Discrepancy and Halftoning

Warning: Discrete Maths ahead!

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1. Introduction to discrepancy theory.
2. Halftoning as linear discrepancy problem.
3. Dependent randomized rounding.

Introduction to Discrepancy Theory

Discrepancy Problems:

a) Geometric discrepancy problem.

b) Combinatorial discrepancy problem.

c) Linear discrepancy and rounding problems.

d) Game theoretic discrepancy problems: Liar games/noisy channels.

Literature:

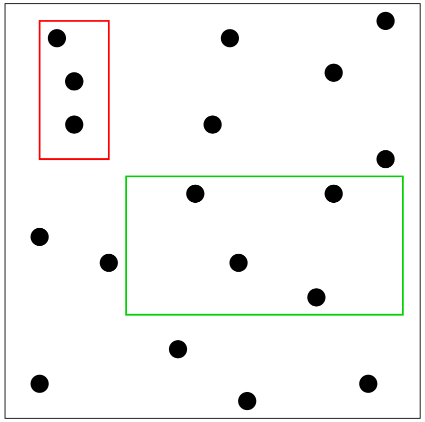
J. Beck, V. T. Sós, *Discrepancy Theory*, in *Handbook of Combinatorics*, 1995.

J. Matoušek, *Geometric Discrepancy*, 1999.

B. Chazelle, *The Discrepancy Method*, 2000.

Geometric Discrepancies

Distribute n points *evenly* in the d–dimensional unit cube:
Each axis-parallel rectangle R shall contain $n\,\mathrm{vol}(R)$ of the points.



$d = 2$
$n = |P| = 18$

$\mathrm{vol}(R_1) = \frac{2}{9}: \; |P \cap R| = 4 = n\,\mathrm{vol}(R_1)$
$\mathrm{vol}(R_2) = \frac{1}{18}: \; \big||P \cap R| - n\,\mathrm{vol}(R_2)\big| = |3 - 1| = 2$

Observation: A perfect solution does not exist.

$\Rightarrow$ Minimize the discrepancy $\mathrm{disc}(P) := \sup_R \big||P \cap R| - n\,\mathrm{vol}(R)\big|$.

Geometric Discrepancy — 2

Theorem:

$\forall P \subseteq [0,1]^d, |P| = n : \mathrm{disc}(P) = \Omega_d((\log n)^{\frac{d-1}{2}})$ [Roth (1954)]
$\exists P \subseteq [0,1]^d, |P| = n : \mathrm{disc}(P) = O_d((\log n)^{d-1})$ [Halton, Hammersley (1960)]

Numerical Integration: For given $f : [0,1]^d \to \mathbb{R}$, estimate the integral by $\frac{1}{n} \sum\limits_{p \in P} f(p)$ for some n–point set $P \subseteq [0,1]^d$.

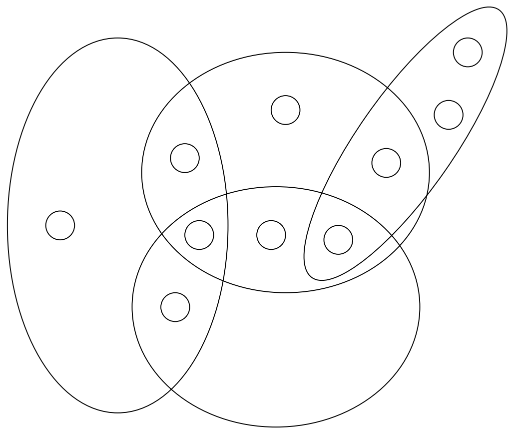
Koksma-Hlawka Inequality: $\left| \int\limits_{[0,1]^d} f(x)\, dx - \frac{1}{n} \sum\limits_{p \in P} f(p) \right| \leq \frac{1}{n}\, \mathrm{disc}(P)\, V(f)$.

Low-discrepancy point sets yield good estimates for the integral.

Combinatorial Discrepancy

Def: *Hypergraph* $\mathcal{H} = (V, \mathcal{E})$
– V finite set, 'vertices',
– $\mathcal{E} \subseteq 2^V$, '(hyper)edges'.

Color the vertices of a hypergraph with k colors in a balanced manner:
Each hyperedge E shall contain $\frac{1}{k}|E|$ vertices in each color.



