

Computer-generated theorems on Nash-solvability of bimatrix games based on excluding certain 2×2 subgames^{*}

Endre Boros¹, Khaled Elbassioni², Vladimir Gurvich¹, Kazuhisa Makino³, and Vladimir Oudalov¹

¹ RUTCOR, Rutgers University, 640 Bartholomew Road, Piscataway NJ 08854-8003
{boros,gurvich,voudalov}@rutcor.rutgers.edu

² Max-Planck-Institut für Informatik, Stuhlsatzenhausweg 85, 66123 Saarbrücken, Germany;
elbassio@mpi-sb.mpg.de

³ Graduate School of Information Science and Technology, University of Tokyo, Tokyo, 113-8656, Japan; makino@mist.i.u-tokyo.ac.jp

Abstract. In 1964 Shapley observed that a matrix has a saddle point whenever every 2×2 submatrix of it has one. In contrast, a bimatrix game may have no Nash equilibrium (NE) even when every 2×2 subgame of it has one. Nevertheless, Shapley's claim can be generalized for bimatrix games in many ways as follows. We partition all 2×2 bimatrix games into fifteen classes $S = \{c_1, \dots, c_{15}\}$ depending on the preference pre-orders of the two players. A subset $t \in S$ is called a NE-theorem if a bimatrix game has a NE whenever it contains no subgame from t . We suggest a general method for getting all minimal (that is, strongest) NE-theorems based on the procedure of joint generation of transversal hypergraphs given by a special oracle. By this method we obtain all (six) minimal NE-theorems.

1 Introduction, main concepts and results

1.1 Bimatrix games and Nash equilibria

Let X_1 and X_2 be finite sets of strategies of players 1 and 2. Pairs of strategies $x = (x_1, x_2) \in X_1 \times X_2 = X$ are called *situations*. A *bimatrix game* $U = (U_1, U_2)$ is a pair of real-valued matrices $U_i : X \rightarrow \mathbb{R}$, $i = 1, 2$, with common set of entries X . Value $U_i(x)$ is interpreted as utility function (also called profit or payoff) of player $i \in \{1, 2\}$ in the situation x . A situation $x = (x_1, x_2) \in X_1 \times X_2 = X$ is called a Nash equilibrium (NE) if

$$U_1(x'_1, x_2) \leq U_1(x_1, x_2) \quad \forall x'_1 \in X_1 \quad \text{and} \quad U_2(x_1, x'_2) \leq U_2(x_1, x_2) \quad \forall x'_2 \in X_2;$$

in other words, if no player can make a profit by choosing a new strategy if the opponent keeps the old one. A bimatrix game U is called a *zero sum or matrix game* if $U_1(x) + U_2(x) = 0$ for every $x \in X$. In this case the game is well-defined by one of two matrices, say, by U_1 , and a NE is called a *saddle point* (SP).

^{*} The second author gratefully acknowledges the partial support of DIMACS, the NSF Center for Discrete Mathematics and Theoretical Computer Science

1.2 Locally minimal SP-free matrix and NE-free bimatrix games

Standardly, we define a subgame as the restriction of U to a subset $X' = X'_1 \times X'_2 \subseteq X_1 \times X_2 = X$, where $X'_1 \subseteq X_1$ and $X'_2 \subseteq X_2$. In 1964 Shapley [8] noticed that a matrix has a saddle point whenever each of its 2×2 submatrices has one. Obviously, in this case, every submatrix has a SP, too. In other words, all minimal SP-free matrices are of size 2×2 . Moreover, all locally minimal SP-free matrices are of size 2×2 , too; in other words, every SP-free matrix of larger size has a row or column whose elimination still results in an SP-free submatrix; see [1]. Other generalizations of Shapley's theorem can be found, for example, in [6, 7]. Let us also notice that a 2×2 matrix has no SP if and only if one of its diagonals is strictly larger than the other.

The "naive generalization" of Shapley's claim to bimatrix games fails: a 3×3 game might have no NE even if each its 2×2 subgame has one; moreover, for each $n \geq 3$ a $n \times n$ bimatrix game might have no NE even if every its subgame has one; see Example 1 in [6] or [1] and also examples given below. However, all locally minimal NE-free games admit the following explicit characterization [1].

For the sake of brevity, let us denote situation (x_1^i, x_2^j) by $x_{i,j}$, where $X_1 = \{x_1^1, x_1^2, \dots\}$ and $X_2 = \{x_2^1, x_2^2, \dots\}$.

