatorial discrepancies are connected with each other. An excellent reference on geometric discrepancies, their connection to combinatorial ones and applications is the book of Matoušek [Mat99].

The notion of linear discrepancy of matrices describes how well a solution of a linear program can be rounded to an integer solution (*lattice approximation problem*). Due to work of Beck and Spencer [BS84] and Lovász et al. [LSV86], the linear discrepancy can be bounded (in a constructive manner) by combinatorial discrepancies.

Further applications are found in *computational geometry* and the theory of *communication complexity*. For these and other applications of discrepancies in theoretical computer science we refer to the new book of Chazelle [Cha00].

Discrepancy problems can be formulated as integer linear programs. Let $X = \{1, \ldots, n\} =: [n]$ and $\mathcal{E} = \{E_1, \ldots, E_m\}$. Then the following integer linear program (here given as a $0, 1$ ILP) solves the discrepancy problem for $\mathcal{H}$:

minimize 2λ
subject to

$$\begin{aligned} \sum_{i \in E_j} x_i - \tfrac{1}{2}|E_j| &\leq \lambda, \quad j = 1, \ldots, m \\ -\sum_{i \in E_j} x_i + \tfrac{1}{2}|E_j| &\leq \lambda, \quad j = 1, \ldots, m \\ x_i &\in \{0,1\}, \quad i = 1, \ldots, n \\ \lambda &\geq 0. \end{aligned}$$

The problem in using the linear relaxation of this ILP is that there always exists the trivial solution $\mathbf{x} = (x_1, \ldots, x_n) = \frac{1}{2}\mathbf{1}_n$. Therefore, a fruitful connection between solutions of the $[0,1]$–relaxation and the original problem is not to be expected.

On the other hand, randomized rounding strategies for this trivial solution yield random colorings and, vice versa, generators of random colorings can be interpreted as randomized rounding strategies. Thus both problems are strongly connected. It also turns out that the tools used and the difficulties occurring in both problems are very similar. Thus we think that the methods of this paper might have a broader application and are not restricted to the discrepancy problem.

Note that when applying a randomized rounding strategy to the above ILP, we do not need to care about feasibility (as for most randomized rounding problems). The reason is that any infeasibility inflicted by the rounding, i.e., any violation of the constraints, simply is a discrepancy.

1.2 Algorithmic Aspects of Randomized Coloring and Randomized Rounding

Discrepancy is an NP–hard problem. It is even NP–hard to decide whether a zero discrepancy coloring exists or not. Efficient algorithms finding an optimal coloring therefore are not to be expected. Indeed, very little is known about the algorithmic aspect of discrepancy. For some restrictions of the problem a nice solution exist, e. g. for hypergraphs having vertex degree at most t. Beck and Fiala [BF81] gave a polynomial time algorithm leading to a coloring having discrepancy less than $2t$.

A common algorithmic approach for the general case (and in fact the only one known to us) are random colorings obtained by independently choosing a random color for each vertex. Via a Chernoff-bound analysis (see e.g. Alon and Spencer [AS00]) this yields colorings having discrepancy $O(\sqrt{n \log m})$, where as above n shall always denote the number of vertices and m the number of hyperedges. More precisely, they show

Theorem 1. *A random coloring obtained by independently choosing a random color for each vertex has discrepancy*

$$\mathrm{disc}(\mathcal{H}, \chi) \leq \sqrt{2n \ln(4m)}$$

with probability at least $\frac{1}{2}$.

Note that this yields a randomized algorithm computing a coloring of the claimed discrepancy by repeatedly generating and testing a random coloring until the discrepancy guarantee of the theorem is satisfied. This algorithm has expected run-time $O(n(R+m)$, where R is the complexity of generating a random bit. To get rid of the random aspect, several so-called derandomization techniques have been developed. We refer to [Sri01] for a survey. Random constructions show that (at least for suitable values of n and m) there are hypergraphs having discrepancy $\Omega(\sqrt{n \log \frac{m}{n}})$. Thus this approach cannot be improved significantly in the general case.

