# On effectivity functions of game forms *

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**Abstract**

To each game form $g$ an effectivity function (EFF) $E_g$ is assigned. An EFF $E$ is called *formal (formal-minor)* if $E = E_g$ (respectively, $E \leq E_g$) for a game form $g$.

(i) An EFF is formal iff it is superadditive and monotone.

(ii) An EFF is formal-minor iff it is weakly superadditive.

Theorem (ii) looks more sophisticated, yet, it is simpler than Theorem (i) and instrumental in its proof. In addition, (ii) has important applications in social choice, game, and even graph theories. Constructive proofs of (i) were given by Moulin, in 1983, and by Peleg, in 1998. Both constructions are elegant, yet, sets of strategies $X_i$ of players $i \in I$ might be doubly exponential in size of the input EFF $E$. In this paper, we suggest a third construction such that $|X_i|$ is only linear in the size of $E$.

Also, we extend Theorems (i, ii) to tight and totally tight game forms.

**Keywords**: game form, tight, totally tight, effectivity function, monotone, superadditive, weakly superadditive, dual-minor, self-dual

**Journal of Economic Literature Classification**: C.62, C.72

## 1 Introduction

The effectivity function (EFF) is an important concept of voting theory that describes the distribution of power between the voters and candidates. This concept was introduced in the early 80s by Abdou [1, 2], Moulin and Peleg

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[23], [22] Chapter 7, [24], [25] Chapter 6. We also refer the reader to the book "Effectivity Functions in Social Choice" by Abdou and Keiding [3] for numerous applications of EFFs in the voting and game theories.

An EFF can be viewed as a Boolean function whose set of variables is the mixture of the voters (players) and candidates (outcomes); see Section 2.1.

A game form $g$ can be viewed as a game in normal form in which no payoffs are defined yet and only an outcome $g(x)$ is associated with each strategy profile $x$. To every game form $g$ an EFF $E_g$ can be naturally assigned; see Section 4.

Some important properties of a game form $g$ depend only on its EFF $E_g$.

For example, the core $C(g, u)$ is not empty for any payoff $u$ if and only if EFF $E_g$ is stable; see [22] Chapter 7, [25] Chapters 6, [3] Chapter 3; furthermore, a *two-person* game $(g, u)$ has a Nash equilibrium in pure strategies for any $u$ if and only if EFF $E_g$ is self-dual; see [12, 14], and also Sections 8.3 and 10.

It is a natural and important problem to characterize the EFFs related to game forms. Already in [23] it was mentioned that these EFFs are monotone and superadditive. The inverse statement holds too, yet, it is more difficult.

An EFF $E$ will be called *formal* (respectively, formal-minor) if $E = E_g$ (respectively, $E \leq E_g$) for a game form $g$. The following two claims hold:

(i) An EFF is formal if and only if is superadditive and monotone;

(ii) An EFF is formal-minor if and only if it is weakly superadditive.

In both cases the EFFs must satisfy some natural "boundary conditions"; see Sections 2.2 and 2.3 for definitions and more details.

Theorem (ii) looks more sophisticated, yet, it is simpler and instrumental in the proof of (i). In addition, (ii) has important applications in social choice, game, and even graph theories; see [22] Chapter 7 and [4, 5, 6].

Constructive proofs of (i) were given by Moulin, in 1983, and by Peleg, in 1998. (In fact, Peleg proved a slightly more general statement that includes, in particular, the case of infinite sets of outcomes.) Both constructions are interesting and elegant, yet, in both, the set of strategies $X_i$ of each player $i \in I$ in $g$ is doubly exponential in size of the input EFF $E$. In this paper, we suggest a third construction such that $|X_i|$ is only linear in the size of $E$.

Furthermore, an EFF $E$ will be called T-formal (TT-formal) if $E = E_g$ for a tight (totally tight (TT)) game form $g$; see Sections 8 and 9 for definitions. Obviously, the families of TT-formal, T-formal, and formal EFFs are nested, since every TT game form is tight; see Section 9.

Moulin's results readily imply that an EFF is T-formal if and only if it is maximal, superadditive, monotone, and satisfies the boundary conditions.

In this paper, we add to this list one more property, which also holds for each TT-formal EFF, and show that the extended list of properties is a characterization of the two-person TT-formal EFFs, leaving the $n$-person case open.