Given an integer $n \geq 2$ and a bimatrix game U with $|X_1| \geq n$ and $|X_2| \geq n$, let us say that U has the *canonical strong improvement n -cycle* C_n^0 if each situation $x_{1,1}, x_{2,2}, \dots, x_{n-1,n-1}, x_{n,n}$ (respectively, $x_{1,2}, x_{2,3}, \dots, x_{n-1,n}, x_{n,1}$) is a unique largest in its row with respect to U_2 (in its column with respect to U_1) and is the second largest, not necessarily, unique, in its column with respect to U_1 (in its row with respect to U_2). Any other strong improvement n -cycle C_n is obtained from the canonical one C_0 by arbitrary permutations of the rows of X_1 and columns of X_2 .

It is easy to see that if an $n \times n$ bimatrix game U has a strong improvement cycle then U has no NE, yet, every proper subgame obtained from U by elimination of either one row or one column has a NE. In other words, U is a *locally minimal* NE-free bimatrix game. Moreover, the inverse holds, too.

Theorem 1. ([1]). *A bimatrix game U is a locally minimal NE-free game if and only if U is of size $n \times n$ for some $n \geq 2$ and it contains a strong improvement n -cycle.* \square

Thus, locally minimal NE-free games can be arbitrary large. Several examples are given in Figures 2 - 6, where each game has the canonical strong improvement cycle. Although it seems difficult to characterize or recognize the minimal NE-free games (see [1]), yet, the above characterization of the locally-minimal ones will be sufficient for us.

1.3 Pre-orders

Given a set Y and a mapping $P : Y^2 \rightarrow \{<, >, =\}$ that assigns one of these three symbols to every ordered pair $y, y' \in Y$, we say that y is *less or worse* than y' if $y < y'$, respectively, y is *more or better* than y' if $y > y'$, and finally, y and y' are *equivalent or they make a tie* if $y = y'$. Furthermore, P is called a *pre-order* if the following standard properties (axioms) hold for all $y, y', y'' \in Y$:

symmetry: $y < y' \Leftrightarrow y' > y$, $y = y' \Leftrightarrow y' = y$, and $y = y$;

transitivity: $y < y' \ \& \ y' < y'' \Rightarrow y < y''$, $y < y' \ \& \ y' = y'' \Rightarrow y < y''$,
 $y = y' \ \& \ y' < y'' \Rightarrow y < y''$, $y = y' \ \& \ y' = y'' \Rightarrow y = y''$,

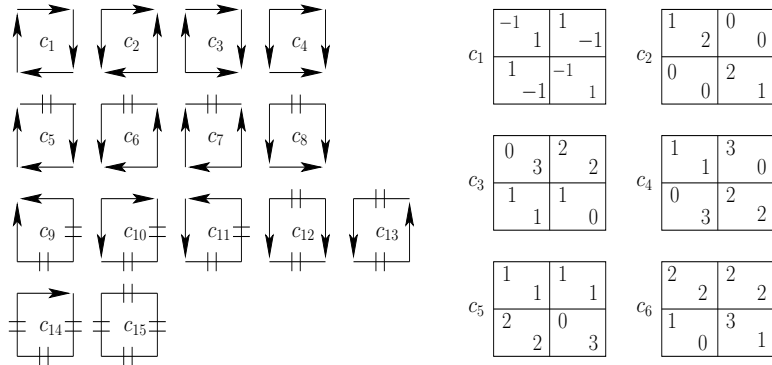


Fig. 1. Fifteen 2-squares.

A pre-order without ties is called a (*linear or complete*) *order*.

We use standard notation: $y \leq y'$ if $y < y'$ or $y = y'$ and $y \geq y'$ if $y > y'$ or $y = y'$. Obviously, transitivity and symmetry still hold:

$$\begin{aligned} y \leq y' \ \& \ y' < y'' \Rightarrow y < y'', \quad y < y' \ \& \ y' \leq y'' \Rightarrow y < y'', \\ y \leq y' \ \& \ y' \leq y'' \Rightarrow y \leq y'', \quad \text{and } y \leq y' \Leftrightarrow y' \geq y. \end{aligned}$$

In Figures 1-6 we use the following notation: an arrow from y to y' for $y > y'$, a line with two dashes for $y = y'$, and an arrow with two dashes for $y \geq y'$.

1.4 Configurations; fifteen 2-squares

Let us notice that to decide whether a situation $x = (x_1, x_2) \in X_1 \times X_2 = X$ is a NE in U , it is sufficient to know only two pre-orders: in the row x_1 with respect to U_2 and in column x_2 with respect to U_1 .

Given X_1 and X_2 , let us assign a pre-order P_{x_i} over X_{3-i} to each $x_i \in X_i$; $i = 1, 2$, and call the obtained preference profile $P = \{P_{x_1}, P_{x_2} \mid x_1 \in X_1, x_2 \in X_2\}$ a *configuration or bi-pre-order*.

