Enumerating Spanning and Connected Subsets in Graphs and Matroids

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Abstract

We show that enumerating all minimal spanning and connected subsets of a given matroid is quasi-polynomially equivalent to the well-known hypergraph transversal problem, and thus can be solved in incremental quasi-polynomial time. In the special case of graphical matroids, we improve this complexity bound by showing that all minimal 2-vertex connected edge subsets of a given graph can be generated in incremental polynomial time.

1 Introduction

The well-known *spanning trees enumeration* problem for an undirected graph $G = (V, E)$ calls for listing all minimal subsets X of edges such that the subgraph $G' = (V, X)$ is connected A. Shioura and Uno [1997], Read and Tarjan [1975]. In this paper, we study the following natural extension of this enumeration problem:

> **Minimal 2-Connected Spanning Subgraphs**: *Given a 2-vertex connected undirected graph $G = (V, E)$, enumerate all minimal edge sets $X \subseteq E$ such that $G' = (V, X)$ is still 2-vertex connected.*

We shall also consider the generalization of this problem for matroids. We follow the standard terminology of matroid theory, see e.g., Oxley [1992] or Welsh [1976] for a thorough introduction. Given a matroid M on ground set E, a subset $T \subseteq E$ is called *connected* if for every pair of distinct elements x, y of T there is a circuit of M within T containing x and y. Connectedness defines an

*Our friend and colleague, Leo Khachiyan passed away with tragic suddenness while we were working on this problem.

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equivalence relation whose equivalence classes are called *connected components* of M. A subset $X \subseteq E$ is said to *span* the matroid M if $r(X) = r(E)$, where $r : E \to \mathbb{Z}_+$ denotes the rank function of M. Minimal spanning subsets are called the bases of M. In particular, in the cycle matroid of $G(V, E)$ spanning trees are the bases.

Let us note that the family of spanning and connected subsets in a matroid form a monotone system, i.e., if $X \subseteq E$ is spanning and connected, then so are all supersets $X' \supseteq X$ of it (see Section 2). In contrast, connected subsets do not form, in general, a monotone family. Let us denote by $\mathcal{F}_M$ the family of all minimal spanning and connected subsets of the matroid M.

Let us remark that in the cycle matroid M of a graph $G = (V, E)$, the family $\mathcal{F}_M$ is formed by 2-vertex connected edges sets $X \subseteq E$ (see e.g., Theorem 3 on page 70 in Welsh [1976]). Hence the following enumeration problem generalizes naturally the problem of enumerating spanning trees of graphs:

> **Minimal Connected Spanning Subsets in Matroids***: Given a connected matroid M on ground set E, generate the family $\mathcal{F}_M$.*

It is clear that in both problems the size of the output may be exponential in terms of the input sizes. For such problems the efficiency of the enumeration method is measured both in the input and output sizes (see e.g., Lawler et al. [1980]). In particular, it is said that the enumeration procedure runs in incremental *(quasi-) polynomial time*[1], if generating k elements of the target hypergraph can be done in (quasi-) polynomial time in the size of the input and k, for an arbitrary integer k.

In general, a *monotone enumeration problem* consists of generating all minimal subsets of a finite set satisfying a given monotone property. An important example is so-called *hypergraph transversal* problem (or equivalently, the monotone Boolean dualization) which consists of generating all minimal transversals of a given hypergraph. This problem has many applications in various areas (see e.g., Eiter and Gottlob [1995]). The currently best known algorithm solves it in incremental quasi-polynomial time, Fredman and Khachiyan [1996]. We prove the following two theorems.

Theorem 1 *The problem of generating all minimal spanning and connected subsets of a matroid is quasi-polynomially equivalent to the hypergraph transversal problem. In particular, it can be solved in incremental quasi-polynomial time.*

In the special case, when M is a graphical matroid we show that $\mathcal{F}_M$ can be generated more efficiently.

Theorem 2 *All minimal 2-vertex connected spanning subgraphs of a given graph can be enumerated in incremental polynomial time.*

The proofs of Theorems 1 and 2 are in Sections 1 and 2, respectively.

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[1]A function $f(x)$ is called quasi-polynomial if $f(x) = O(2^{polylog(x)})$.

2 Proof of Theorem 1

Let us show first that, as we remarked in the Introduction, spanning and connected subsets of a matroid form a monotone family.