$|V| = 10$ vertices
$|\mathcal{E}| = 4$ hyperedges

Find a good solution for $k = 2$*!*

Combinatorial Discrepancy

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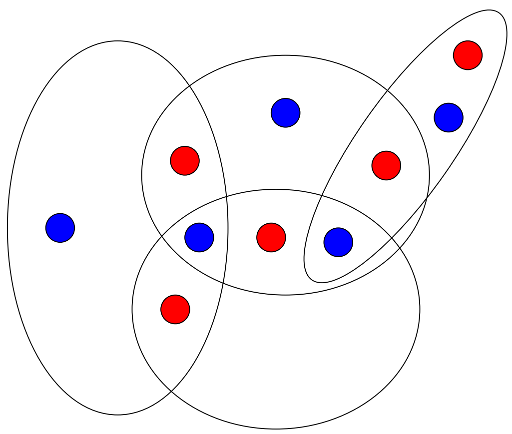

$|V| = 10$ vertices
$|\mathcal{E}| = 4$ hyperedges
$k = 2$ colors
Perfect solution: Each edge $E \in \mathcal{E}$ contains $\frac{1}{2}|E|$ red and blue vertices.

Combinatorial Discrepancy

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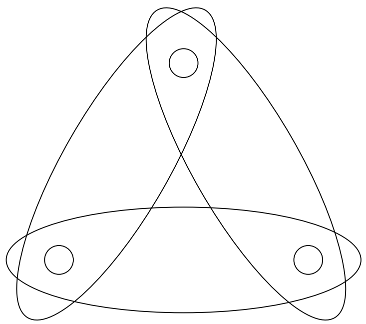

$|V| = 3$ vertices
$|\mathcal{E}| = 3$ hyperedges
$k = 2$ colors
No perfect solution exists!

Combinatorial Discrepancy — 2

Observation: In general, a perfect solution does not exist.

Combinatorial Discrepancy Problem: Find a coloring $\chi : V \to [k]$[1] minimizing

$$\mathrm{disc}(\mathcal{H}, \chi) \quad := \quad \max_{i \in [k]} \max_{E \in \mathcal{E}} \left| |\chi^{-1}(i) \cap E| - \tfrac{1}{k}|E| \right|. \quad \text{Discrepancy of } \chi$$

Theorem: For all $\mathcal{H}$ and k, there are $\chi : V \to [k]$ such that

$$\mathrm{disc}(\mathcal{H}, \chi) \quad = \quad O\Big(\sqrt{\tfrac{1}{k}|V| \log |\mathcal{E}|}\Big).$$

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[1] $[k] := \{1, \ldots, k\}$

Linear Discrepancy: Rounding Problems

Given: $x \in [0,1]^n$,
$A \in \mathbb{R}^{m \times n}$.

Find $y \in \{0,1\}^n$ such that $(Ay)_i \approx (Ax)_i$ for all $i \in [m]$. 'small rounding errors'

Linear Discrepancy of A: $\operatorname{lindisc}(A) := \max_{x \in [0,1]^n} \min_{y \in \{0,1\}^n} \|A(y-x)\|_\infty$.

Theorem[1] [2]**:** $\operatorname{lindisc}(A) \leq 2 \max_{x \in \{0,\frac{1}{2}\}^n} \min_{y \in \{0,1\}^n} \|A(y-x)\|_\infty$.

[1] Hidden in: Beck, Spencer (1984)

[2] Explicit in: Lovász, Spencer, Vesztergombi (1986)

Summary: Introduction to Discrepancy Theory

Discrepancy Problems:

a) Geometric: All rectangles shall contain the right number of points.

b) Combinatorial: Each hyperedge has the same number of vertices in each color.

c) Linear: All rounding errors shall be small.

Common Characteristics:

- Achieve several objectives simultaneously / in a balanced manner.
- Approximate something large/continuous through smaller/discrete objects ('quantization').