Naturally, every bimatrix game $U = (U_1, U_2)$ defines a unique configuration $P = P(U)$, where P_{x_i} is the pre-order over X_{3-i} defined by U_i ; $i = 1, 2$. Clearly, each configuration is realized by infinitely many bimatrix games. Yet, it is also clear that to get all NE in game U it is enough to know its configuration $P(U)$.

For brevity, we will refer to a 2×2 configuration as a *2-square*. Up to permutations and transpositions, there exist fifteen 2-squares. They are listed in Figure 1 together with the corresponding bimatrix games. Four 2-squares c_1, c_2, c_3, c_4 have no ties; another four, c_5, c_6, c_7, c_8 and the next five, $c_9, c_{10}, c_{11}, c_{12}, c_{13}$, have, respectively, one and two ties each; finally, c_{14} and c_{15} have 3 and 4 ties.

Fifteen 2-squares have 0, 2, 1, 1, 1, 2, 1, 2, 3, 2, 2, 2, 3, and 4 NE, respectively. Thus, only c_1 has no NE. Shapley's theorem asserts that each c_1 -free zero-sum game (or configuration) has a NE. Let us note that 2-squares $c_1 - c_6$ are frequent in the literature. For example, the non-zero-sum bimatrix games realizing c_2 and c_4 may represent classical "family dispute" and "prisoner's dilemma"; respectively, c_5 and c_6 illustrate the concepts of the "promise" and "threat".

1.5 Dual or transversal hypergraphs

Let C be a finite set whose elements we denote by $c \in C$. A *hypergraph* H (on the ground set C) is a family of subsets $h \subseteq C$ that are called the *edges* of H . A hypergraph H is called *Sperner* if containment $h \subseteq h'$ holds for no two distinct edges of H . Given two hypergraphs T and E on the common ground set C , they are called *transversal* or *dual* if the following properties hold:

- (i) $t \cap e \neq \emptyset$ for every $t \in T$ and $e \in E$;
- (ii) for every subset $t' \subseteq C$ such that $t' \cap e \neq \emptyset$ for each $e \in E$ there exists an edge $t \in T$ such that $t \subseteq t'$;
- (iii) for every subset $e' \subseteq C$ such that $e' \cap t \neq \emptyset$ for each $t \in T$ there exists an edge $e \in E$ such that $e \subseteq e'$.

Property (i) means that edges of E and T are transversal, while (ii) and (iii) mean that T contains all minimal transversals to E and E contains all minimal transversals to T , respectively. It is well-known, and not difficult to see, that (ii) and (iii) are equivalent whenever (i) holds. Although for a given hypergraph \mathcal{H} there exist infinitely many dual hypergraphs, yet, only one of them, which we will denote by \mathcal{H}^d , is Sperner. Thus, within the family of Sperner hypergraphs duality is well-defined; moreover, it is an involution, that is, equations $T = E^d$ and $E = T^d$ are equivalent. It is also easy to see that dual Sperner hypergraphs have the same set of elements. For example, the following two hypergraphs are dual:

$$E' = \{(c_1), (c_2, c_3), (c_5, c_9), (c_3, c_5, c_6)\}, \quad (1)$$

$$T' = \{(c_1, c_2, c_5), (c_1, c_3, c_5), (c_1, c_2, c_6, c_9), (c_1, c_3, c_9)\}; \quad (2)$$

as well as the following two:

$$E = \{(c_1), (c_2, c_3), (c_5, c_9), (c_3, c_5, c_6), (c_2, c_4, c_5, c_6)\}, \quad (3)$$

$$T = \{(c_1, c_2, c_5), (c_1, c_3, c_5), (c_1, c_2, c_3, c_9), (c_1, c_2, c_6, c_9), (c_1, c_3, c_4, c_9), (c_1, c_3, c_6, c_9)\}. \quad (4)$$

1.6 Hypergraphs of examples and theorems

Let $C = \{c_1, \dots, c_{15}\}$. We call a subset $e \subseteq C$ a *NE-example* if there is a NE-free configuration P such that e is the set of types of 2-squares in P ; respectively, a subset $t \subseteq C$ is called a *NE-theorem* if a configuration has a NE whenever it contains no 2-squares from t . Obviously, $e \cap t \neq \emptyset$ for any NE-example e and NE-theorem t , since otherwise e is a counterexample to t . Moreover, it is well-known and easy to see that the hypergraphs of all inclusion-minimal (that is, strongest) NE-examples E_{NE} and NE-theorems T_{NE} are transversal. Let us consider c_1 and four configurations in Figure 2. It is easy to verify that all five contain canonical strong cycles and hence, they are locally minimal (in fact, minimal) NE-free configurations. These five configurations are chosen because they contain few types of 2-squares; the corresponding sets are given in Figure 2; they form the hypergraph E defined by (3). Figure 2 shows that each edge of E is a NE-example.