To see this, assume that M is a matroid on base set E, and consider a spanning and connected subset $X \subseteq E$. We show that for an arbitrary element $f \in E \setminus X$ the set $X \cup f$ is again spanning and connected. Clearly $X \cup f$ is spanning. According to Welsh [1976], to see that $X \cup f$ is also connected it is enough to show that $r(Y)+r(Z) \geq r(X \cup f)+1$ holds for an arbitrary partition $Y \cup Z = X \cup f$ with $|Y| \geq 1$, $|Z| \geq 1$. Note that, since X is spanning, $r(X)+1 = r(X \cup f)+1$. Without loss of generality assume that $f \in Z$. If $|Z| = 1$ we have $r(Y)+r(Z) = r(X)+r(f) = r(X)+1$, since we assumed that all singletons of M have rank 1. In case $|Z| \geq 2$, we have $r(Y)+r(Z) \geq r(Y)+r(Z \setminus f) \geq r(X)+1$, since $r(Z) \geq r(Z \setminus f)$ and the sets Y and $Z \setminus f$ form a partition of X, with $|Y| \geq 1$, $|Z \setminus f| \geq 1$, completing the proof of our claim.

We start the proof of Theorem 1 by showing that the problem of enumerating all minimal spanning and connected subsets in M includes, as a special case, the well-known hypergraph transversal problem.

Let $\mathcal{H}$ be a hypergraph on n vertices and m edges. We denote by $\mathbf{v_1}, \dots, \mathbf{v_n}$ column vectors of the incidence matrix of $\mathcal{H}$.

Let $H = \{\mu(\mathbf{v_1}), \dots, \mu(\mathbf{v_n})\}$, let $A = \{\mathbf{a_1}, \dots, \mathbf{a_m}\}$, let $B = \{\mathbf{b_1}, \dots, \mathbf{b_m}\}$ and let $C = \{\mathbf{c_1}, \mathbf{c_2}\}$, where $(2m+2)$-dimensional vectors $\mu(\mathbf{v_i})$, $\mathbf{a_i}$, $\mathbf{b_i}$, $\mathbf{c_1}$ and $\mathbf{c_2}$ are defined as follows: $\mu(\mathbf{v_i}) = (\mathbf{v_i^T}, 0, \dots, 0, 1, 0)^T$, $\mathbf{a_i} = (0, \dots, \overset{i}{1}, \dots, \overset{m+i}{1}, \dots, 0)^T$, $\mathbf{b_i} = (0, \dots, \overset{m+i}{1}, \dots, 0)^T$, $\mathbf{c_1} = (0, \dots, 0, 1, 1)^T$ and $\mathbf{c_2} = (0, \dots, 0, 1)^T$.

We next construct a binary matroid M on a ground set $H \cup A \cup B \cup C$ (see Example 2.1).

Example 2.1 *Consider the hypergraph $\mathcal{H}$ defined by the incidence matrix*

$$\mathcal{H} = (\mathbf{v_1}, \dots, \mathbf{v_5}) = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Then a binary matroid M on $H \cup A \cup B \cup C$ is represented by a matrix

$$\left(\begin{array}{ccccc|cccc|cccc|cc}
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
\hline
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{array}\right)$$

Claim 1 *Let X be a spanning and connected subset in M. Then $A \cup B \cup C \subseteq X$ and $\{\mathbf{v_i} \mid \mu(\mathbf{v_i}) \in X\}$ is a transversal of $\mathcal{H}$.*

Proof: First we show that for each $i = 1, \ldots, 2m+2$ at least two vectors of X have *ith* coordinate is 1. Since X is spanning and the matrix representing M has a full row rank, there is at least one vector in X whose *ith* coordinate is 1. Suppose that there is exactly one such vector $\mathbf{x} \in X$. Then we consider partition $Y = \{\mathbf{x}\}$ and $Z = X \setminus \{\mathbf{x}\}$. We have $r(Y) = 1$ and $r(Z) < r(M)$, since all vectors of Z has *ith* equal to 0.

The above claim implies that X contains all vectors of E, F and G in order to have two 1's in rows from $m+1$ to $2m+3$. To contain two 1's in first m rows, X must contain a vector of H whose *ith* coordinate is 1, where $i = 1, \ldots, m$. Consequently first m rows of $H \cap X$ form a transversal of $\mathcal{H}$. □

Claim 2 *Let $\{\mathbf{v_i} \mid i \in T\}$ be a transversal of $\mathcal{H}$, where $T \subseteq \{1, \ldots, n\}$. Then $X \stackrel{\text{def}}{=} \{\mu(\mathbf{v_i}) \mid i \in T\} \cup A \cup B \cup C$ is a connected and spanning subset in M.*

Proof: First note that $r(M) = 2m+2$. We observe that X is spanning, since $r(E, F, G) = 2m+2$.