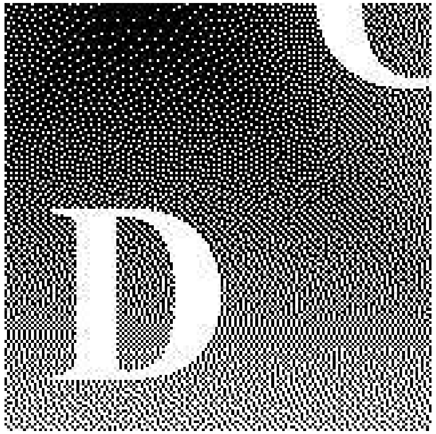
Discrepancy and Halftoning

1. Introduction to discrepancy theory
    - Different problems
    - Common aspects: Balance, quantization
2. Halftoning as linear discrepancy problem
    - Halftoning problem
    - Error measure
3. Dependent randomized rounding

Digital Halftoning Problem

Convert a continuous-tone image into a binary one that *looks similar*.


Input image


Output image

Input: $X = (x_{ij}) \in [0,1]^{m \times n}$ Continuous tone image
Output: $Y = (y_{ij}) \in \{0,1\}^{m \times n}$ Binary image
Aim: X and Y represent *similar* images.

Problem: What is similarity (in maths terms)?

Digital Halftoning as Linear Discrepancy Problem

Input: $X = (x_{ij}) \in [0,1]^{m \times n}$ Continuous tone image
Output: $Y = (y_{ij}) \in \{0,1\}^{m \times n}$ Binary image

Error measure: Average error in the 2×2 boxes.

$$d(X,Y) := \frac{1}{|\mathcal{R}|} \sum_{R \in \mathcal{R}} \left| \sum_{(i,j) \in R} (x_{ij} - y_{ij}) \right|,$$

where $\mathcal{R} := \big\{ \{i, i+1\} \times \{j, j+1\} \,\big|\, i, j \ldots \big\}$ is the set of all 2×2 boxes.

Claim[3]**:** Y is a good halftoning of X if $d(X,Y)$ is small.
$\Rightarrow$ Models the halftoning problem as L_1 linear discrepancy problem.

[3] T. Asano, N. Katoh, K. Obokata, and T. Tokuyama, SIAM J. Computing (2003).

Some Comments on this Error Measure

Average error in the 2×2 boxes: $d(X,Y) := \frac{1}{|\mathcal{R}|} \sum_{R \in \mathcal{R}} \left| \sum_{(i,j) \in R} (x_{ij} - y_{ij}) \right|$.

$\oplus$ General approach: Splits the problem into two.

- What is a good error measure?
- How to compute low-error approximations?

$\oplus$ This error measure: Simple $\Rightarrow$ hope that we can prove something.

$\ominus$ Will model the human visual system not perfectly.

Theorem [AMOT]: For any X, a Y such that $d(X,Y) \leq 0.75$ can be computed in quadratic time. [Lower bound: 0.5.]

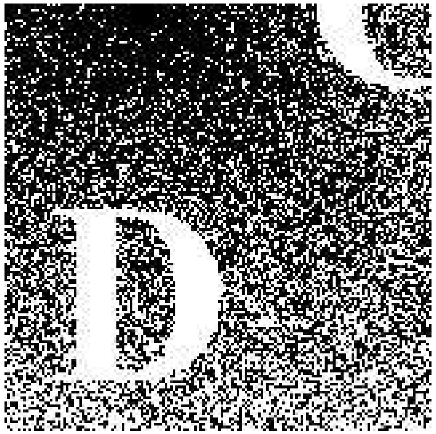
Discrepancy and Halftoning

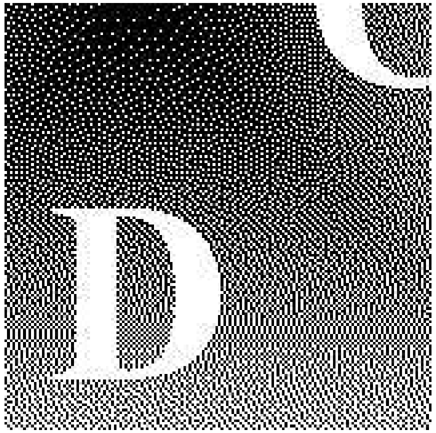
1. Introduction to discrepancy theory
2. Halftoning as linear discrepancy problem
   - Halftoning is a rounding problem
   - Error measure: Average rounding error in 2×2 boxes
     $\Rightarrow L_1$ linear discrepancy problem
3. Dependent randomized rounding
   - Randomized rounding (random dither) is unsuitable
   - Dependent randomized rounding does better.