Let us consider the dual hypergraph T given by (4). We will prove that every edge $t \in T$ is a NE-theorem, thus, showing that the “research is complete”, that is, $E = E_{NE}$ and $T = T_{NE}$ are the hypergraphs of all strongest NE-examples and theorems.

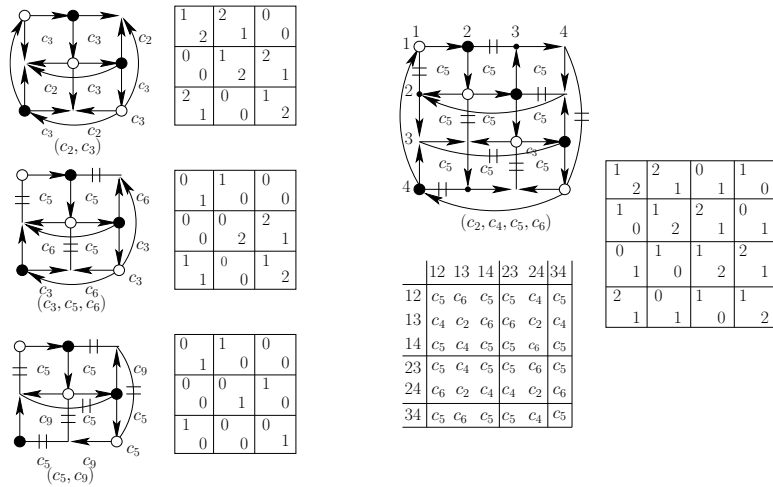


Fig. 2. NE-examples.

Remark 1. Given a family of NE-examples E' , the dual hypergraph T' should be viewed as a hypergraph of conjectures rather than theorems. Indeed, some inclusion-minimal examples might be missing in E' ; moreover, some examples of E' might be reducible. In this case some conjectures from the dual hypergraph $T' = E'^d$ will fail, being too strong. For instance, let us consider E' given by (1) in which the NE-example (c_2, c_4, c_5, c_6) is missing. (In fact, it is not that easy to obtain a minimal 4×4 example without computer.) Respectively, conjecture (c_1, c_3, c_9) appears in $T' = E'^d$. This conjecture is too strong, so it fails. In $T = T_{NE}$ we substitute for it three weaker (but correct) NE-theorems (c_1, c_3, c_9, c_2) , (c_1, c_3, c_9, c_4) , and (c_1, c_3, c_9, c_6) . Thus, if it seems too difficult to prove a conjecture, one should look for new examples.

1.7 Joint generation of examples and theorems

Of course, this approach can be applied not only to NE-free bimatrix games.

In general, given a set of objects Q (in our case, configurations), list C of subsets (properties) $Q_c \subseteq Q$, $c \in C$ (in our case, c -free configurations), the target subset $Q_0 \subseteq Q$ (configurations that have a NE), we introduce a pair of hypergraphs $E = E(Q, Q_0, C)$ and $T = T(Q, Q_0, C)$ (examples and theorems) defined on the ground set C as follows:

- (i) every set of properties assigned to an edge $t \in T$ (a theorem) implies Q_0 , that is, $q \in Q_0$ whenever q satisfies all properties of t , or in other words, $\bigcap_{c \in t} Q_c \subseteq Q_0$; in contrast,
- (ii) each set of properties corresponding to the complement $C \setminus e$ of an edge $e \in E$ (an example) does not imply Q_0 , i.e., there is an object $q \in Q \setminus Q_0$ satisfying all properties of $C \setminus e$, or in other words, $\bigcap_{c \notin e} Q_c \not\subseteq Q_0$.

If hypergraphs E and T are dual then we can say that “our understanding of Q_0 in terms of C is perfect”, that is, every new example $e' \subset C$ (theorem $t' \subseteq C$) is a superset of some old example $e \in E$ (theorem $t \in T$).

Without loss of generality we can assume that examples of $e \in E$ and theorems $t \in T$ are inclusion-wise minimal in C ; or in other words both hypergraphs E and T are Sperner.

Given Q, Q_0 and C , we try to generate hypergraphs E and T jointly [5]. Of course, the oracle may be a problem: Given a subset $C' \subseteq C$, it may be difficult to decide whether C' is a theorem (i.e., if $q \in Q_0$ whenever q satisfies all properties of C') or an example (i.e., if there is a $q \in Q \setminus Q_0$ satisfying all properties of $C \setminus C'$). However, the stopping criterion, $E^d = T$, is well-defined and, moreover, it can be verified in quasi-polynomial time [3].

