Matrix (Quasi-)Rounding

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1. Matrix rounding problems
2. Rectangle rounding and Tusnády's problem
3. Quasi-rounding
4. An open problem

Matrix Rounding Problems

Given: $X \in \mathbb{R}^{m \times n}$,
$\mathcal{R} \subseteq 2^{[m] \times [n]}$ 'regions' in $[m] \times [n]$.[1]

Task: Find $Y \in \mathbb{Z}^{m \times n}$ such that

- $|x_{ij} - y_{ij}| < 1$ for all i, j 'Rounding'
- the rounding error $d(X, Y, R) = \left| \sum_{(i,j) \in R} (x_{ij} - y_{ij}) \right|$ is small for all $R \in \mathcal{R}$.

  $\Rightarrow$ Minimize $d(X, Y, \mathcal{R}) := \max_{R \in \mathcal{R}} \left| \sum_{(i,j) \in R} (x_{ij} - y_{ij}) \right|$.

Remark: Matrix rounding problem $=$ *Linear discrepancy* problem of $\mathcal{H} = ([m] \times [n], \mathcal{R})$.

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[1] $[n] := \{1, \ldots, n\}$.

Example [Baranyai '75]

Lemma: For any matrix X there is a rounding Y such that the rounding error in each row, each column and the whole matrix is less than one:

$$\left|\sum_j (x_{ij} - y_{ij})\right| < 1, \qquad \forall i,$$

$$\left|\sum_i (x_{ij} - y_{ij})\right| < 1, \qquad \forall j,$$

$$\left|\sum_i \sum_j (x_{ij} - y_{ij})\right| < 1.$$

Lemma was used to prove partitioning theorems for complete uniform hypergraphs:

- K_n^r is f–factorizable iff $r|fn$ and $\frac{fn}{r}|\binom{n}{r}$.
- Chromatic index $\chi'(K_n^r) = \left\lceil \binom{n}{r} / \left\lfloor \frac{n}{r} \right\rfloor \right\rceil$.

Matrix Rounding with Small Errors in Rectangles

Given: $X \in \mathbb{R}^{n \times n}$,[2]
$\mathcal{R} = \{I \times J \mid I, J \subseteq [n] \text{ intervals}\}$ 'regions' are rectangles.

Task: Find $Y \in \mathbb{Z}^{n \times n}$ such that

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- the rounding error $d(X,Y,R) = \left| \sum_{(i,j) \in R} (x_{ij} - y_{ij}) \right|$ is small for all $R \in \mathcal{R}$.

  $\Rightarrow$ Minimize $d(X,Y,\mathcal{R}) := \max_{R \in \mathcal{R}} \left| \sum_{(i,j) \in R} (x_{ij} - y_{ij}) \right|$.

What is $r_n = \sup_X \min_Y d(X,Y,\mathcal{R})$?

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[2] For simplicity, we assume $m = n$ in the remainder.

Simple Lower Bound: $\Omega(\log n)$ [3]

Let $X = \frac{1}{n}\mathbf{1}_{n\times n}$ [the $n \times n$ matrix with all entries $\frac{1}{n}$]

Lemma: For all $Y \in \{0,1\}^{n\times n}$, $d(X,Y,\mathcal{R}) = \Omega(\log n)$.

'Proof': Assume you have a better Y. Then

$$P = \{\tfrac{1}{n}(i,j) \,|\, y_{ij} = 1\}$$

is an $(n \pm \log n)$–point set in $[0,1]^2$ such that in each axis parallel rectangle $R \subseteq [0,1]^2$, $|P \cap R|$ deviates from the fair value $n\,\mathrm{vol}(R)$ by less than $\Omega(\log n)$.

Such a set P does not exist [Schmidt '72].

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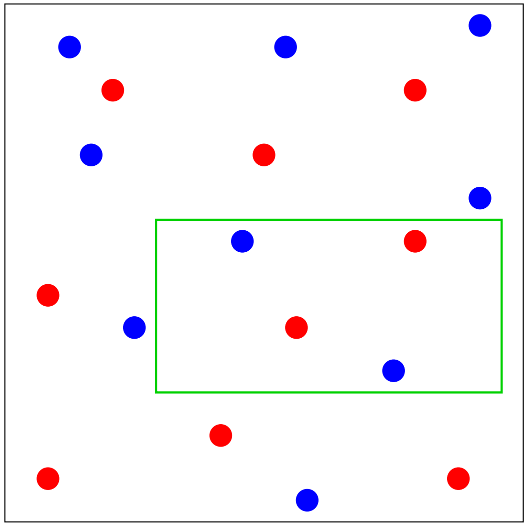
[3] For the discrepancy lovers, others please ignore this (and only this) slide.

Tusnády's Problem

Color n given points in $[0,1]^2$ with two colors such that the difference of red and blue points (discrepancy) in each axis-parallel rectangle is small.

Let t_n minimal s.t. this is possible for all $P \subset [0,1]^2$, $|P| = n$, with all discrepancies at most t_n.



Beck '81: $t_n = O(\log^4 n)$ *
Beck '89: $t_n = O(\log^{3.5+\varepsilon} n)$
Bohus '90: $t_n = O(\log^3 n)$ *
Matoušek ??: $t_n = O(\log^{2.5+\varepsilon} n)$
Srinivasan '97: $t_n = O(\log^{2.5} n)$
Beck '81 via Schmidt '72: $t_n = \Omega(\log n)$
(*: constructive)

Rectangle Rounding vs. Tusnády's Problem

Theorem [folklore?]: $t_n = \Theta(r_n)$.

