Generating Randomized Roundings with Cardinality Constraints and Derandomizations

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1. Introduction to randomized rounding and its derandomization.
2. Cardinality constraints.
3. Randomized roundings with cardinality constraints.

Randomized Rounding

- Let $x \in [0, 1]$.[1]
- A random variable y is a *randomized rounding of* x if

$$\begin{aligned} \Pr(y = 1) &= x, \\ \Pr(y = 0) &= 1 - x. \end{aligned}$$

- Note: $E(y) = x$. [In statistics: "Unbiased rounding"]

[1] Restriction to $[0, 1]$ for presentation purposes only. In general: y is a RR of x, if $y - \lfloor x \rfloor$ is a RR of $x - \lfloor x \rfloor$ as defined above.

Independent Randomized Rounding

- Let $x \in [0,1]^n$.
- $y = (y_1, \ldots, y_n)$ is an *independent randomized rounding of* x if y_j is a randomized rounding of x_j independently for all $j \in [n]$.
- If $A \in [0,1]^{m \times n}$, then
  - $E(Ay) = Ax$,
  - W. h. p.: $\forall i\colon\ |(Ay)_i - (Ax)_i| = O(\max\{\sqrt{(Ax)_i \log m}, \log m\})$.
    Chernoff bound
- This is classical randomized rounding
  - Raghavan, Thompson (1987, 1988)
  - Numerous applications since then

Cardinality Constraints

- Cardinality Constraint: $\sum_{i \in I} x_i = k \quad (k \in \mathbb{N})$
  Exactly k of the x_i, $i \in I$, shall become one.
- Occur frequently in applications:
  Example: "Choose exactly one path from s to t."
- Do not go well with independent RR:
  - $\Pr(\sum_{i \in I} y_i = \sum_{i \in I} x_i)$ can be as small as $|I|^{-1/2}$.
  - "Repeat RR until constraint satisfied" does not work (in addition to being inefficient).
- **Aim:** Generate RRs that satisfy cardinality constraints.

Applications of Hard Constraints

- Statistics: Controlled Rounding.
  - Round a table s.t. row/column totals are preserved.
- Algorithmics:
  - Max-Cut with given sizes of parts [Ageev, Sviridenko]
  - Integer splittable flow problem [Srinivasan]
  - Half-toning [D.]
- Discrete Mathematics:
  - Random Graphs with given degrees [Gandhi et al.]

RR with Constraints: Previous Results

Srinivasan (2001): Let $x \in [0,1]^n$.
Generate RR y of x such that $\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} x_i$.[1]
Time complexity: Linear.

Gandhi, Khuller, Parthasarathy, Srinivasan (2002):

- Bipartite graph $G = (V, E)$.
- Edge weights $x_e \in [0,1]$, $e \in E$.
- Generate RR y of x s.t. $\sum_{e \ni v} y_e = \sum_{e \ni v} x_e$[1] for all v.
- Time complexity: $O(|V||E|)$.

D. (2004): Dependent RR exist iff hard constraints form a totally unimodular system.

[1] Always assume x-sums to be integral.

New Result (STACS '06)

Let $x \in [0,1]^n$
$C \in \{0,1\}^{m \times n}$
$\mathcal{S} \subseteq 2^{[n]}$ closed under taking subsets.

Definition: y is an *independent-looking $(C, \mathcal{S})$-RR* of x if

- y is a randomized rounding of x (component-wise).
- Cy is a randomized rounding of Cx (component-wise).
- $\forall S \in \mathcal{S} \, \forall b \in \{0,1\} : \Pr(\forall j \in S : y_j = b) \leq \prod_{j \in S} \Pr(y_j = b)$.

[Negative correlation. Implies Chernoff bounds.]

New Result (STACS '06)

Theorem: You can generate independent-looking $(C, \mathcal{S})$-RRs for all $x \in [0, 1]^n$ if you can for all $x \in \{0, \frac{1}{2}\}^n$.

More precise: If all x_i have binary length at most ℓ, then an independent-looking $(C, \mathcal{S})$-RR of x can be generated from ℓ independent-looking $(C, \mathcal{S})$-RRs of some $\{0, \frac{1}{2}\}^n$.

Advantage: Often the $\{0, \frac{1}{2}\}$-case is very simple!

Example 1: Sequence Rounding

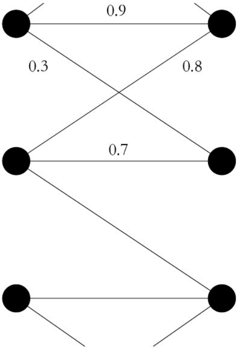
Problem: Let $x \in [0,1]^n$. Find a RR y of x such that for all intervals $I \subseteq [n]$, $\sum_{i \in I} y_i$ is randomized rounding of $\sum_{i \in I} x_i$ [no negative correlation required].

Solution: Solve the case $x \in \{0, \frac{1}{2}\}^n$ and use the theorem!

$$\begin{aligned} x &= (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, 0, \tfrac{1}{2}, 0, 0, \tfrac{1}{2}), \\ y^{(1)} &= (0,1,0,0,1,0,0,0), \\ y^{(2)} &= (1,0,1,0,0,0,0,1). \end{aligned}$$

Alternatingly, round up and down.
Choose one of these two solutions each with probability $\frac{1}{2}$.
[Try general case without reduction to $\{0, \frac{1}{2}\}$!]