Via the transfer sketched in the previous subsection, all of the above holds for general rounding problems as well. In particular, no randomized rounding strategy for linear problem of n variables and m constraints can guarantee a violation in the constraints of less than $\Omega(\sqrt{n \log \frac{m}{n}})$.

A central problem with random colorings (random rounding) is therefore how to take into account the special structure of the hypergraph (the ILP). One way to deal with this is to use random colorings as above, but to tighten the analysis using the structural information. Limited dependencies of 'bad' events play a crucial role here. Two papers in this context are Schmidt et al. [SSS95] and Srinivasan [Sri96].

A second approach is to use a different kind of random colorings, i.e., to design the random experiment in a way that it exploits the structure of the hypergraph. This is what we do in this paper.

1.3 Our results

We analyze a way of generating random colorings not by independently coloring the vertices, but by enforcing some dependencies. Thus we are able to exploit structural information about the hypergraph.

This proves to be effective in several ways. Firstly, it allows to generate random colorings having smaller discrepancy. Being a fairly general class of hypergraphs that have some common structure, we analyze hypergraphs of d–dimensional boxes. Our randomized colorings beat the ordinary random colorings in terms of discrepancy by a factor roughly $2^{d/2}$.

A second advantage is that we also obtain some more information about the random coloring. For example, we may prescribe that our colorings should be *fair*, that is, have equal-sized color classes. This can be useful in some applications, e. g. the recursive method to construct balanced multi-colorings of [DS01] uses fair colorings. A nice thing from the technical point of view is that we get these fair colorings without extra technical difficulties. Usually, working with fair colorings is more difficult, since the hypergeometric distribution is harder to analyze than the binomial one (cf. Chvátal [Chv79] and Uhlmann [Uhl66]).

A third point concerns the complexity of generating the colorings. Due to the dependencies the number of random bits needed to generate our random colorings is smaller than for ordinary random colorings. For the hypergraphs of d–dimensional boxes we reduce the number of random bits needed by a factor of 2^d. This is important, if generating random bits is costly.

Finally, computing the discrepancy of our random colorings can be done faster compared to ordinary random colorings. The reason is that (depending on the hypergraph, of course) the number of relevant hyperedges, i.e., those for which $\chi(E)$ has to be computed, is reduced. Since a typical randomized algorithm computes a low-discrepancy coloring by repeatedly generating a random coloring and then computing its discrepancy until a satisfactory solution is found, this fact also speeds up computing low-discrepancy colorings.

2 Structured Randomized Coloring

As mentioned in the introduction, our aim is to generate random colorings that to not independently color the vertices, but on the contrary use suitable dependencies that reflect the structure of the hypergraph. To do so, we partition the vertex set into classes. Within such a class, we will have perfect dependence, that is, each vertex determines the color of all other vertices in the class. For two vertices lying in different classes, their colors shall be chosen independently. The problem of this very general approach is of course to catch the structure of the hypergraph through the partition and the dependencies in the partition classes. We show an example of how to do so in the next section and proceed by fixing the general framework.

Let $\mathcal{P} = \{P_1, \ldots, P_r\}$ be a partition of the vertex set. Let $\chi_{P_i} : P_i \to \{-1, +1\}$ be colorings such that $|\chi_{P_i}(E \cap P_i)| \le 1$ holds for all edges $E \in \mathcal{E}$. For a hyperedge $E \in \mathcal{E}$ set

$$I(\mathcal{P}, E) := \{i \in [r] | \chi_{P_i}(E \cap P_i) \neq 0\},$$

hence $\chi_{P_i}(E \cap P_i) \in \{-1, +1\}$ for all $i \in I(\mathcal{P}, E)$. We generate a random coloring like this: For each $i \in [r] := \{1, \ldots, r\}$ we 'flip a coin', i.e., independently and uniformly choose a random sign $\varepsilon_i \in \{-1, +1\}$. Let $\chi : X \to \{-1, +1\}$ denote the union of the $\varepsilon_i \chi_{P_i}$, that is, we have $\chi(x) = \varepsilon_i \chi_{P_i}(x)$ for all $x \in P_i$. We call χ a *structured random coloring* with respect to $\mathcal{P}$ and the $\chi_{P_i}, i \in [r]$. Here is a discrepancy estimate for such a coloring.