Remark 2. Let us notice that containment $\bigcap_{c \in t} Q_c \subseteq Q_0$ might be strict. In other words, theorem t gives sufficient but not always necessary conditions for $q \in Q_0$. We can also say that theorems $t \in T$ give all optimal “inscribed approximations” of $Q_0 \subseteq Q$ in terms of C .

Remark 3. In [4], this approach was illustrated by a simple model problem in which Q is the set of 4-gons, Q_0 is the set of squares, C is a set of six properties of a 4-gon. Two dual hypergraphs of all minimal theorems T and examples E were constructed. In [2], the same approach was applied to a more serious problem related to families of Berge graphs.

1.8 Strengthening NE-theorems; main results

We will prove all six NE-theorems $t \in T_{NE}$. Formally, they cannot be strengthened, since t' is not a NE-theorem whenever $t' \subset t \in T_{NE}$ and the containment $t' \subset t$ is strict. Still, we can get stronger claims in slightly different terms.

Let us notice that for any t the class of t -free configurations (games) is hereditary. Indeed, if a configuration (game) is t -free then every subconfiguration (subgame) of it is t -free, too. Hence, we can restrict ourselves by the locally minimal NE-free examples, which are characterized by Theorem 1.

Now, let us consider NE-theorems (c_1, c_2, c_5) , (c_1, c_3, c_5) , and (c_1, c_2, c_6, c_9) . Formally, since 2-square c_1 has no NE, it must be eliminated. Yet, in a sense, it is the only exception. More precisely, we can strengthen the above three NE-theorems as follows.

Theorem 2. *The 2-square c_1 is a unique locally minimal NE-free configuration that is also (c_2, c_5) - or (c_3, c_5) -, or (c_2, c_6, c_9) -free.*

Furthermore, theorems (c_1, c_3, c_9, c_2) , (c_1, c_3, c_9, c_4) , (c_1, c_3, c_9, c_6) can be strengthened, too. In fact, we will characterize explicitly the configurations that are locally minimal NE-free and also (c_3, c_9) -free. This family is sparse but still infinite. In particular, we obtain the following result. Let $C(P)$ denote the set of all types of 2-squares of configuration P ; furthermore, let $C' = \{c_2, c_4, c_5, c_6, c_7, c_8, c_{13}, c_1\}$ and $C'' = C' \cup \{c_{12}\}$.

Theorem 3. *Let P be a locally minimal NE-free $n \times n$ configuration that is also (c_3, c_9) -free. Then*

- (i) n is even unless $n = 1$; (ii) if $n = 2$ then P is c_1 ;
- (iii) if $n = 4$ then P is a unique (c_2, c_4, c_5, c_6) -configuration in Figure 2;
- (iv) if $n = 6$ then $C(P) = C'$;
- (v) if $n = 8$ then $C' \subseteq C(P) \subseteq C''$ and there exist P with $C(P) = C'$;
- (vi) finally, if $n \geq 10$ is even then $C(P) = C''$.

It is clear that this statement implies the remaining three NE-theorems: (c_1, c_3, c_9, c_2) , (c_1, c_3, c_9, c_4) , and (c_1, c_3, c_9, c_6) .

2 Proof of Theorems 2 and 3

As we already mentioned, we can restrict ourselves to the locally minimal NE-free configurations. By Theorem 1, each such configuration P is of size $n \times n$ for some $n \geq 2$ and P contains a strong improvement cycle C_n . Without loss of generality we can assume that $C_n = C_n^0$ is canonical. In particular,

$$x_{i,i+1} \geq x_{i,j}, \quad x_{i,i+1} > x_{j,i+1}, \quad \text{for } j \neq i, \quad x_{j,j} \geq x_{i,j}, \quad x_{j,j} > x_{j,i+1}, \quad \text{for } j \neq i+1. \quad (5)$$

Furthermore, if $n = 2$ then 2-square c_1 is a unique NE-free configuration (in fact, c_1 is a strong 2-cycle). Hence, we will assume that $n \geq 3$. Additionally, we assume that P is t -free and consider successively the following subsets $t : (c_2, c_5), (c_3, c_5), (c_2, c_6, c_9)$, and (c_3, c_9) . Theorem 2 will follow, since in the first three cases we get a contradiction. For $t = (c_3, c_9)$ we will characterize the corresponding configurations explicitly, thus proving Theorem 3.

2.1 Locally minimal NE-free and (c_2, c_5) -free configurations

Let us consider C_n^0 in Figure 3 (where $n = 7$). By (5), $x_{i,i} > x_{i,j}$ (with respect to U_2) whenever $j \neq i$; in particular, $x_{i,i} > x_{i,i-1}$ for $i \in [n] = \{1, \dots, n\}$, where standardly, $0 \equiv n$. Similarly, $x_{i,i} \geq x_{j,i}$ whenever $j \neq i-1$ (with respect to U_1); in particular, $x_{i,i} \geq x_{i+1,i}$ for $i \in [n] = \{1, \dots, n\}$, where standardly, $n+1 \equiv 1$. Moreover, the latter n inequalities are also strict, since otherwise c_5 would appear.

