Optimization II **Winter 2009/10**

Lecture 7 — November 23

Lecturer: Khaled Elbassioni

7.1 A fast approximation scheme for maximum multicommodity flows

In the this lecture, we will show how to derive the result of Garg and Könemann [GK98] for multicommodity flows from the framework presented in the previous lecture.

Recall that the maximum multicommodity flow problem, on a graph $G = (V, E)$ with edge capacities $c_e > 0$, $e \in E$, can be cast as the following packing LP:

$$z^* = \max\{\sum_{I\in\mathcal{I}} f_I : \sum_{I\in\mathcal{I},\ e\in I} \frac{f_I}{c_e} \leq 1 \ \ \forall e \in E,\ f_I \geq 0 \ \ \forall I \in \mathcal{I}\},$$

where $\mathcal{I}$ is the set of paths between a source-sink pair, and f_I is a variable representing the flow on path I. We apply the framework of the previous lecture to obtain an algorithm for finding an ε-approximate solution of the above LP. Let $A \in \mathbb{R}_+^{E\times\mathcal{I}}$ be the constraint matrix, i.e., $a_{e,I} = \frac{1}{c_e}$ if edge $e \in E$ belongs to path $I \in \mathcal{I}$, and $a_{e,I} = 0$ otherwise. Then we can rewrite the above LP as $Z^* = \max\{e^T f \mid Af \leq 1,\ f \geq 0\}$.

When we attempt to apply the framework of the previous lecture, we face one problem: the number of columns corresponds to the number of paths in the graph, and hence could be exponential in general. To handle this, we keep track only of the paths with positive flows, and replace the operation of sampling from the columns by a call to a shortest path oracle. We will still maintain weights on the rows (which can be thought of as sampling probabilities), at time t, proportional to

$$p_e(t) = (1+\varepsilon)^{A_e F(t-1)} = (1+\varepsilon)^{\frac{Y_e(t-1)}{c_e}},$$

where A_e is the row corresponding to edge e in the matrix A, $F(t-1) \in \mathbb{R}_+^{\mathcal{I}}$ is the (not necessarily feasible) flow vector at time $t-1$, and $Y_e(t-1) \overset{\text{def}}{=} \sum_{t'=0}^{t-1}\sum_{I\in\mathcal{I},\ e\in I} F_I(t')$ is the total flow routed on edge e upto time $t-1$.

Denote by $\Delta F(t) \in \mathbb{R}_+^{\mathcal{I}}$ the change in the flow vector at time t. The following property will be crucial for the analysis:

$$\max_{e\in E}\{A_e \Delta F(t)\} = 1, \tag{7.1}$$

(recall conditions (i) and (iii) in the previous lecture) and hence will be maintained throughout the algorithm.

For a path $I \in \mathcal{I}$, we denote by $c_I = \min\{c_e : e \in I\}$ the bottleneck capacity on that path. The algorithm starts with $F_I(0) = 0$ for all $I \in cI$. As long as $M(t) \overset{\text{def}}{=} \max_{e\in E}\{\frac{Y_e(t)}{c_e}\} < T \overset{\text{def}}{=} \frac{\ln m}{\varepsilon^2}$, where $m = |E|$, we pick a path $I(t)$, shortest with respect to edge lengths

$$\ell^t(e) \overset{\text{def}}{=} \frac{p_e(t)}{|p(t)|c_e}, \text{ for } e \in E, \tag{7.2}$$

and push a flow of $c_{I(t)}$ on this path. This will guarantee (7.1), at any time t, since we essentially increase $F_{I(t)}(t-1)$ (and hence $Y_e(t-1)$ by $\delta(t) = c_{I(t)}$ for every $e \in I(t)$; this corresponds to setting $\delta(t) = \frac{1}{u_j}$ in Exercise 3 of the previous lecture). At the end we have to scale the flows by $M(t)$ to get a feasible flow. We are now ready to describe the algorithm.

Algorithm 1 FPTAS FOR MAXIMUM MULTICOMMODITY FLOWS

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1. $F_I(0) := 0$ for all $I \in \mathcal{I}$; $t := 0$; and $T := \frac{\ln m}{\varepsilon^2}$
2. while $M(t) < T$ do
3.     $t := t + 1$
4.     Let $I(t)$ be a shortest path in $G$ with edge lengths given by (7.2)
5.     $F_{I(t)}(t) := F_{I(t)}(t-1) + c_{I(t)}$;
6. return $f(t) = \frac{F(t)}{M(t)}$
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Theorem 7.1. *The above procedure terminates in at most mT iterations with a feasible flow. At termination, it holds that*

$$\sum_{I \in \mathcal{I}} f_I(t) \geq (1 - 2\varepsilon) z^*. \tag{7.3}$$

Proof: Feasibility is obvious since we scale by $\max_{e \in E}\{A_e F(t)/c_e\}$. Note also that the update step (5) implies that property (7.1) is satisfied, since $c_{I(t)}$ is the minimum capacity along path $I(t)$. This implies in particular the bound on the termination time, since any edge e can be the maximizer of $\frac{Y_e(t')}{c_e}$ for at most T iterations t', before termination.