Lemma 1. *Let χ be a structured random coloring with respect to $\mathcal{P}$ and the $\chi_{P_i}, i \in [r]$. For any hyperedge $E \in \mathcal{E}$ we have*

$$P(|\chi(E)| > \lambda) < 2e^{-\frac{\lambda^2}{2|I(\mathcal{P},E)|}}.$$

The proof is not very difficult. We still state it here, as it reveals why our structured random colorings are superior to the ordinary ones.

Proof. For each $i \in I(\mathcal{P}, E)$ define a random variable $Z_i = \chi_{P_i}(E \cap P_i) = \sum_{x \in E \cap P_i} \chi_{P_i}(x)$. Set $Z = \sum_{i \in I(\mathcal{P},E)} Z_i$. Note that $Z = \chi(E)$. Since the Z_i are mutually independent $-1, 1$ random variable, we may apply the Chernoff bound (cf. [AS00], Corollary A.1.2) and get

$$P(|Z| > \lambda) < 2e^{-\frac{\lambda^2}{2|I(\mathcal{P},E)|}}.$$

□

Comparing Lemma 1 with the analogous estimate for ordinary random colorings

$$P(|\chi(E)| \geq \lambda) \leq 2e^{-\frac{\lambda^2}{2|E|}},$$

we see that in our version we replaced the cardinality $|E|$ of the hyperedge by the possibly smaller number of P_i such that $\chi_{P_i}(E \cap P_i) \neq 0$. We thus reduced the relevant size of the hyperedges.

There is a second way structured random colorings can improve discrepancy bounds, namely by reducing the number of relevant hyperedges. Set

$$E_{\mathcal{P}} := \bigcup \{(E \cap P_i) \,|\, \chi_{P_i}(E \cap P_i) \neq 0\}$$

for all $E \in \mathcal{E}$ and $\mathcal{E}_{\mathcal{P}} := \{E_{\mathcal{P}} | E \in \mathcal{E}\}$. From the definition of structured random colorings it is clear that any structured random coloring χ with respect to $\mathcal{P}$ and the $\chi_P, P \in \mathcal{P}$ fulfills $\chi(E) = \chi(E_{\mathcal{P}})$. In particular, we have

$$\operatorname{disc}(\mathcal{H}, \chi) = \operatorname{disc}((X, \mathcal{E}_{\mathcal{P}}), \chi).$$

Depending on the partition $\mathcal{P}$ and the colorings χ_{P_i}, the mapping $E \mapsto E_{\mathcal{P}}$ is not injective, and hence $|\mathcal{E}_{\mathcal{P}}| < |\mathcal{E}|$. In this case we only need to consider the smaller number $|\mathcal{E}_{\mathcal{P}}|$ of hyperedges. Since the discrepancy bound depends on the number of hyperedges just logarithmically, this effect is less important compared to the reduction of the relevant sizes of the hyperedges. It can however be useful, as it makes the computation of $\operatorname{disc}(\mathcal{H}, \chi)$ easier.

This observation together with Lemma 1 yields

Theorem 2. *Let* $s_0 := \max_{E\in\mathcal{E}} |I(\mathcal{P}, E)|$ *and* $m_0 := |\mathcal{E}_{\mathcal{P}}|$. *Then a structured random coloring with respect to* $\mathcal{P}$ *and the* $\chi_P, P \in \mathcal{P}$ *has discrepancy at most*

$$\mathrm{disc}(\mathcal{H}, \chi) \le \sqrt{2s_0 \ln(4m_0)}$$

with probability at least $\frac{1}{2}$.

Proof. Omitted. □

There are two more points to add concerning structured random colorings. One is that we may get information about χ through properties of the colorings $\chi_P, P \in \mathcal{P}$. For example, if each $\chi_P, P \in \mathcal{P}$ has equal-sized color classes, then this also holds for χ. Conversely of course, we may enforce certain properties on χ by choosing suitable colorings $\chi_P, P \in \mathcal{P}$.

Secondly, it is easily seen that to generate a structured random coloring with respect to $\mathcal{P}$ and $\chi_P, P \in \mathcal{P}$ we need only $|\mathcal{P}|$ random bits instead of n random bits needed for ordinary random colorings.