Independent Randomized Rounding (Random Dither)

For each $i \in [m]$, $j \in [n]$ independently round with probabilities

$$\Pr(y_{ij} = 1) = x_{ij},$$
$$\Pr(y_{ij} = 0) = 1 - x_{ij}.$$


Independent RR


Error Diff.

Expected error: $Ed(X, Y) \leq 0.8294$.
Run-time: linear.

Why is Randomized Rounding so Bad?

An example: Let Y be an independent RR of $X = \begin{pmatrix} 0.4 & 0.4 \\ 0.4 & 0.4 \end{pmatrix}$.

$i = \Sigma Y$[1]	$\lvert\Sigma(X-Y)\rvert =: e_i$	$\Pr(\Sigma Y = i) =: p_i$	$p_i e_i$
0	1.6	0.1296	0.20736
1	0.6	0.3456	0.20736
2	0.4	0.3456	0.13824
3	1.4	0.1536	0.21504
4	2.4	0.0256	0.06144
Σ		1.0	0.82944

A 25% contribution to the expected error is caused by the possibility that we round *all* four entries down.

[1] $\Sigma Y := \sum_{i,j} y_{ij}$

Joint Randomized Roundings

Let $x = (x_1, \ldots, x_k) \in [0,1]^k$. Then $(y_1, \ldots, y_k)$ is a *joint randomized rounding* of x if:

- For all $i \in [k]$, y_i is a randomized rounding of x_i.
- $\sum_i y_i$ is a randomized rounding of $\sum_i x_i$:

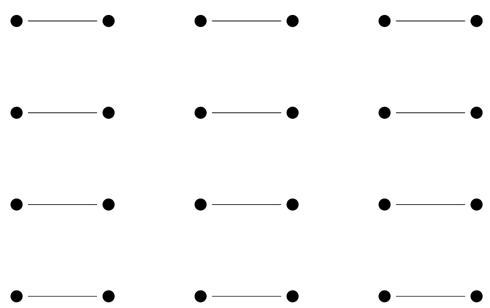
$$\begin{aligned} \Pr(\textstyle\sum_i y_i = \lfloor \sum_i x_i \rfloor + 1) &= \{\textstyle\sum_i x_i\}^{4} \\ \Pr(\textstyle\sum_i y_i = \lfloor \sum_i x_i \rfloor) &= 1 - \{\textstyle\sum_i x_i\}. \end{aligned}$$

Note: This is *dependent randomized rounding*!

4 $\{r\} := r - \lfloor r \rfloor$.

Simple Application to Our Rounding Problem

- Partition X into horizontal pairs,
- round each pair independently as joint RR.

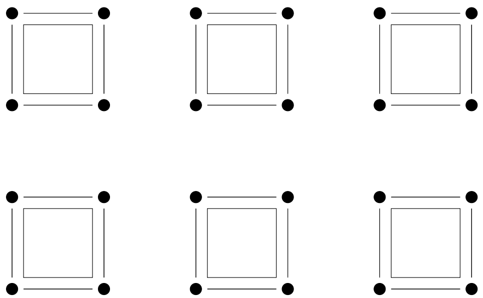


Expected error: $Ed(X,Y) \leq 0.7111$.
Run-time: linear.

More Dependencies

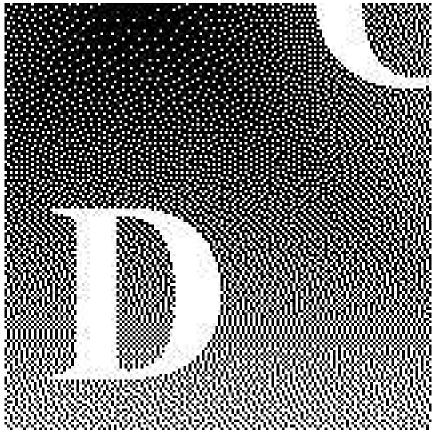
Partition X into 2×2 blocks. Round each block independently such that

- each horizontal and vertical pair in the block is a joint RR,
- the whole block is a joint RR.   5 dependencies per block

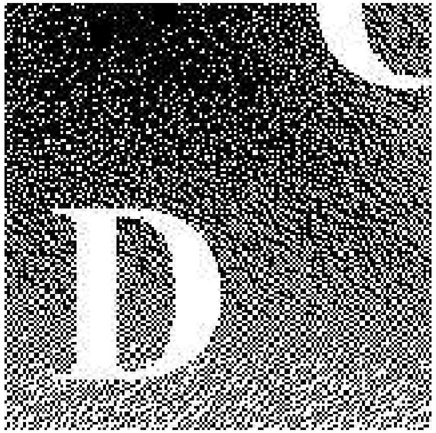


Expected error: $Ed(X, Y) \leq 0.63$.
Run-time: linear.