By similar arguments we show that $x_{i,i+1} > x_{i,i+2}$ and $x_{i,i+1} > x_{i-1,i+1}$ for $i = 1, \dots, n-1$; see Figure 3.

Next, let us notice that $x_{i,i} = x_{i-2,i}$ for $i = 2, \dots, n$. Indeed, $x_{i,i} \geq x_{i-2,i}$, since C_n is a strong cycle, and c_2 would appear in case $x_{i,i} > x_{i-2,i}$.

Furthermore, $x_{i,i+2} \geq x_{i,i+3}$ for $i = 1, \dots, n-3$, since otherwise $x_{i,i+2}, x_{i,i+3}, x_{i+2,i+2}, x_{i+2,i+3}$ would form a c_5 .

Next, let us notice that $x_{i,i+3} = x_{i+1,i+3}$ for $i = 1, \dots, n-3$. Indeed, $x_{i,i+3} \leq x_{i+3,i+3} = x_{i+1,i+3}$, and if $x_{i,i+3} < x_{i+1,i+3}$ then $x_{i,i+1}, x_{i,i+3}, x_{i+3,i+1}, x_{i+3,i+3}$ would form a c_2 , by (5).

Similarly, by induction on j , we show that $x_{i,i+j} \geq x_{i,i+j+1}$ and $x_{i,i+j} = x_{i+1,i+j}$ for $1 \leq i \leq n-3$ and $2 \leq i+j \leq n-1$.

In particular, $x_{n,n} = x_{n-2,n} = x_{n-3,n} = \dots = x_{2,n} = x_{1,n}$ in contradiction with the strict inequality $x_{n,n} > x_{1,n}$ obtained before. \square

2.2 Locally minimal NE-free and (c_2, c_6, c_9) - or (c_3, c_5) -free configurations

These two cases are easy. Let us consider C_n^0 in Figures 4 (a) and (b) (where $n = 3$), corresponding respectively to the two cases. By definition, in both cases $x_{2,2} > x_{2,1}$ $x_{1,1} \geq x_{2,1}$. In case (b) we already got a contradiction, since four above situations form c_3 or c_5 .

In case (a) we have to proceed a little further. Clearly, $x_{2,3} \geq x_{2,1}$, $x_{1,2} \geq x_{1,3}$, $x_{2,3} > x_{1,3}$, and again we get a contradiction, since situations $x_{1,1}, x_{1,3}, x_{2,1}, x_{2,3}$ form c_9 if two equalities hold, c_6 if exactly one, and c_2 if none. \square

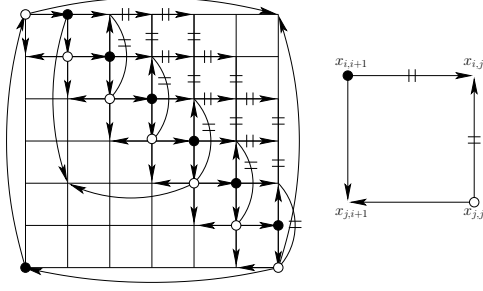


Fig. 3. Locally minimal NE-free and (c_2, c_5) -free configurations do not exist, except c_1 .

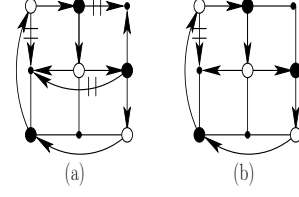


Fig. 4. Locally minimal NE-free and (c_2, c_6, c_9) - or (c_3, c_5) -free configurations do not exist, except c_1 .

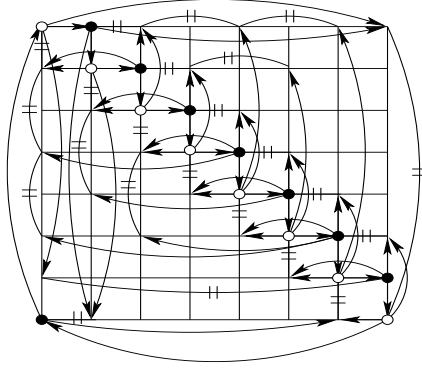


Fig. 5. Locally minimal NE-free and (c_3, c_9) -free configurations.