To show (7.3), we analyze, as usual, the increase in the potential function $\Phi(t) = |p(t+1)|$:

$$\begin{aligned}
|p(t+1)| &= \sum_{e \in E} p_e(t+1) = \sum_{e \in I(t+1)} (1+\varepsilon)^{\frac{Y_e(t-1)}{c_e} + \frac{C_{I(t+1)}}{c_e}} + \sum_{e \notin I(t+1)} (1+\varepsilon)^{\frac{Y_e(t-1)}{c_e}} \\
&= \sum_{e \in I(t+1)} p_e(t)(1+\varepsilon)^{\frac{C_{I(t+1)}}{c_e}} + \sum_{e \notin I(t+1)} p_e(t) \\
&\leq |p(t)| \left(1 + \varepsilon \sum_{e \in I(t+1)} \frac{p_e(t) c_{I(t+1)}}{|p(t)| c_e} \right) \quad \text{(By Fact (I)-Exercise 2-Lecture 6)} \\
&\leq |p(t)| e^{\varepsilon \sum_{e \in I(t+1)} \frac{p_e(t) c_{I(t+1)}}{|p(t)| c_e}}. \qquad (7.4)
\end{aligned}$$

Iterating, we get

$$|p(t+1)| \leq |p(0)| e^{\varepsilon \sum_{t'=0}^{t-1} \sum_{e \in I(t'+1)} \frac{p_e(t') c_{I(t'+1)}}{|p(t')| c_e}} = |p(0)| e^{\varepsilon \sum_{t'=1}^{t} \ell(I(t')) c_{I(t')}},$$

where $\ell(I(t')) = \ell^{t'}(I(t'))$ is the length of the shortest path picked at time t' (with respect to edge lengths at time t'). This gives

$$(1+\varepsilon)^{\frac{Y_e(t)}{c_e}} \leq |p(0)| e^{\varepsilon \sum_{t'=1}^{t} \ell(I(t')) c_{I(t')}} \quad \text{for all } e \in E.$$

Taking logs and using $|p(0)| = m$, we conclude that

$$\frac{Y_e(t)}{c_e} \ln(1+\varepsilon) \leq \ln m + \varepsilon \sum_{t'=1}^{t} \ell(I(t'))c_{I(t')} \quad \text{for all } e \in E. \tag{7.5}$$

We will relate the flow $e^T F(t) = \sum_{t'=1}^{t} c_{I(t')}$ we obtain by the algorithm at time t to the optimal flow z^* by the following claim.

Claim 1. $\sum_{t'=1}^{t} \ell(I(t'))c_{I(t')} \leq \frac{e^T F(t)}{z^*}$.

Proof: Let $f^* \in \mathbb{R}_+^{\mathcal{I}}$ be a maximum multicommodity flow. Then by feasibility of f^*, we have $\sum_{I\in\mathcal{I},\, e\in I} \frac{f_I^*}{c_e} \leq 1$ for all $e \in E$, and taking convex combinations with multipliers $p_e(t')$, we get that

$$\sum_{e\in E} \frac{p_e(t')}{|p(t')|} \sum_{I\in\mathcal{I}: e\in I} \frac{f_I^*}{c_e} \leq 1,$$

which is equivalent to

$$\sum_{I\in\mathcal{I}} f_I^* \ell(I) \leq 1.$$

Since $I(t')$ is a shortest path at time t' we have that $\ell(I(t')) \leq \ell(I)$, for all $I \in \mathcal{I}$. Thus,

$$z^* \ell(I(t')) = \sum_{I\in\mathcal{I}} f_I^* \ell(I(t')) \leq \sum_{I\in\mathcal{I}} f_I^* \ell(I) \leq 1.$$

Multiplying both sides of this inequality by $c_{I(t')}$ and summing up over $t' \leq t$ finishes the proof. □

Using the above claim, we can deduce from (7.5) that

$$\frac{Y_e(t)}{c_e} \ln(1+\varepsilon) \leq \ln m + \varepsilon \frac{e^T F(t)}{z^*} \quad \text{for all } e \in E.$$

Dividing both sides by $M(t)$, arranging, and noting that at termination $M(t) \geq T$, we get that for all $e \in E$,

$$\frac{e^T F(t)}{M(t) z^*} \geq \frac{\ln(1+\varepsilon)}{\varepsilon} \cdot \frac{Y_e(t)}{M(t) c_e} - \frac{\ln m}{\varepsilon M(t)} \geq \frac{\ln(1+\varepsilon)}{\varepsilon} \cdot \frac{Y_e(t)}{M(t) c_e} - \frac{\ln m}{\varepsilon T}.$$

In particular for the edge e such that $M(t) = \frac{Y_e(t)}{c_e}$, we have at termination,

$$\frac{e^T F(t)}{M(t) z^*} \geq \frac{\ln(1+\varepsilon)}{\varepsilon} - \frac{\ln m}{\varepsilon T} = \frac{\ln(1+\varepsilon)}{\varepsilon} - \varepsilon \geq 1 - 2\varepsilon,$$

$$\frac{e^T F(t)}{M(t) z^*} \geq \frac{\ln(1+\varepsilon)}{\varepsilon} - \frac{\ln m}{\varepsilon T}.$$

Using $T = \frac{\ln m}{\varepsilon^2}$, we finally get

$$\frac{e^T f(t)}{z^*} \geq \frac{\ln(1+\varepsilon)}{\varepsilon} - \varepsilon \geq 1 - 2\varepsilon,$$

where the last inequality follows from Fact (IV) of Exercise 2 of the previous lecture. □

Bibliography

[GK98] Naveen Garg and Jochen Könemann. Faster and simpler algorithms for multicommodity flow and other fractional packing problems. In *39th Annual IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 300–309, 1998.