# 2 Basic properties

## 2.1 Effectivity functions as Boolean functions of players and outcomes

Given a set of players (or voters) $I = \{1, \dots, n\}$ and a set of outcomes (or candidates) $A = \{a_1, \dots, a_p\}$, subsets $K \subseteq I$ are called *coalitions* and subsets $B \subseteq A$ *blocks.* An *effectivity function* (EFF) is defined as a mapping $E : 2^I \times 2^A \to \{0, 1\}$. We say that coalition $K \subseteq I$ is effective (respectively, not effective) for block $B \subseteq A$ if $E(K, B) = 1$ (respectively, $E(K, B) = 0$).

Since $2^I \times 2^A = 2^{I \cup A}$, we can view an EFF $E$ is a Boolean function whose variables $I \cup A$ are a mixture of the players and outcomes.

An EFF describes the distribution of power of voters and of candidates.

For two EFFs $E$ and $E'$ on the same variables $I \cup A$, obviously, the implication $E' = 1$ whenever $E = 1$ is equivalent with the inequality $E \leq E'$.

The "complementary" function, $\mathcal{V}(K, B) \equiv E(K, A \setminus B)$, is called the *veto function*; by definition, $K$ is effective for $B$ if and only if $K$ can veto $A \setminus B$. Both names are frequent in the literature [1, 2, 15, 16, 17, 22, 23, 24, 25].

## 2.2 Boundary conditions

The complete ($K = I,\ B = A$) and empty ($K = \emptyset,\ B = \emptyset$) coalitions and blocks will be called *boundary* and play a special role. From now on, we assume that the following *boundary conditions* hold for all considered EFFs:

$E(K, \emptyset) = 0$ and $E(K, A) = 1 \ \forall\ K \subseteq I$;

$E(I, B) = 1$ unless $B = \emptyset$; $E(\emptyset, B) = 0$ unless $B = A$;

$E(I, \emptyset) = 0,\ \ E(\emptyset, A) = 1.$

**Remark 1** *In fact, the value of $E(\emptyset, A)$ is irrelevant. However, in Section 8 we will define self-duality (maximality) of an EFF by the equation*
$E(K, B) + E(I \setminus K, A \setminus B) \equiv 1$ *for all $K \subseteq I,\ B \subseteq A$. Thus, formally, since $E(I, \emptyset) = 0$, we have to set $E(\emptyset, A) = 1$, otherwise self-duality will never hold.*

## 2.3 Monotonicity and the minimum monotone majorant of an effectivity function

An EFF is called *monotone* if the following implication holds:

$E(K, B) = 1,\ K \subseteq K' \subseteq I,\ B \subseteq B' \subseteq A \ \Rightarrow\ E(K', B') = 1.$

It is easy to see that the above definition is in agreement with the standard concept of monotonicity for Boolean functions.

A (*monotone*) Boolean function is given by the set of its (*minimal*) true vectors. Respectively, a (*monotone*) EFF $E$ is given by the list $\{(K_j, B_j) \mid j \in J\}$ of all (inclusion-minimal) pairs $K_j \subseteq I$ and $B_j \subseteq A$ such that $E(K_j, B_j) = 1$.

Let us remark that $\mathcal{K}_E = \{K_j \mid j \in J\}$ and $\mathcal{B}_E = \{B_j \mid j \in J\}$ are multi-hypergraphs whose edges, labeled by $J$, might be not pairwise distinct.

It is also clear that for each EFF $E$ there is a unique minimum monotone EFF $E^M$ such that $E^M \geq E$. This EFF is defined by formula:

$E^M(K^M, B^M) = 1$ iff $E(K, B) = 1$ for some $K \subseteq K^M \subseteq I$, $B \subseteq B^M \subseteq A$

and is called the *minimum monotone majorant* of $E$.

# 3 Superadditive and weakly superadditive EFFs

## 3.1 Superadditivity

An EFF $E$ is called *2-superadditive* if the following implication holds:

$$E(K_1, B_1) = E(K_2, B_2) = 1,\ K_1 \cap K_2 = \emptyset \ \Rightarrow \ E(K_1 \cup K_2, B_1 \cap B_2) = 1.$$

More generally, an EFF $E$ is called *$k$-superadditive* if, for every set of indices $J$ of cardinality $|J| = k \geq 2$, the following implication holds:

if $E(K_j, B_j) = 1 \, \forall \, j \in J$ and coalitions $\{K_j \mid j \in J\}$ are pairwise disjoint (that is, $K_{j'} \cap K_{j''} = \emptyset \ \forall j', j'' \in J \mid \ j' \neq j''$) then

$$E\left( \bigcup_{j \in J} K_j, \bigcap_{j \in J} B_j \right) = 1.$$

Let us notice that, in particular, $\bigcap_{j \in J} B_j \neq \emptyset$, since otherwise the boundary condition $E(K, \emptyset) = 0$ would fail. By induction on $k$, it is easy to show that 2-superadditivity implies $k$-superadditivity for all $k \geq 2$; see, for example, [22] or [3]. An EFF satisfying these properties is called *superadditive*.