We show that $r(Y) + r(Z) \geq r(M) + 1 = 2m+3$ for every partition Y, Z of X with $|Y| \geq 1$ and $|Z| \geq 1$.

Suppose all vectors of $A \cup B \cup C$ belong to Y. Then $r(Y) = 2m+2$ and $r(Z) \geq 1$, since Z is nonempty. Thus the vectors of $A \cup B \cup C$ must be split between Y and Z.

Without loss of generality suppose Y contains k vectors of $A \cup B \cup C$ including $\mathbf{c_1}$, where $1 \leq k \leq 2m+1$. Consequently $r(Y) \geq k$ and $r(Z) \geq 2m+2-k$. Let $I_i \subseteq \{1, \ldots, m\}$ denote the set of coordinates of $\mathbf{v_i}$ equal to 1. Observe that $\mu(\mathbf{v_i}) = \mathbf{c_1} + \mathbf{c_2} + \sum_{j \in I_i}(\mathbf{a_j} + \mathbf{b_j})$ is the only combination of vectors in $A \cup B \cup C$ producing $\mu(\mathbf{v_i})$. Depending whether Z contains a vector of $\mu(T)$ we have two cases:

Case 1: Z contain at least one vector of $\{\mu(\mathbf{v_i}) \mid i \in T\}$. Since vectors of $\{\mu(\mathbf{v_i}) \mid i \in T\}$ cannot be obtained without $\mathbf{c_1}$, $r(Z) \geq 2m+3-k$. Thus $r(Y) + r(Z) \geq 2m+3$.

Case 2: Y contains all vectors of $\{\mu(\mathbf{v_i}) \mid i \in T\}$. Since T is a transversal of $\mathcal{H}$, we have $\bigcup_{i \in T} I_i = \{1, \ldots, m\}$. As Y does not contain all vectors of $A \cup B \cup C$, there is a vector in $\{\mu(\mathbf{v_i}) \mid i \in T\}$ which cannot be obtained as a combination of vectors in Y. Thus $r(Y) \geq k+1$ and $r(Y) + r(Z) \geq 2m+3$. □

Thus generating all minimal spanning and connected subsets in M is at least as hard as generating all minimal transversals of a hypergraph $\mathcal{H}$.

To complete the proof of Theorem 1, we next show that generating all minimal spanning and connected subsets of a matroid can be reduced in quasi-polynomial time to the hypergraph transversal problem. We achieve this, by proving that for every matroid M, the family $\mathcal{F}_M$ is exactly the set of minimal solutions of a polymatroid inequality with polynomially bounded right hand side. For such inequalities, it is known that the generation of minimal feasible sets can be done in incremental quasi-polynomial time Boros et al. [2003].

Let $f(X)$ be a set function defined on the subsets of E as:

$$f(X) = |E|\, r(X) - 1.$$

The *Dilworth truncation of* $f(X)$ is the set function $f_*(X)$ defined as follows:

$$f_*(\emptyset) = 0,$$

$$f_*(X) = \min\{f(X_1) + \ldots + f(X_k) \mid X_1, \ldots, X_k \text{ is a partition of } X\} \text{ for } X \neq \emptyset.$$

Clearly, $f(X)$ is a nondecreasing submodular function. Thus $f_*(X)$ is submodular and can be evaluated in polynomial time and $poly(|E|)$ calls to the oracle Lovasz [1983]. We next show that $f_*(X)$ is nondecreasing function, implying that $f_*(X)$ is a polymatroid function.

Claim 3 $f_*(X)$ *is nondecreasing.*

Proof: We will show that $f_*(X \cup e) \geq f_*(X)$, where $X \subseteq E$, $e \in E \setminus X$. Let $X_1, X_2, \ldots, X_k$ be an optimal partition for $X \cup e$, i.e., $f_*(X \cup e) = f(X_1) + f(X_2) + \ldots + f(X_k)$. Without loss of generality assume that $e \in X_1$. There are two cases:

Case 1: $X_1 \setminus e \neq \emptyset$. Then $X_1 \setminus e, X_2, \ldots, X_k$ is a partition of X. Hence

$$f_*(X) \leq f(X_1 \setminus e) + \sum_{i=2}^{k} f(X_i) \leq f(X_1) + \sum_{i=2}^{k} f(X_i) = f_*(X \cup e),$$

where the last inequality in the chain follows from $f(X_1 \setminus e) = |E| r(X_1 \setminus e) - 1 \leq |E| r(X_1) - 1 = f(X_1)$.