3 Higher-Dimensional Boxes

In this section we show how the method described above can be applied to an actual example, namely hypergraphs of higher-dimensional boxes. They display some regularity that can be exploited. On the other hand, this is still a fairly general class of hypergraphs. For similarly geometrically defined hypergraphs, so-called cylinder intersections, a discrepancy result was used to prove bounds on multi-party communication complexities by Babai et al. [BHK98].

We say that a hypergraph $\mathcal{H} = (X, \mathcal{E})$ is a *hypergraph of* d*–dimensional boxes* for some $d \in \mathbb{N}$, if there is a decomposition $X = X_1 \times \cdots \times X_d$ such that each hyperedge $E \in \mathcal{E}$ has a representation $E = E_1 \times \cdots \times E_d$ respecting the decomposition of X, i.e., such that $E_i \subseteq X_i$ holds for all $i \in [d]$. Obviously, the E_i then are uniquely determined by the decomposition $X = X_1 \times \cdots \times X_d$. Let us agree to call any set $E_1 \times \cdots \times E_d$ such that $E_i \subseteq X_i$ for all $i \in [d]$ a *box*.

For an arbitrary number r we denote by $\lceil r \rceil_2$ the smallest even integer not being smaller than r. We show

Theorem 3. *Let* $\mathcal{H} = (X, \mathcal{E})$ *be a hypergraph of* d*–dimensional boxes. Let* $X = X_1 \times \cdots \times X_d$ *be a corresponding decomposition. Set* $n := |X|$, $n_i := |X_i|$ *for* $i \in [d]$ *and* $m := |\mathcal{E}|$. *Then there are structured random colorings* $\chi : X \to \{-1, 1\}$ *having discrepancy at most*

$$\mathrm{disc}(\mathcal{H}, \chi) \le 2^{-\frac{d-1}{2}} \sqrt{\lceil n_1 \rceil_2 \cdots \lceil n_d \rceil_2 \ln(4m)}$$

with probability at least $\frac{1}{2}$. Generating these structured random colorings needs $2^{-d}n$ random bits.

Note that Theorem 1 using ordinary random colorings only proves a bound of $\sqrt{2n_1 \cdots n_d \ln(4m)} = \sqrt{2n \ln(4m)}$. This is worse by a factor of $2^{d/2}$ (in the case of even n_i).

Proof. Without loss of generality we may assume that $X_i = [n_i]$. We first consider the case that all $n_i, i = 1, \dots, d$, are even.

Set $\mathcal{P} := \{\{2x_1 - 1, 2x_1\} \times \cdots \times \{2x_d - 1, 2x_d\} \mid \forall i \in [d] : x_i \in [\frac{n_i}{2}]\}$, that is, we partition the vertex set into small cubes of size 2^d in a rather canonical way.

The coloring corresponding to each small cube shall be such that adjacent (in the Hamming distance sense) corners always receive opposite colors. More formally, for a given cube $P \in \mathcal{P}$ we define a coloring $\chi_P : P \to \{-1, 1\}$ by

$$\chi_P(x) = 1 \iff \sum_{i \in [d]} x_i \text{ is even}$$

for all $x = (x_1, \dots, x_d)$.

Let $E \in \mathcal{E}$ and $P \in \mathcal{P}$. As both E and P are boxes, so is $E \cap P$. From the definition of χ_P we see that any subbox S of P such that $|S| \neq 1$ fulfills $\chi(S) = 0$. Hence $|\chi_P(E \cap P)| \leq 1$ for all $E \in \mathcal{E}$ and $P \in \mathcal{P}$. We may therefore define random structured colorings with respect to $\mathcal{P}$ and the $\chi_P, P \in \mathcal{P}$ as introduced in Section 2. Let χ be such a coloring.

We have $|I(\mathcal{P}, E)| \leq |\mathcal{P}| = 2^{-d}n$. Applying Theorem 2 with $s_0 = 2^{-d}n$, we get the bound

$$\operatorname{disc}(\mathcal{H}, \chi) \leq 2^{-\frac{d-1}{2}} \sqrt{n \ln(4m)},$$

which finishes the proof in the case that all $n_i, i = 1, \dots, d$, are even.