## 3.2 Weak superadditivity

Furthermore, an EFF $E$ is called *weakly superadditive* if for every set of indices $J$ the following implication holds:

if $E(K_j, B_j) = 1 \, \forall \, j \in J$ and coalitions $\{K_j;\ j \in J\}$ are pairwise disjoint then

$$\bigcap_{j \in J} B_j \neq \emptyset.$$

**Remark 2** *In [3], a family $\{(K_j, B_j) \mid j \in J\}$ is called an* upper cycle *whenever the above implication fails. Thus, an EFF is weakly superadditive if and only if it has no upper cycle. It was also shown in [3] that the upper and lower acyclicity are necessary for stability of EFFs; see also [6] for an alternative proof, definitions, and more details.*

Let us also remark that weak superadditivity, in contrast to superadditivity, cannot be reduced to the case $|J| = 2$.

**Example 1** *An EFF $E$ such that*

$E(\{1\},\{a_2,a_3\}) = E(\{2\},\{a_3,a_1\}) = E(\{3\},\{a_1,a_2\}) = 1$

*is not weakly superadditive, yet, EFF $E$ might be weakly 2-superadditive.*

Finally, let us note that superadditivity implies weak superadditivity; indeed, otherwise boundary conditions $E(K,\emptyset) = 0$ would not hold.

**Example 2** *However, the inverse implication fails. An EFF $E$ such that*

$E(\{1\},\{a_2,a_3\}) = E(\{2\},\{a_3,a_1\}) = 1,$ *while* $E(\{1,2\},\{a_3\}) = 0$

*is not superadditive but might be weakly superadditive.*

## 3.3 On complexity of verifying (weak) superadditivity

It is a CoNP-complete problem to verify whether a *monotone* EFF $E$ is weakly superadditive; see [5] Theorem 12, Lemma 28, and Remarks 10 and 29.

In contrast, one can easily verify in cubic time whether a (monotone) EFF $E = \{(K_j, B_j) \mid j \in J\}$ is superadditive. Indeed, as we know, superadditivity of $E$ is equivalent with its 2-superadditivity and the latter can be verified in cubic time just according to the definition.

## 3.4 On (weak) superadditivity of a minorant of an EFF

**Proposition 1** *If an EFF $E$ is weakly superadditive and $E' \leq E$ then EFF $E'$ is weakly superadditive, too.*

**Proof**. Let $J$ be a set of indices and $E'(K_j, B_j) = 1$ for each $j \in J$, where coalitions $\{K_j : j \in J\}$ are pairwise disjoint. Then $E(K_j, B_j) = 1$ for each $j \in J$, too, since $E \geq E'$. Hence, $\bigcap_{j \in J} B_j \neq \emptyset$, since $E$ is weakly superadditive. Thus, $E'$ is weakly superadditive, too. $\square$

However, the above arguments do not extend to superadditivity, since

$$E\left(\bigcup_{j\in J} K_j, \bigcap_{j \in J} B_j\right) = 1 \text{ and } E' \leq E \quad \not\Rightarrow \quad E'\left(\bigcup_{j\in J} K_j, \bigcap_{j \in J} B_j\right) = 1.$$

**Example 3** *Let us consider EFFs $E$ and $E'$ such that*

$E(\{1\},\{a_2,a_3\}) = E(\{2\},\{a_3,a_1\}) = E'(\{1\},\{a_2,a_3\}) = E'(\{2\},\{a_3,a_1\}) = 1;$
$1 = E(\{1,2\},\{a_3\}) > E'(\{1,2\},\{a_3\}) = 0.$

*Obviously, EFF $E'$ is not superadditive, while EFF $E$ might be superadditive and inequality $E' < E$ might hold. Moreover, both $E$ and $E'$ can be monotone.*

### 3.5 On superadditivity and weak superadditivity of the minimum monotone majorant of an EFF

It is clear that superadditivity of an EFF $E$ does not imply even weak 2-superadditivity of a majorant $E' \geq E$. Indeed, let us consider, for example, the "absolutely minimal" EFF $E$ defined by formula: $E(K, B) = 1$ if and only if $B = A$. (Recall that $E(\emptyset, A) = 1$, by the boundary conditions.) Obviously, $E$ is superadditive and inequality $E \leq E'$ holds for every EFF $E'$.

However, both superadditivity and weak superadditivity of an EFF $E$ are inherited by the minimum monotone majorant $E' = E^M$ of $E$.