Case 2: $X_1 = \{e\}$. Consider the partition $X_2, \ldots, X_k$ of X, which again gives

$$f_*(X) \leq \sum_{i=2}^{k} f(X_i) \leq f(e) + \sum_{i=2}^{k} f(X_i) = f_*(X \cup e),$$

where the last inequality in the chain follows from the fact that $r(e) = 1$, thus $f(e) = |E| - 1 \geq 0$, for all $e \in X$. □

Now consider the polymatroid inequality

$$f_*(X) \geq |E|\, r(M) - 1.$$

Note that the right hand side of the above inequality is bounded by $|E|^2$. We prove that minimal connected spanning subsets are exactly minimal solutions to the above polymatroid inequality.

Claim 4 X *is connected in* M *if and only if* $f_*(X) \geq f(X)$.

Proof: Let X be connected subset in M. Consider a partition of X into at least $k \geq 2$ sets. Since the rank function is submodular and by the definition of connectivity we have $r(A) + r(E \setminus A) > r(E)$ for every proper subset A of E, we obtain $r(X_1) + r(X_2) + \ldots + r(X_k) \geq r(X_1) + r(X_2 \cup \ldots \cup X_k) \geq r(X) + 1$. Hence

$$f(X_1) + f(X_2) + \ldots + f(X_k) \geq |E|\, r(X) + |E| - k > |E|\, r(X) - 1 = f(X).$$

Comparing that with the trivial partition $X = X_1$ for $k = 1$, we conclude that $f_*(X) = f(X)$.

On the other hand, if X is not connected, then we can decompose X into two disjoint sets X_1 and X_2 such that $r(X_1) + r(X_2) = r(X)$. Hence $f(X_1) + f(X_2) = |E|\, r(X) - 2$, and consequently, $f_*(X) < |E| r(X) - 1 = f(X)$. □

Claim 5 *X is connected and spanning subset in M if and only if $f_*(X) \geq |E|\, r(M) - 1$.*

Proof: If X is connected and spanning, the claim follows from Claim 4 and the fact that $r(X) = r(M)$.

Conversely, suppose X satisfies $f_*(X) \geq |E|\, r(M) - 1$ and also suppose that X is not spanning. Then since $r(X) < r(M)$ for the trivial partition $X = X_1$, we obtain $f(X_1) = |E| r(X_1) - 1 < |E| r(M) - 1$, which implies $f_*(X) < |E| r(M) - 1$, a contradiction. Thus X must be spanning and by Claim 4 X must also be connected. □

3 Proof of Theorem 2

We prove Theorems 2 by using a generic approach discussed below. Let E be a finite set and let $\pi : 2^E \to \{0, 1\}$ be a monotone Boolean function, i.e., one for which $X \subseteq Y$ implies $\pi(X) \leq \pi(Y)$. Suppose that an efficient algorithm is available for evaluating $\pi(X)$ in $poly(|E|)$ time for any $X \subseteq E$. Our goal is to enumerate all (inclusionwise) minimal subsets $X \subseteq S$ for which $\pi(X) = 1$. Let

$$\mathcal{F} \overset{\text{def}}{=} \{X \mid X \subseteq E \text{ is a minimal set satisfying } \pi(X) = 1\}.$$

We define the supergraph $\mathcal{G} = (\mathcal{F}, \mathcal{E})$ to be the directed graph on vertex set $\mathcal{F}$ in which two vertices $X, X' \in \mathcal{F}$ are connected by an arc $XX' \in \mathcal{E}$ if we can obtain X' from X by applying the following procedure:

(p1) Delete an element e from X (since X is an minimal set satisfying $\pi(X) = 1$, this implies $\pi(X \setminus e) = 0$).

(p2) Add a minimal set $Y \subseteq E \setminus X$ which restores the property $\pi((X \setminus e) \cup Y) = 1$.

(p3) Assuming a fixed linear order on the elements of E, delete the lexicographically first minimal set $Z \subseteq X \setminus e$ to restore the minimality of $X' = (X \setminus (Z \cup e)) \cup Y$ with respect to the property $\pi(X') = 1$.

The following result is well known.