Proof ingredients:

- Beck, Lovász, Spencer, Vesztergombi: Rounding arbitrary matrices yields at most twice the errors as rounding $\{0, \frac{1}{2}, 1\}$ matrices.
  [$\mathrm{herlindisc}(\mathcal{H}) \leq 2\,\mathrm{herdisc}(\mathcal{H})$]
- $t_n = \log^{\Theta(1)} n$.

Open Problem: Close the gap $r_n = \Theta(t_n) = \begin{cases} O(\log^{2.5} n) \\ \Omega(\log n) \end{cases}$.

Quasi-Rounding: Can we do better without rounding?

Theorem [D. 2004]:

- For any $X \in \mathbb{R}^{n \times n}$ there is a $Y \in \mathbb{Z}^{n \times n}$ such that $d(X, Y, \mathcal{R}) \leq 4 \log n$.
- Y can be chosen such that $|x_{ij} - y_{ij}| < 2$ for all $i, j \in [n]$. quasi-rounding
- Y can be computed in time $O(n^2 \log n)$.

Quasi-Rounding: Can we do better without rounding?

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- Y can be computed in time $O(n^2 \log n)$.

Consequence:
(1) $r_n = \Theta(t_n) = \Theta(\log n)$ is the right answer to Tusnády's problem *or*
(2) the relaxation from $|x_{ij} - y_{ij}| < 1$ to $|x_{ij} - y_{ij}| < 2$ makes "rounding" easier.

Answer: Two !

Theorem [D., Güntürk, Yılmaz 2006$^+$]:

- For any $X \in \mathbb{R}^{n\times n}$ there is a $Y \in \mathbb{Z}^{n\times n}$ such that $d(X, Y, \mathcal{R}) < \mathbf{2}$.
- Y can be chosen such that $|x_{ij} - y_{ij}| < 2$ for all $i, j \in [n]$. quasi-rounding
- Y can be computed in (linear) time $O(n^2)$.

Proof: One paragraph (next slide).

Proof: Any X can be quasi-rounded to Y with $d(X, Y, \mathcal{R}) < 2$

- Traverse X in any increasing order
  [(i, j) before (i', j') iff $i < i'$ or $j < j'$]

Proof: Any X can be quasi-rounded to Y with $d(X, Y, \mathcal{R}) < 2$

- Traverse X in any increasing order
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- Choose $y_{ij} \in \mathbb{Z}$ such that $-\frac{1}{2} \leq \sum_{a=1}^{i} \sum_{b=1}^{j} (x_{ab} - y_{ab}) < \frac{1}{2}$.

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- Result: Error at most $\frac{1}{2}$ in rectangles R containing $(1, 1)$:
  $d(X, Y, [1..i] \times [1..j]) \le \frac{1}{2}$ for all i, j.

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- Result: Error at most $\frac{1}{2}$ in rectangles R containing $(1,1)$:
  $d(X,Y,[1..i] \times [1..j]) \leq \frac{1}{2}$ for all i,j.
- An arbitrary rectangle $R \in \mathcal{R}$ can be written as union/difference of four such rectangles: $[a..i] \times [b..j] = [1..i] \times [1..j] \setminus [1..a-1] \times [1..j] \cup \ldots$.
  $\Rightarrow \quad -2 < \sum_{(a,b)\in R} (x_{ab} - y_{ab}) < 2$
  $\Leftrightarrow \quad d(X,Y,R) < 2$.

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- Traverse X in any increasing order
  [(i, j) before (i', j') iff $i < i'$ or $j < j'$]
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- Result: Error at most $\frac{1}{2}$ in rectangles R containing $(1, 1)$:
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  $\Rightarrow \quad -2 < \sum_{(a,b) \in R} (x_{ab} - y_{ab}) < 2$
  $\Leftrightarrow \quad d(X, Y, R) < 2$.
- Quasi-rounding: $|x_{ij} - y_{ij}| = d(X, Y, \{(i, j)\}) < 2$.

Summary and Discussion

- Rectangle rounding problem “equivalent” to Tusnády’s problem (and open):

$$r_n = \Theta(t_n) = \begin{cases} O(\log^{2.5} n) \\ \Omega(\log n) \end{cases}$$

- Rectangle quasi-rounding: For any $X \in \mathbb{R}^{n\times n}$ there is a $Y \in \mathbb{Z}^{n\times n}$ such that
  - $\left| \sum_{(i,j)\in R} (x_{ij} - y_{ij}) \right| < \mathbf{2}$ for all rectangles $R \in \mathcal{R}$,
  - $|x_{ij} - y_{ij}| < 2$ for all $i, j \in [n]$. quasi-rounding

$\Rightarrow$ Quasi-rounding can be significantly easier than rounding.

- Future work: Understand this phenomenon and/or exploit it!

An Open Problem

- Rounding with low error in 2×2 rectangles:
  - $\mathcal{R}_2 = \big\{\{i, i+1\} \times \{j, j+1\} \,\big|\, i, j \in [n-1]\big\}$.
  - $r^{(2)} = \sup_X \min_Y d(X, Y, \mathcal{R}_2)$.
- What is $r^{(2)}$?
  - Trivial: $r^{(2)} \leq 2$.
  - Asano, Matsui, Tokuyama (2000): $r^{(2)} \leq 1.75$.
  - Asano, Tokuyama (2001): $r^{(2)} \leq 1.66$.
  - D. (2005): $r^{(2)} \leq 1.5$.
  - Asano, Matsui, Tokuyama (2000): $r^{(2)} \geq 1$.