**Proposition 2** *If EFF $E$ is (weakly) superadditive then its minimum monotone majorant $E^M$ is (weakly) superadditive, too.*

**Proof**. Let $J$ be a set of indices and $E^M(K_j^M, B_j^M) = 1$ for each $j \in J$, where coalitions $\{K_j^M : j \in J\}$ are pairwise disjoint. Then, by definition of $E^M$, equality $E(K_j, B_j) = 1$ holds for some $K_j \subseteq K_j^M$, $B_j \subseteq B_j^M$, and $j \in J$. In particular, these coalitions $\{K_j : j \in J\}$ are pairwise disjoint, too.

If $E$ is weakly superadditive then $\bigcap_{j \in J} B_j \neq \emptyset$. Hence, $\bigcap_{j \in J} B_j^M \neq \emptyset$ and, thus, $E^M$ is weakly superadditive, too.

If $E$ is superadditive then $E(\bigcup_{j \in J} K_j, \bigcap_{j \in J} B_j) = 1$. Hence, by containments $K_j \subseteq K_j^M$ and $B_j \subseteq B_j^M$ for $j \in J$, by monotonicity of $E^M$, and by inequality $E^M \geq E$, we conclude that $E^M(\bigcup_{j \in J} K_j^M, \bigcap_{j \in J} B_j^M) = 1$ and, thus, $E^M$ is superadditive, too. □

Yet, the inverse implication holds only for weak superadditivity.

**Proposition 3** *An EFF $E$ is weakly superadditive whenever its minimum monotone majorant $E^M$ is weakly superadditive.*

**Proof**. Let $J$ be a set of indices and $E(K_j, B_j) = 1$ for each $j \in J$, where coalitions $\{K_j : j \in J\}$ are pairwise disjoint. Then, $E^M(K_j, B_j) = 1$, too, by inequality $E^M \geq E$. Hence, $\bigcap_{j \in J} B_j \neq \emptyset$, by weak superadditivity of $E^M$. Thus, EFF $E$ is weakly superadditive, too. □

**Corollary 1** *An EFF $E$ is weakly superadditive if and only if its minimum monotone majorant $E^M$ is weakly superadditive.*

**Proof**. It follows immediately from Propositions 2 and 3. □

However, Proposition 3 does not extend to the case of superadditivity.

**Example 4** *An EFF $E$ such that*
$E(\{1\}, \{a_3\}) = E(\{2\}, \{a_3\}) = E(\{1\}, \{a_2, a_3\}) = E(\{2\}, \{a_3, a_1\}) = 1$, *and* $E(\{1, 2\}, \{a_3\}) = 0$, *is not superadditive, while $E^M$ might be superadditive.*

# 4 Game forms and their effectivity functions

Let $X_i$ be a finite set of strategies of the player $i \in I$ and $X = \prod_{i \in I} X_i$. A *game form* is defined as a mapping $g : X \to A$ that assigns an outcome $a \in A$ to each strategy profile $x = (x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n = X$. We will assume that mapping $g$ is surjective, that is, $g(X) = A$; yet typically, $g$ is not injective, that is, the same outcome might be assigned to several distinct strategy profiles.

A game form can be viewed as a game in normal form in which payoffs are not specified yet. Given a game form $g$, let us introduce an EFF $E_g$ as follows:

$E_g(K, B) = 1$ for a coalition $K \subseteq I$ and block $B \subseteq A$ if and only if there is a strategy $x_K = (x_i : \ i \in K)$ of coalition $K$ such that the outcome $g(x_K, x_{I \setminus K})$ is in $B$ for every strategy $x_{I \setminus K} = (x_i : \ i \notin K)$ of the complementary coalition.

**Remark 3** *The EFF $E_g$ was introduced in [23], where it is called $\alpha$-EFF of $g$ and, respectively, notation $\alpha$-$E_g$ is applied. The EFF $\beta$-$E_g$ is also defined in [23]. Yet, we find it more convenient to substitute $E_g$ and $E_g^d$ for $\alpha$-$E_g$ and $\beta$-$E_g$, where the dual EFF $E_g^d$ will be introduced in Section 8.*

Let us recall that the boundary values $E_g(\emptyset, B)$ are not defined yet. By the boundary conditions, we set $E_g(\emptyset, A) = 1$ and $E_g(\emptyset, B) = 0$ whenever $B \neq A$.

Let us also notice that $E_g(I, \emptyset) = 0$ and $E_g(I, B) = 1$ for all non-empty $B \subseteq A$, since $g$ is surjective. Thus, all boundary conditions hold for EFF $E_g$.