For the general case we consider the hypergraph $\mathcal{H}_1 = ([\lceil n_1 \rceil_2] \times \cdots \times [\lceil n_d \rceil_2], \mathcal{E})$. Since $\mathcal{H}$ is a subhypergraph of $\mathcal{H}_1$, any coloring χ_1 for $\mathcal{H}_1$ by restriction yields a coloring $\chi = (\chi_1)_{|X}$ for $\mathcal{H}$. The claim follows from $\operatorname{disc}(\mathcal{H}, \chi) \leq \operatorname{disc}(\mathcal{H}_1, \chi_1)$ and applying the case of even cardinality sets to $\mathcal{H}_1$. □

Apart from this improved discrepancy bound, we also gained some information about the coloring itself. For example, all geometric boxes are colored very nicely. We call a box $B \subseteq X$ a *geometric box*, if it can be represented in the form $B = I_1 \times \cdots \times I_d$ for some intervals $I_i \subseteq [n_i], i \in [d]$. As can be seen easily, these boxes fulfill $|\chi(B)| \leq 2^d$ for any structured random coloring χ with respect to $\mathcal{P}$ and $\chi_P, P \in \mathcal{P}$.

Furthermore, our colorings are fair, that is, they are perfectly balanced on the whole vertex set. We have $\chi(X) = 0$, if $|X|$ is even, and $\chi(X) \in \{-1, 1\}$, if $|X|$

is odd (note that any odd cardinality set S cannot have discrepancy $\chi(S) = 0$, no matter what the coloring χ is like).

Fair colorings are important in recursive algorithms and divide-and-conquer procedures. The relation between combinatorial discrepancies and ε–approximations (and thus also the "transfer principle" connecting geometric and combinatorial discrepancies) rely on the concept of fair colorings. We refer to the first chapter of Matoušek [Mat99] for the details. Another example is the recursive method to construct balanced multi-colorings from 2–color discrepancy information (cf. [DS01]).

If $X \in \mathcal{E}$, then fairness can be obtained by recoloring some vertices in the larger color class. This increases the discrepancy by a factor of at most 2. With our structured random colorings, we can get fairness "for free".

To show how such structural knowledge about the random coloring can be used to reduce the number of relevant hyperedges, we examine a special class of box hypergraphs: The hypergraph of *all* d–dimensional boxes in $[n_0]^d$ for some $n_0 \in \mathbb{N}$ is $\mathcal{H}^d_{n_0} := ([n_0]^d, \{S_1 \times \cdots \times S_d | S_i \subseteq [n_0]\})$. The usual random colorings (Theorem 1) fulfill

$$\begin{aligned} \operatorname{disc}(\mathcal{H}^d_{n_0}) &\le \sqrt{2n_0^d \ln(4\, 2^{n_0 d})} \\ &= \sqrt{2\ln 2}\, n_0^{\frac{d+1}{2}} \sqrt{d}\,(1 + o(1)) \\ &\approx 1.18\, n_0^{\frac{d+1}{2}} \sqrt{d}\,(1 + o(1)) \end{aligned}$$

with probability at least $\frac{1}{2}$. In the following theorem we improve this bound and also show that less than $3^{n_0 d/2}$ of the $2^{n_0 d}$ hyperedges are relevant. For convenience let us concentrate on the case that n_0 is even. The general result can be obtained from similar reasoning as in the proof of Theorem 3.

Theorem 4. *Let $n_0, d \in \mathbb{N}$, n_0 even, $d \ge 2$ and set $n := n_0^d$. There are structured random colorings χ for $\mathcal{H}^d_{n_0}$ that have*

$$\operatorname{disc}(\mathcal{H}^d_{n_0}, \chi) \le 1.05\; 2^{-d/2} n_0^{\frac{d+1}{2}} \sqrt{d}$$

with probability at least $\frac{1}{2}$. Generating these colorings needs $2^{-d}n$ random bits. To compute their discrepancy, only $2^{-d}3^{n_0 d/2}$ hyperedges have to be regarded.