2. Example: Bipartite Edge Weight Rounding



- Randomly round edge weights in a bipartite graph.
- Hard constraints: $\forall v \in V$ $\sum_{e \ni v} y_e$ is a RR of $\sum_{e \ni v} x_e$.

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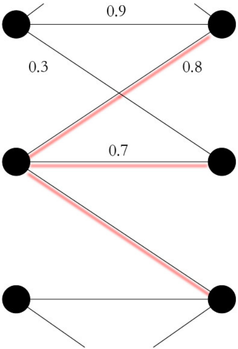
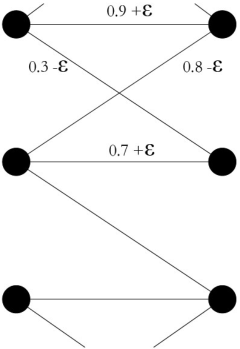


- Randomly round edge weights in a bipartite graph.
- Hard constraints: $\forall v \in V$ $\sum_{e \ni v} y_e$ is a RR of $\sum_{e \ni v} x_e$.
- Negative correlation on stars:

$$\mathcal{S} = \{S \subseteq E \big| |\bigcap S| = 1\}$$

Bipartite Edge Weight Rounding: GKPS's Solution



1. Find a simple cycle (ignore integral edges).
2. Find $\varepsilon^+ > 0$, $\varepsilon^- < 0$ such that an alternating ε-augmentation
   - keeps all weights in $[0, 1]$
   - moves at least one weight to 0 or 1.
3. With probability $\varepsilon^+/(\varepsilon^+ - \varepsilon^-)$ set $\varepsilon = \varepsilon^+$ else $\varepsilon = \varepsilon^-$.

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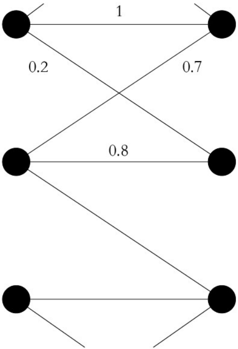

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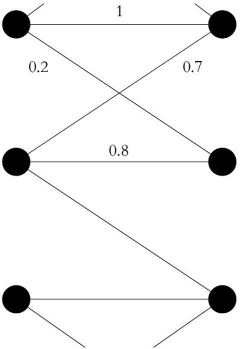

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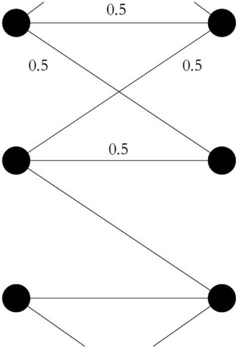
One iteration rounds one weight to an integer in time $O(|V|)$.
Total time: $O(|V||E|)$.
Plus: Difficult Proof!

Bipartite Edge Weight Rounding: $\{0, \frac{1}{2}\}$ Case



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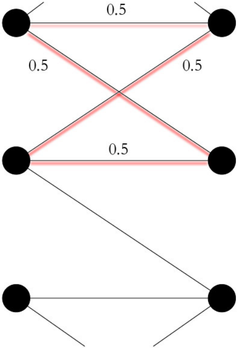

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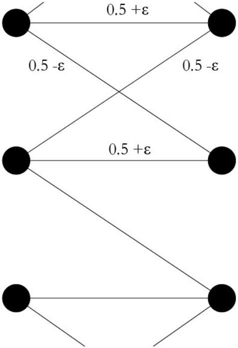

1. Find a simple cycle (ignore integral edges).
2. $\varepsilon^+ = \frac{1}{2}$, $\varepsilon^- = -\frac{1}{2}$.
3. With probability $\frac{1}{2}$ set $\varepsilon = \varepsilon^+$ else $\varepsilon = \varepsilon^-$.

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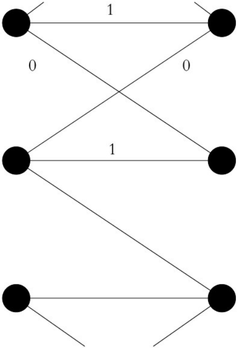

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2. $\varepsilon^+ = \frac{1}{2}$, $\varepsilon^- = -\frac{1}{2}$.
3. With probability $\frac{1}{2}$ set $\varepsilon = \varepsilon^+$ else $\varepsilon = \varepsilon^-$.
4. ε-augment the cycle [here: $\varepsilon = \varepsilon^+ = \frac{1}{2}$].

Each iteration rounds all weights in the cycle! Total time: $O(|E|)$.

Total time for bit-length–ℓ weights: $O(\ell|E|)$.

Summary

- If you can generate RRs with hard constraints and negative correlation for $\{0, \frac{1}{2}\}$ vectors, then for all vectors having finite binary length.
  - $\{0, \frac{1}{2}\}$–problem is simpler: Interval rounding.
  - $\{0, \frac{1}{2}\}$–problem is faster: Bipartite edge weight rounding.
- Not shown: The same is true for derandomizations. [Before, no derandomizations for Srinivasan's or GKPS's roundings.]
- Not shown: Similar results hold for $x \in \mathbb{Q}^n$ instead of finite bit-length. [STACS 2007 ?]