Proposition 1 (Khachiyan et al. [2005]) *For any monotone Boolean function $\pi : 2^E \to \{0,1\}$, and any linear ordering of E, the supergraph $\mathcal{G} = (\mathcal{F}, \mathcal{E})$ is strongly connected.*

Since the supergraph $\mathcal{G} = (\mathcal{F}, \mathcal{E})$ is strongly connected, we can generate $\mathcal{F}$ by first computing an initial vertex $X^o \in \mathcal{F}$ and then performing a traversal (say, breadth-first search) of $\mathcal{G}$. Clearly step (**p1**) can be performed in $poly(|E|)$ time. As $\pi(\cdot)$ can be evaluated in $poly(|E|)$ time, computing an initial vertex of $\mathcal{G}$ and X' in (**p3**) can also be done in polynomial time.

Proposition 2 (Khachiyan et al. [2005]) *All elements of $\mathcal{F}$ can be enumerated in incremental polynomial time whenever we can enumerate all sets Y in* (**p2**) *in incremental polynomial time.*

Now we apply the method described above to enumerating all minimal 2-vertex connected spanning subgraphs.

For a graph $G = (V, E)$ and $X \subseteq E$, we define a Boolean function π as follows:

$$\pi(X) = \begin{cases} 1, & \text{if (V,X) is 2-vertex connected;} \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\mathcal{F} = \{X \mid X \subseteq E \text{ is a minimal set satisfying } \pi(X) = 1\}$$

is a family of edge sets of all minimal 2-vertex connected spanning subgraphs of G. Clearly π is monotone. Proposition 3 below implies that this can be accomplished in incremental polynomial time.

Before stating Proposition 3, we recall that a maximal connected subgraph without a cutvertex is called a *block*. Thus, every block of a connected graph H is either a maximal 2-vertex connected subgraph, or a bridge (with its ends). Different blocks overlap in at most one vertex, which is a cutvertex of H. Hence, every edge of the graph lies in a unique block.

Let A denote the set of cutvertices of H and let $\mathcal{B}$ denote the set of its blocks. We then have a natural bipartite graph on vertex set $A \cup \mathcal{B}$ in which two vertices $B \in \mathcal{B}$, $a \in A$ are connected if a is a cutvertex of H belonging to B. We call such graph a *block graph* of H. Observe that the block graph of a connected graph is a tree.

Proposition 3 *Let $X \in \mathcal{F}$ and let $e \in X$ (see Figure 1). Then all sets Y in* **(p2)** *can be enumerated with polynomial delay.*

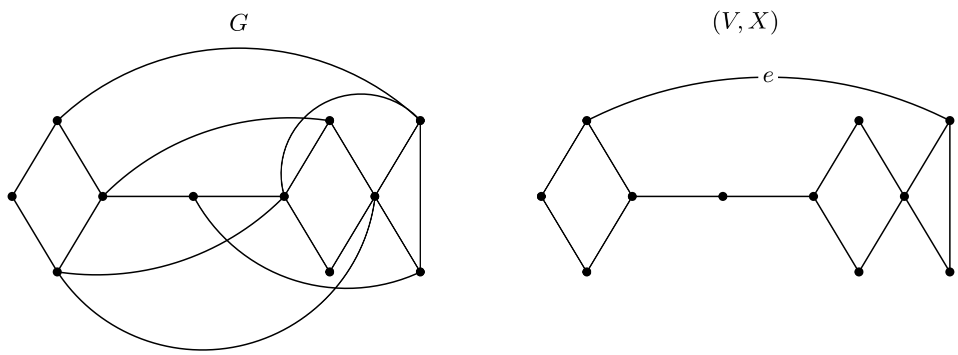

Figure 1: 2-vertex connected graph $G = (V, E)$ and a minimal 2-connected spanning subgraph (V, X) of G.

Proof: First we show that the block graph of $(V, X \setminus e)$ is a path such that endpoints of e belong to its ends. As we observed above the block graph of $(V, X \setminus e)$ is a tree. Suppose it has a leaf B that does not contain an endpoint of e. Let a be a cutvertex of $(V, X \setminus e)$ adjacent to B in the block graph. But removing the vertex a from the 2-vertex connected graph (V, X) disconnects vertices of B from other vertices, a contradiction. Thus the block graph of $(V, X \setminus e)$ has only two leaves, each containing one endpoint of e.

We denote by $B_1, \ldots, B_r$ the blocks of $(V, X \setminus e)$ and by $a_1, \ldots, a_{r-1}$ its cutvertices. Without loss of generality we assume that the block graph of $(V, X \setminus e)$ is a path $B_1 a_1 B_2 \ldots a_{r-1} B_r$ (see Figure 2).