Proof. Set $\mathcal{P} := \{\{2x_1 - 1, 2x_1\} \times \cdots \times \{2x_d - 1, 2x_d\} \,|\, x_1, \cdots, x_d \in [\frac{n_0}{2}]\}$ and define $\chi_P, P \in \mathcal{P}$ as in the proof of Theorem 3. Let χ be a random coloring with respect to $\mathcal{P}$ and $\chi_P, P \in \mathcal{P}$. As above we have $|I(\mathcal{P}, E)| \le 2^{-d} n_0^d$.

Now let us bound the number of hyperedges that are relevant for the discrepancy of χ with respect to $\mathcal{H}$. We first compute $|\mathcal{E}_\mathcal{P}|$. Let $E = S_1 \times \cdots \times S_d$. Assume that for some $i \in [d]$ and $x \in [\frac{n_0}{2}]$ we have $\{2x - 1, 2x\} \subseteq S_i$. Then no box

$P = \{2x_1 - 1, 2x_1\} \times \cdots \times \{2x_d - 1, 2x_d\}$ such that $x_i = x$ intersects E in exactly one vertex. From some elementary properties of boxes and the definition of χ_P we derive $\chi_P(E \cap P) = 0$. Thus $E_{\mathcal{P}} = (S_1 \times \cdots \times (S_i \setminus \{2x-1, 2x\}) \times \cdots \times S_d)_{\mathcal{P}}$. By induction we see that $\pi : E \mapsto E_{\mathcal{P}}$ is a projection of $\mathcal{E}$ onto $\mathcal{E}$. Therefore we need to counts its fixed points only to get $|\mathcal{E}_{\mathcal{P}}|$. We just exhibited that a necessary (and sufficient) condition for a hyperedge $E = S_1 \times \cdots \times S_d$ to be a fixed point is

$$\forall i \in [d] \, \forall x \in [\tfrac{n_0}{2}] : |S_i \cap \{2x - 1, 2x\}| \leq 1.$$

For each $i \in [d], x \in [\frac{n_0}{2}]$ we therefore have exactly three possibilities: $S_i \cap \{2x - 1, 2x\}$ is empty or $\{2x - 1\}$ or $\{2x\}$. This makes $|\mathcal{E}_{\mathcal{P}}| = 3^{n_0 d/2}$ fixed points.

Still, not all hyperedges in $\mathcal{E}_{\mathcal{P}}$ are relevant. From the structure of χ we derive a further reduction: Note that for all $i \in [d]$,

$$\gamma_i : \mathcal{E} \to \mathcal{E}; S_1 \times \cdots \times S_i \times \cdots \times S_d \mapsto S_1 \times \cdots \times ([n_0] \setminus S_i) \times \cdots \times S_d$$

is a fixed-point-free bijection of $\mathcal{E}$ that leaves the set $\mathcal{E}_{\mathcal{P}}$ of reduced hyperedges invariant and preserves discrepancy: We have

$$\chi(E) = -\chi(\gamma_i(E))$$

for all hyperedges $E \in \mathcal{E}$. In particular, the group $\langle \gamma_1, \ldots, \gamma_d \rangle \simeq \mathbb{Z}_2^d$ acts on $\mathcal{E}$ and $\mathcal{E}_{\mathcal{P}}$ in such a way that all orbits have length 2^d. As all elements of an orbit have the same discrepancy with respect to χ, it is enough to consider just one representative from each orbit. Let $\mathcal{E}_0 \subseteq \mathcal{E}$ be system of representatives of the orbits in $\mathcal{E}_{\mathcal{P}}$, that is, $\mathcal{E}_0$ contains exactly one element of each orbit in $\mathcal{E}_{\mathcal{P}}$. Since $|\mathcal{E}_0| = 2^{-d}|\mathcal{E}_{\mathcal{P}}|$, we reduced the number of relevant hyperedges by another factor of 2^d. From Theorem 2 we finally get (with probability at least $\frac{1}{2}$)

$$\begin{aligned}
\operatorname{disc}(\mathcal{H}, \chi) &= \operatorname{disc}((X, \mathcal{E}_0), \chi) \\
&\leq \sqrt{2\, 2^{-d} n_0^d \ln(4\, 2^{-d} 3^{n_0 d/2})} \\
&\leq \sqrt{\ln 3}\, 2^{-d/2} n_0^{\frac{d+1}{2}} \sqrt{d} \\
&\leq 1.05\ 2^{-d/2} n_0^{\frac{d+1}{2}} \sqrt{d}.
\end{aligned}$$