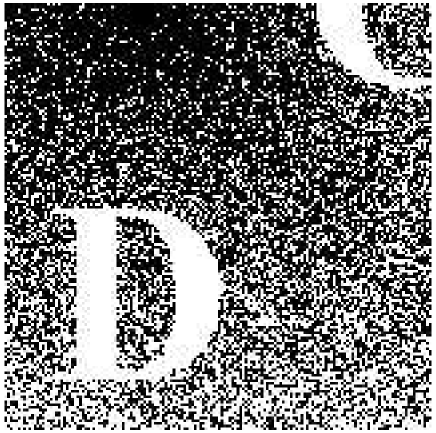
Images



Error Diffusion



Dependent RR



Independent RR

- Dependent RR is less grainy than independent RR.
- Dependent RR produces less artefacts than error diffusion.

Existence of Joint RRs: General Framework

Numbers $x_i \in [0,1], i \in I$
Hypergraph $\mathcal{H} = (I, \mathcal{E})$ "dependency hypergraph"
For $E \in \mathcal{E}$ put $x_E := \sum_{i \in E} x_i$.

Find random variables $y_i, i \in I$, such that "$\mathcal{H}$–randomized rounding"

y_i is a RR of x_i for all $i \in [n]$,

y_E is a RR of x_E for all $E \in \mathcal{E}$.

First matrix rounding example (pairs):

$$I = [m] \times [n]$$
$$\mathcal{E} = \{\{(i,j),(i,j+1)\} \,|\, i \in [m], j \in [n-1] \text{ odd}\}.$$

Existence of Joint RRs: Characterization

Def: A dependency hypergraph $\mathcal{H} = (I, \mathcal{E})$ is *realizable*, if for all $x_i \in [0,1], i \in I$, an $\mathcal{H}$–randomized rounding exists.

Existence of Joint RRs: Characterization

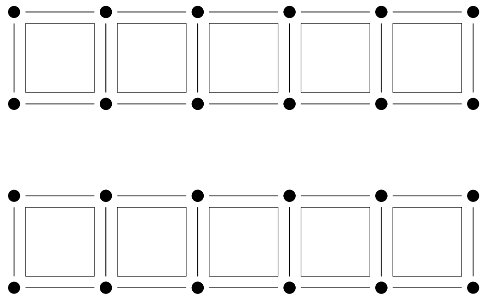
Def: A dependency hypergraph $\mathcal{H} = (I, \mathcal{E})$ is *realizable*, if for all $x_i \in [0,1], i \in I$, an $\mathcal{H}$–randomized rounding exists.

Theorem: $\mathcal{H}$ is realizable if and only if $\mathcal{H}$ is unimodular.

Def: $\mathcal{H}$ is *unimodular* iff each square submatrix of the incidence matrix of $\mathcal{H}$ has determinant -1, 0 or 1.

Corollary

- A rounding Y of X such that all $(y_{i,j}, y_{i,j+1}, y_{i+1,j}, y_{i+1,j+1})$, i odd, j arbitrary, are rounded as "blocks" is realizable.



- This Y satisfies $Ed(X,Y) \leq 0.5463$.[5]

- **Not a corollary (but true):** Such roundings can be computed in linear time.

[5] In fact, we have $Ed(X,Y) \leq 0.5368$. This is sharp for these dependencies. [H. Schnieder, Diploma thesis, Kiel 2003]

Summary

- Discrepancy Problems:
  - uniformly distributed point sets
  - balanced coloring of hypergraphs
  - rounding problems

- Halftoning: An L_1 linear discrepancy problem

- Dependent randomized rounding: Return of random dither?
  - Low avg. error in 2×2 boxes
  - Combines advantages of random dither (no artefacts) and error diffusion (low graininess).