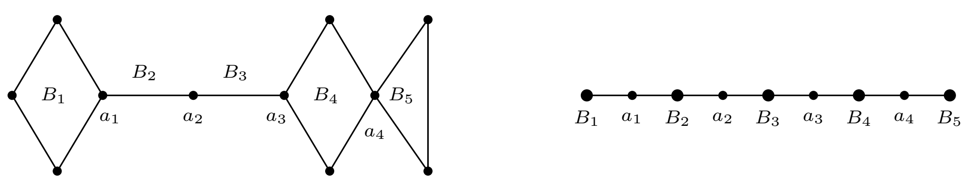

Figure 2: $(V, X \setminus e)$ and its block graph

Let $f = uv$ be an edge of $E \setminus X$, such that u belongs to the block B_i and v belongs to B_j, where $i < j$. We define $\alpha(f) = \begin{cases} i, & \text{if } v \in B_i \setminus a_i \ ; \\ i+1, & \text{if } v = a_i, \end{cases}$ and $\beta(f) = \begin{cases} j, & \text{if } v \in B_j \setminus a_{j-1} \ ; \\ j-1, & \text{if } v = a_{j-1}. \end{cases}$

Then we construct a directed multigraph D on vertex set $B_1, \ldots, B_r$ whose edge set is defined as follows:

- for each $i = 1, \ldots, r-1$, we add an arc $B_{i+1} B_i$,

- for each edge $f \in E \setminus X$, such that $\alpha(f) < \beta(f)$, we add an arc $B_{\alpha(f)} B_{\beta(f)}$ (see Figure 3).

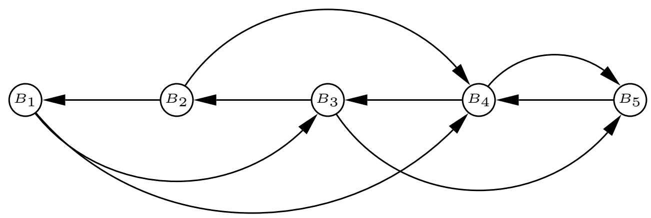

Figure 3: Directed multigraph D.

Now we show that the enumeration of sets Y in **(p2)** is equivalent to the enumeration of minimal directed B_1-B_r paths in D.

For every cutvertex a_k there is an edge $f \in Y$ such that $\alpha(f) \leq k < \beta(f)$. By minimality of Y, edges of $E \setminus X$ whose both endpoints belong to the same block cannot be in Y. We conclude that $Y = \{f_1, \ldots, f_s\}$ such that

$$1 = \alpha(f_1) < \alpha(f_2) \leq \beta(f_1) < \alpha(f_3) \leq \ldots < \alpha(f_s) \leq \beta(f_{s-1}) < \beta(f_s) = r.$$

Thus Y corresponds to a directed path $B_{\alpha(f_1)}\ B_{\beta(f_1)}\ B_{\beta(f_1)-1} \ldots B_{\alpha(f_2)+1}\ B_{\alpha(f_2)}$ $B_{\beta(f_2)}\ B_{\beta(f_2)-1} \ldots B_{\alpha(f_3)+1}\ B_{\alpha(f_3)} B_{\beta(f_3)} \ldots B_{\beta(f_s)}$ (see Figure 4).

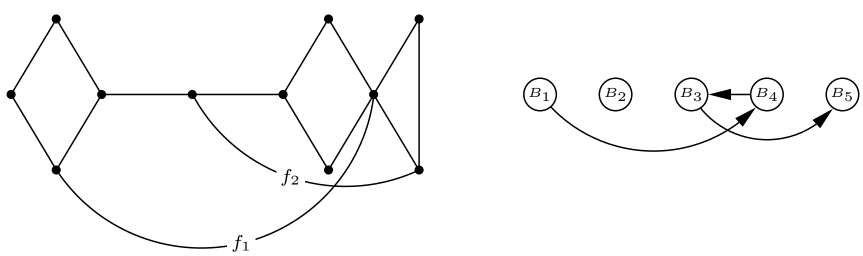

Figure 4: $Y = \{f_1, f_2\}$ and corresponding directed path $B_1 B_4 B_3 B_5$.

Since all minimal directed paths between two vertices can be enumerated via backtracking with polynomial delay Read and Tarjan [1975], Proposition 3 follows. □

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