□

We should remark that the size reduction yields a change in the order of magnitude in terms of d, namely the additional $2^{-d/2}$ factor, whereas counting the relevant edges (less than $(7/8)^n$ of the total number of edges) only improves the constant by about 11%. Recall however, that reducing the number of relevant hyperedges does reduce the complexity of checking whether a structured random coloring fulfills the discrepancy bound of the theorem or not.

4 Summary and Conclusion

In this paper we presented a new way of generating random colorings for the discrepancy problem of hypergraphs. This allows to use structural information about the hypergraph and thus

- improves discrepancy guarantees,
- allows to prescribe additional properties regarding the coloring, e.g. fairness,
- reduces the number of random bits needed to generate the coloring,
- reduces the number of relevant hyperedges, and thus reduces the complexity of computing the discrepancy of the random coloring and the expected complexity of computing a low-discrepancy random coloring.

Since generating random colorings for a discrepancy problem is equivalent to generating random roundings for the trivial solution of the corresponding ILP-relaxation, we believe that these methods can be applied to a broader range of ILPs as well.

References

[AS00] N. Alon and J. Spencer. *The Probabilistic Method.* John Wiley & Sons, Inc., 2nd edition, 2000.

[BF81] J. Beck and T. Fiala. "Integer making" theorems. *Discrete Applied Mathematics*, 3:1–8, 1981.

[BHK98] L. Babai, T. P. Hayes, and P. G. Kimmel. The cost of the missing bit: Communication complexity with help. In *Proceedings of the 30th STOC*, pages 673–682, 1998.

[BS84] J. Beck and J. Spencer. Integral approximation sequences. *Math. Programming*, 30:88–98, 1984.

[Cha00] B. Chazelle. *The Discrepancy Method.* Princeton University, 2000.

[Chv79] V. Chvátal. The tail of the hypergeometric distribution. *Discrete Math.*, 25:285–287, 1979.

[DS01] B. Doerr and A. Srivastav. Recursive randomized coloring beats fair dice random colorings. In A. Ferreira and H. Reichel, editors, *Proceedings of the 18th Annual Symposium on Theoretical Aspects of Computer Science (STACS) 2001*, volume 2010 of *Lecture Notes in Computer Science*, pages 183–194, Berlin–Heidelberg, 2001. Springer Verlag.

[LSV86] L. Lovász, J. Spencer, and K. Vesztergombi. Discrepancies of set-systems and matrices. *Europ. J. Combin.*, 7:151–160, 1986.

[Mat99] J. Matoušek. *Geometric Discrepancy.* Springer-Verlag, Berlin, 1999.

[Rot64] K. F. Roth. Remark concerning integer sequences. *Acta Arithmetica*, 9:257–260, 1964.

[Sri96] A. Srinivasan. An extension of the Lovász local lemma, and its applications to integer programming. In *Proceedings of the Seventh Annual ACM-SIAM Symposium on Discrete Algorithms (Atlanta, GA, 1996)*, pages 6–15, New York, 1996. ACM.

[Sri01] A. Srivastav. Derandomization in combinatorial optimization. In P. Pardalos, S. Rajasekaran, J. Reif, and J. D. P. Rolim, editors, *Handbook of Randomization*. Kluver, to appear in 2001.

[SSS95] J. P. Schmidt, A. Siegel, and A. Srinivasan. Chernoff-Hoeffding bounds for applications with limited independence. *SIAM J. Discrete Math.*, 8:223–250, 1995.

[Uhl66] W. Uhlmann. Vergleich der hypergeometrischen mit der Binomial-Verteilung. *Metrika*, 10:145–158, 1966.

[vdW27] B. L. van der Waerden. Beweis einer Baudetschen Vermutung. *Nieuw Arch. Wsk.*, 15:212–216, 1927.