2.3 Locally minimal NE-free and (c_3, c_9) -free configurations

Let us consider C_n^0 in Figure 5 (where $n = 8$). By (5), for all i we have:

$$\begin{aligned} x_{i,i} &> x_{i,i+1}, x_{i,i} > x_{i,i-1}, x_{i,i} \geq x_{i+1,i}, x_{i,i} \geq x_{i-2,i}; \\ x_{i,i+1} &> x_{i+1,i+1}, x_{i,i+1} > x_{i-1,i+1}, x_{i,i+1} \geq x_{i,i+2}, x_{i,i+1} \geq x_{i,i-1}. \end{aligned}$$

Furthermore, it is not difficult to show that

$$x_{i,i} = x_{i+1,i} \text{ and } x_{i,i+1} = x_{i,i+2}, \quad (6)$$

since otherwise c_3 appears, while

$$x_{i,i} > x_{i-2,i} \text{ and } x_{i,i+1} > x_{i,i-1}, \quad (7)$$

since otherwise c_9 appears; see Figure 5.

Standardly, we prove all four claims in (6) and (7) by induction introducing situations in the following (alternating diagonal) order:

$$x_{2,1}, x_{1,3}, \dots, x_{i,i-1}, x_{i-1,i+1}, \dots, x_{n,n-1}, x_{n-1,1}, x_{1,n}, x_{n,2}.$$

Furthermore, $x_{1,1} = x_{2,1} \geq x_{4,1}$ unless $n < 5$; moreover, $x_{2,1} = x_{4,1}$, since otherwise situations $x_{2,1}, x_{4,1}, x_{2,4}$, and $x_{4,4}$ form c_3 .

Similarly, we prove that $x_{1,3} = x_{1,5}$ unless $n < 5$.

Then let us recall that $x_{4,5} \geq x_{4,1}$ and conclude that $x_{4,5} > x_{4,1}$, since otherwise situations $x_{1,1}$, $x_{4,1}$, $x_{1,5}$, and $x_{4,5}$ form c_9 .

In general, it is not difficult to prove by induction that

$$x_{i,i} = x_{i+1,i} = x_{i+3,i} = \dots = x_{i+2j-1,i}, \text{ while } x_{i-1,i} > x_{i,i} > x_{i+2j,i}; \quad (8)$$

$$x_{i,i+1} = x_{i,i+2} = x_{i,i+4} = \dots = x_{i,i+2j}, \text{ while } x_{i,i} > x_{i,i+1} > x_{i,i+2j+1}. \quad (9)$$

In both cases each sum is taken \pmod{n} (in particular, $n = 0$) and $1 \leq j < n/2$ (in particular, j takes values 1, 2, and 3 for $n = 7$ and $n = 8$).

If $n > 1$ is odd we immediately get a contradiction, since in this case, by (8), $x_{1,1} = x_{n-1,1}$, while, by (7), $x_{1,1} > x_{n-1,1}$ for all $n > 1$. Yet, for each even n , the family F_n of all locally minimal NE-free and (c_3, c_9) -free configurations is not empty.

Up to an isomorphism, F_2 (respectively, F_4) consists of a unique configuration: c_1 in Figure 1 (respectively, (c_2, c_4, c_5, c_6) in Figure 2). Two larger examples, from F_6 and F_8 , are given in Figures 6 (a) and (b), respectively.

We already know that each configuration $P \in F_{2k}$ must satisfy (5) - (9). Yet, P has one more important property:

$$x_{i,i+2j+1} \neq x_{i,i+2j'+1}, \quad x_{i+2j,i} \neq x_{i+2j',i} \quad (10)$$

for all $i \in [n]$ and for all positive distinct $j, j' < n/2$. Indeed, it is easy to see that otherwise c_9 appears; see Figure 6(a).

Let us denote by G_n the family of all configurations satisfying (5) - (10). We already know that $F_n \subseteq G_n$ and $F_n = G_n = \emptyset$ if $n > 1$ is odd. Let us show that $F_n = G_n$ for even n . Obviously, G_4 consists of a unique configuration (c_2, c_4, c_5, c_6) in Figure 2 and $G_2 = \{c_1\}$. Examples of configurations from G_6 and G_8 are given in Figures 6 (a) and (b). It is easy to verify that each configuration of G_n contains eight 2-squares $C' = \{c_2, c_4, c_5, c_6, c_1, c_7, c_8, c_{13}\}$ whenever $n \geq 6$; see Figure 6 (a). Moreover, c_{12} appears, too, when $n \geq 10$.