**Proposition 4** *EFF $E_g$ is monotone and superadditive for every game form $g$.*

This statement was shown already by Abdou [1, 2], Moulin and Peleg [23].

**Proof**. First, let us consider monotonicity. If $E_g(K, B) = 1$ then, by definition, coalition $K$ has a strategy $x_K = (x_i : \ i \in K)$ enforcing $B$. Furthermore, if $K \subseteq K'$ and $B \subseteq B'$ then $K'$ has a strategy $x_{K'} = (x_i : \ i \in K')$ enforcing $B'$. Indeed, $g(x) \in B \subseteq B'$ whenever coalitionists of $K$ play in accordance with $x_K$, while players of $K' \setminus K$ apply arbitrary strategies. In this case, $E(K', B') = 1$, too. Hence, $E_g$ is monotone.

Now, let us prove superadditivity. Let $E(K_1, B_1) = E(K_2, B_2) = 1$ and $K_1 \cap K_2 = \emptyset$. By definition of $E_g$, coalition $K_j$ has a strategy $x_{K_j}$ enforcing $B_j$, where $j = 1$ or 2. Since coalitions $K_1$ and $K_2$ are disjoint, they can apply these strategies $x_{K_1}$ and $x_{K_2}$ simultaneously. Obviously, the resulting strategy $x_K$ of the union $K = K_1 \cup K_2$ enforces the intersection $B = B_1 \cap B_2$. $\square$

# 5 Main theorems

It is natural to ask whether the inverse is true too. A positive answer was given in 1983 by Moulin [22], Theorem 1 of Chapter 7.

**Theorem 1** *An EFF is formal if and only if it is monotone and superadditive.*

In 1998, Peleg [26] proved a slightly more general claim. In particular, his proof works for infinite sets of outcomes $A$. Both proofs are constructive. Yet, the number $|X_i|$ of strategies of a player $i \in I$ is doubly exponential in the size of the (*monotone*) input EFF $E$. In this paper, we suggest a third construction in which $|X_i|$ is only linear in the size of $E$ for every player $i \in I$; more precisely,

$$|X_i| = |A| + deg(i, \mathcal{K}_E) \leq |A| + |J| = p + m.$$

Here the monotone EFF $E = \{(K_j, B_j) \mid j \in J\}$ is given as in Section 3, $\mathcal{K}_E = \{K_j : \; j \in J\}$ is the corresponding multi-hypergraph of the coalitions, and $deg(i, \mathcal{K}_E) = \#\{j \in J \mid i \in K_j\}$ is the degree of player $i$ in $\mathcal{K}_E$.

The following statement will be instrumental in our proof of Theorem 1 and is also of independent interest.

**Theorem 2** *An EFF is formal-minor if and only if it is weakly superadditive.*

In fact, we can immediately extend this statement as follows.

**Theorem 3** *The next four properties of an EFF $E$ are equivalent:*

*(i) $E$ is formal-minor; (ii) $E$ is weakly superadditive;*

*(iii) $E^M$ is formal-minor; (iv) $E^M$ is weakly superadditive.*

**Proof**. Equivalence of (i) and (ii) (as well as of (iii) and (iv), in particular) is claimed by Theorem 2. Furthermore, (i) and (iii) are equivalent, too, by the definition of the minimum monotone majorant $E^M$ and monotonicity of $E_g$. □

Let us remark that Proposition 3 follows from Theorem 3.

We will prove Theorems 1 and 2 in the next two subsections.

In accordance with Section 3.3, it can be verified in polynomial time whether a (monotone) EFF is formal or whether it is superadditive; in contrast, to verify whether a monotone EFF is formal-minor or whether it is weakly superadditive is a CoNP-complete decision problem.

# 6 Main proofs

## 6.1 Proof of Theorem 2

Obviously, an EFF $E$ is formal-minor if and only if $E^M$ is formal-minor. Since $E^M$ is monotone, it can be conveniently specified by the list $\{(K_j, B_j) \mid j \in J\}$ of all inclusion-minimal pairs such that $E^M(K_j, B_j) = 1$.

Clearly, $E = E^M$ whenever EFF $E$ is monotone; otherwise the input size of $E$ might be much larger: $E$ is specified by the list of *all* (not only inclusion-minimal) pairs $\{(K_j, B_j) \mid j \in J'\}$ such that $E(K_j, B_j) = 1$. Yet, we can easily reduce this list $J'$ to $J$ by leaving only inclusion-minimal pairs and eliminating all others. This reduction, obviously, results in $E^M$. Thus, without loss of generality, we can assume that $E = E^M$, or in other words, that the input EFF $E$ is monotone and given by the list $\{(K_j, B_j) \mid j \in J\}$.