On the other hand, no configuration $P \in G_n$ contains c_9 , c_{10} , c_{11} , c_{14} , or c_{15} , since no 2-square in P can have two adjacent equalities. It is also easy to verify that P cannot contain c_3 . Thus, P can contain only nine 2-squares of $C'' = C' \cup \{c_{12}\}$. In particular, each $P \in G_n$ is (c_3, c_9) -free; in other words, $G_n \subseteq F_n$ and, hence, $G_n = F_n$ for each n . This implies Theorem 3 and provides an explicit characterization for family F_n of locally minimal NE-free and (c_3, c_9) -free configurations. \square

Remark 4. Interestingly, for even n each configuration $P \in F_n = G_n$ contains the same set of nine 2-squares C'' if $n \geq 10$; for $P \in G_8$ there are two options: C'' or C' (see example in Figure 6 (b), where c_{12} does not appear); for $P \in G_6$ only C' ; furthermore, G_4 consists of a unique configuration (c_2, c_4, c_5, c_6) in Figure 2 and G_2 only of c_1 ; finally, $F_n = G_n$ is empty if $n > 1$ is odd.

Acknowledgments. We are thankful to Kukushkin who promoted the idea of generalizing Shapley's (1964) theorem to bimatrix games and various concepts of solution.

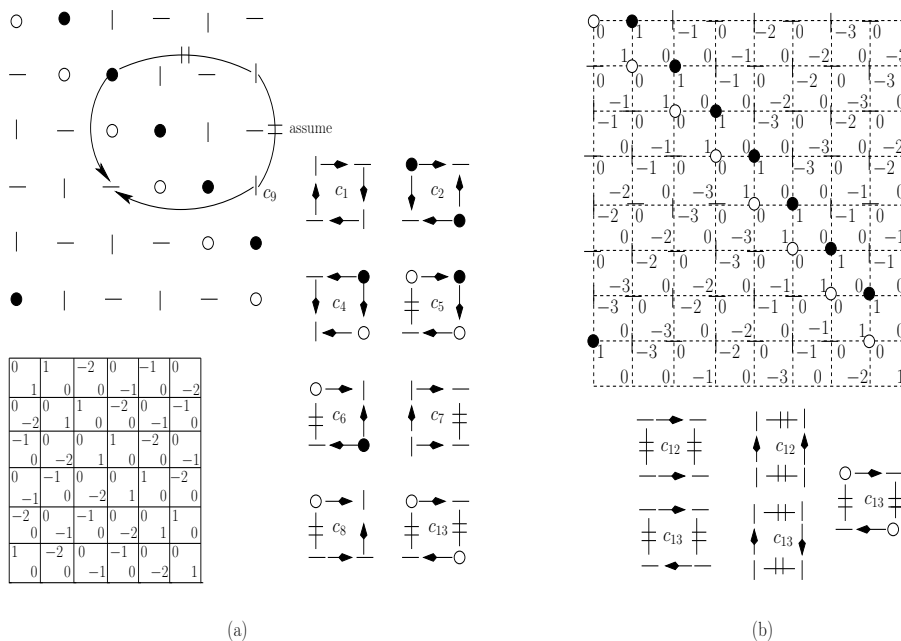


Fig. 6. Two examples from F_6 and F_8 : horizontal (respectively, vertical) bars indicate second largest elements with respect to U_1 (respectively U_2).

References

1. E. Boros, V. Gurvich, and K. Makino, Minimal and locally minimal games and game forms, Rutcor Research Report 28-2007, Rutgers University.
2. V. Chvatal, W.J.Lenhart, and N.Sbihi, Two-colourings that decompose perfect graphs, J. Combin. Theory Ser. B 49 (1990), 1-9.
3. M. Fredman, and L. Khachiyan, On the Complexity of Dualization of Monotone Disjunctive Normal Forms, J. Algorithms 21 (3) (1996), 618-628.
4. V.A. Gurvich and A.D. Gvishiani, Dual set systems and their applications, Izvestiya Akad. Nauk SSSR, ser. Tekhnicheskaya Kibernetika , 4 (1983), 31-39 (in Russian), English translation in Soviet J. of Computer and System Science (formerly Engineering Cybernetics).
5. V. Gurvich and L. Khachiyan, On generating the irredundant conjunctive and disjunctive normal forms of monotone Boolean functions. Discrete Applied Mathematics, 96-97 (1999), 363-373.
6. V. Gurvich and L. Libkin, Absolutely determined matrices. Mathematical Social Sciences, 20 (1990), 1-18.
7. N.S. Kukushkin, Shapley's 2×2 theorem for game forms, Economics Bulletin, <http://economicsbulletin.vanderbilt.edu/2007/volume3/EB-07C70017A.pdf>; see also Technical report, Department of Mathematical Methods for Economic Decision Analysis, Computing Center of Russian Academy of Sciences, <http://www.ccas.ru/mmes/mmeda/ququ/Shapley.pdf>
8. L. S. Shapley, Some topics in two-person games, in Advances in Game Theory (M. Drescher, L.S. Shapley, and A.W. Tucker, eds.), Annals of Mathematical Studies, AM52, Princeton University Press (1964), 1-28.