If $E \leq E_g$ for some game form $g$, then Propositions 1 and 4 imply that $E$ is weakly supperadditive.

For the converse direction, given a monotone weakly superadditive EFF $E$, we want to construct a game form $g$ such that $E \leq E_g$. To do this, let us assign a set of strategies $X_i = \{x_i^j \mid i \in K_j\}$ to each player $i \in I$. In other words, given $i \in I$ and $j \in J$, strategy $x_i^j$ is unique whenever $i \in K_j$ and it is not defined otherwise. Thus, $|X_i| = deg(i, \mathcal{K})$, where $\mathcal{K}$ is the multi-hypergraph of coalitions $\mathcal{K} = \{K_j \mid j \in J\}$.

Given $j \in J$, a (unique) strategy $x_{K_j} = (x_i^j : i \in K_j)$ of coalition $K_j$ is called *proper*. We will call a game form $g$ *proper*, if for each proper strategy $x_{K_j}$ and each strategy $x_{I \setminus K_j}$ of the complementary coalition, inclusion $g(x_{K_j}, x_{I \setminus K_j}) \in B_j$ holds.

Let us show that we can construct a proper game form $g$ such that $E \leq E_g$ whenever EFF $E$ is weakly superadditive. Indeed, if a strategy profile $x = (x_1, \ldots, x_n)$ is proper with respect to several coalitions $\{K_j \mid j \in J' \subseteq J\}$ then, obviously, these coalitions are pairwise disjoint and, hence, $\bigcap_{j \in J'} B_j \neq \emptyset$.

For each strategy profile $x \in X$ let us choose an outcome $a$ from the above intersection and fix $g(x) = a$. If $x$ is proper for no $j \in J$ then choose $g(x) \in A$ arbitrarily. This construction defines a proper game form $g : X \to A$.

The desired inequality $E \leq E_g$ holds for each proper game form $g$. Indeed, let $E(K, B) = 1$; then $E(K_j, B_j) = 1$ for some $j \in J$; then $g(x_{K_j}, x_{I \setminus K_j}) \in B_j$ for each $x_{I \setminus K_j}$ whenever $x_{K_j}$ is the proper strategy of $K_j$. $\square$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | $a_1$ | $a_2$ | | | | |
| | | | $a_2$ | $a_3$ | | | | |
| | | | $a_3$ | $a_4$ | | | | |
| $a_1$ | $a_2$ | $a_4$ | $a_1$ | $a_2$ | $a_1$ | $a_2$ | $a_4$ | $a_1$ |
| $a_1$ | $a_3$ | $a_4$ | $a_1$ | $a_3$ | $a_1$ | $a_3$ | $a_4$ | $a_1$ |
| | | | $a_1$ | $a_4$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ |
| | | | $a_2$ | $a_2$ | $a_4$ | $a_1$ | $a_2$ | $a_3$ |
| | | | $a_3$ | $a_3$ | $a_3$ | $a_4$ | $a_1$ | $a_2$ |
| | | | $a_1$ | $a_4$ | $a_2$ | $a_3$ | $a_4$ | $a_1$ |

Table 1: Two-person EFF $E_g$.

Let us consider an example given by the upper left corner (the first two rows and columns) of Table 1. In this example $I = \{1, 2\}$, $A = \{a_1, a_2, a_3, a_4\}$, and EFF $E$ is given by the list:

$E(1, \{a_1, a_2, a_4\}) = E(1, \{a_1, a_3, a_4\}) =$
$E(2, \{a_1, a_2, a_3\}) = E(2, \{a_2, a_3, a_4\}) = 1.$

Each of the four entries of the desired game form must be an outcome of the corresponding intersection. The obtained EFF $E_g$ is given by the list:

$E_g(1, \{a_1, a_2\}) = E_g(1, \{a_1, a_3\}) = E_g(2, \{a_1\}) = E_g(2, \{a_2, a_3\}) = 1.$

Of course, $E \leq E_g$, however, $E \neq E_g$. Similar observations were made by Moulin; see [22] Theorem 1 of Chapter 7, pp. 166-168.

**Remark 4** *Let $\mathcal{K} = \{K_j \mid j \in J\}$ be a family of coalitions and $\{x_{K_j} \mid j \in J\}$ be a family of their strategies. If the coalitions of $\mathcal{K}$ are pairwise disjoint (pairwise intersect) then the corresponding faces in the direct product $X = \prod_{i \in I} X_i$ must intersect (might be pairwise disjoint). This observation, which is instrumental in the above proof of Theorem 2, was mentioned already in [19] (Lemma 6) and illustrated in case $n = 3$ and $\mathcal{K} = \{\{1, 2\}, \{2, 3\}, \{3, 1\}\}$.*

## 6.2 Proof of Theorem 1

Now we assume that EFF $E = \{(K_j, B_j) \mid j \in J\}$ is monotone and superadditive and want to construct a game form $g$ such that $E = E_g$. In the previous section, we already got a game form $g'$ such that $E \leq E_{g'}$. To enforce the equality, we will have to extend $g'$ to $g$ as follows. To each player $i \in I$, in addition to the *proper* strategies $X'_i = \{x_i^j : \ i \in K_j\}$, we will add $p = |A|$ *backup* strategies $X''_i = \{x_i^b : \ b \in \{0, 1, \ldots, p-1\}\}$. Thus, $X_i = X'_i \cup X''_i$ for all $i \in I$ and $X = \prod_{i \in I} X_i = \prod_{i \in I}(X'_i \cup X''_i)$.

Thus, each strategy profile $x \in X$ defines a unique partition $I = K' \cup K''$, where $K' = K'(x)$ and $K'' = K''(x)$ are the coalitions of all "proper" and "backup" players, respectively, that is, $x_i \in X'_i$ for $i \in K'$ and $x_i \in X''_i$ for $i \in K''$. To obtain the desired game form $g : X \to A$ (such that $E_g = E$), we will define $g(x)$ successively for $|K''(x)| = k(x) = k = 0, 1, \ldots, n$.

Two extreme cases, $k = 0$ and $k = n$ are simple. If $k(x) = 0$, that is, in $x$ all players choose proper strategies , then $g(x) = g'(x)$ is defined as in the previous section. If $k(x) = n$, that is, in $x$ all players choose backup strategies $x_i \in X''_i = \{x_i^{b_i} : \ b_i \in \{0, 1, \ldots, p-1\}\}$ , then

$$g(x) = a_r \in A = \{a_1, \ldots, a_p\}, \ \text{ where } \ r - 1 = \sum_{i=1}^{n} b_i \mod p. \tag{1}$$

Table 3 and the lower right $4 \times 4$ corner of Table 1 provide two examples, with $n = p = 3$ and $n = 2, p = 4$, respectively.

Now, we plan to define $g(x)$ for $k(x) \in \{1, \ldots, n-1\}$.

First, we have to extend the concepts of a proper coalition, strategy, and game form defined in the previous section. Given a strategy profile $x \in X$, let us consider partition $I = K'(x) \cup K''(x)$, where players of $K'$ and $K''$ choose in $x$ their proper and backup strategies, respectively. A coalition $K_j$ is called *proper* if $x_i = x_i^j$ for each $i \in K_j$. By this definition, $K_j \subseteq K'(x)$, that is, each proper coalition is a subcoalition of $K'(x)$. The obtained strategy $x_{K_j} = (x_i^j : \ i \in K_j)$ of coalition $K_j$ is called *proper*, too. If for each such strategy and every strategy $x_{I \setminus K_j}$ of the complementary coalition, inclusion $g(x_{K_j}, x_{I \setminus K_j}) \in B_j$ holds then game form $g$ will be also called *proper*. As before, these conditions are not contradictory whenever EFF $E$ is (weakly) superadditive. Indeed, if several coalitions $\{K_j : j \in J' \subseteq J\}$ are proper with respect to a given strategy profile $x = (x_1, \dots, x_n)$ then, obviously, these coalitions are pairwise disjoint and, hence, $B(x) = \bigcap_{j \in J'} B_j \neq \emptyset$.

Two strategy profiles $x', x'' \in X$ will be called equivalent if the corresponding partitions coincide, or in other words, if $K'(x') = K'(x'') = K$ and, moreover, $x'_i = x''_i$ for every $i \in K$. Obviously, these classes partition $X$.

Given $x \in X$, let $|K''(x)| = k(x) = k$ and $|B(x)| = q(x) = q$; furthermore, let, for simplicity, $K''(x) = \{1, \dots, k\} \subseteq I$ and $B(x) = \{a_1, \dots, a_q\} \subseteq A$.

We generalize formula (1) for arbitrary integral $q \leq p$ and $k \leq n$ as follows:

$$g(x) = a_r \in B(x) = \{a_1, \dots, a_q\}, \text{ where } r - 1 = \sum_{i=1}^{k} b_i \mod q, \tag{2}$$

whenever in the given profile $x \in X$ each player $i \in K''(x)$ chooses a backup strategy $x_i = b_i \in \{0, 1, \dots, p-1\}$.

Several examples are given in Tables 2 and 3, where $p = 4$ or $p = 5$, $q = 3$, $k = 2$ and $p = q = k = 3$, respectively.

| | | | |
|---|---|---|---|
| $a_1$ | $a_2$ | $a_3$ | $a_1$ |
| $a_1$ | $a_1$ | $a_2$ | $a_3$ |
| $a_3$ | $a_1$ | $a_1$ | $a_2$ |
| $a_2$ | $a_3$ | $a_1$ | $a_1$ |

| | | | | |
|---|---|---|---|---|
| $a_1$ | $a_2$ | $a_3$ | $a_1$ | $a_2$ |
| $a_2$ | $a_1$ | $a_2$ | $a_3$ | $a_1$ |
| $a_1$ | $a_2$ | $a_1$ | $a_2$ | $a_3$ |
| $a_3$ | $a_1$ | $a_2$ | $a_1$ | $a_2$ |
| $a_2$ | $a_3$ | $a_1$ | $a_2$ | $a_1$ |

Table 2: $q = 3$, $k = 2$, $p = 4$ and $p = 5$.

| | | |
|---|---|---|
| $a_1$ | $a_2$ | $a_3$ |
| $a_3$ | $a_1$ | $a_2$ |
| $a_2$ | $a_3$ | $a_1$ |

| | | |
|---|---|---|
| $a_2$ | $a_3$ | $a_1$ |
| $a_1$ | $a_2$ | $a_3$ |
| $a_3$ | $a_1$ | $a_2$ |

| | | |
|---|---|---|
| $a_3$ | $a_1$ | $a_2$ |
| $a_2$ | $a_3$ | $a_1$ |
| $a_1$ | $a_2$ | $a_3$ |

Table 3: $p = q = k = 3$.

By the above definition, for every $x \in X$, there are exactly $p^{k(x)}$ strategy profiles equivalent with $x$. Let us define function (game form) $g$ on these profiles in accordance with (2).

In particular, $g(x) = g'(x)$ when $K''(x) = \emptyset$ and $g(x)$ is defined by (1) when $K'(x) = \emptyset$. Table 1 represents an example in which $n = 2$ and $p = 4$.

By construction, each strategy $x_K$ is effective for the block $B(x_K) = \cap_{j \in J'} B_j$, where $J' = J(x_K) \subseteq J$ is defined as follows: $x_K$ is a proper strategy of $K_j$ if and only if $j \in J'$. In particular, $K_j \subseteq K$ for all $j \in J'$. By monotonicity and superadditivity, we also have $E(K, B(x_K)) = 1$. It follows that $E_g \leq E$.

On the other hand, $E_g(K_j, B_j) = 1$ for all $j \in J$, since the proper strategy $x_{K_j} = (x_i^j \mid i \in K_j)$ is effective for $B_j$. Thus, by the above construction, equality $E = E_g$ holds if and only if the input EFF $E$ is monotone and superadditive. □

**Remark 5** *In general, the obtained EFF $E_g$ is the minimum monotone and superadditive majorant of the input EFF $E$.*

*Let us also note that the above construction is computationally efficient: for every strategy profile $x$ the corresponding outcome $g(x)$ is determined in polynomial time. Obviously, the same is true in case of Theorem 2 too.*

### 6.3 Theorem 2 results from Theorem 1

We derived Theorem 1 from Theorem 2. In fact, the latter is of independent interest. For example, it is instrumental in the proof of the Berge and Duchet conjecture in [4]; see also [5, 6]. In these papers, conversely, Theorem 2 was derived from Theorem 1, since the latter was already published by Moulin.

**Remark 6** *In an old joke, a mathematician solves the problem of boiling water in the kettle as follows: "... If water is already in the kettle then pour it out and, by this, the problem is reduced to the previous one".*

An EFF $E$ and its minimum monotone majorant $E^M$ can be weakly superadditive or, respectively, formal-minor only simultaneously. Moreover, $E \leq E_g$ if and only if $E^M \leq E_g$, since $E_g$ and $E^M$ are both monotone. Hence, we can prove Theorem 2 for $E^M$ rather than for $E$. Since EFF $E^M$ is monotone, it is uniquely defined by the set of its minimal "ones" $E^M = \{(K_j, B_j) \mid j \in J\}$.

First, let us assume that $E^M$ is formal-minor, that is, $E^M \leq E_g$ for a game form $g$. Furthermore, let $J' \subseteq J$ be a family of pairwise disjoint coalitions, $K'_{j'} \cap K'_{j''} = \emptyset$ for all $j'.j'' \in J'$ such that $j' \neq j''$. Obviously, $E^M \leq E_g$ implies that $E_g(K_j, B_j) = 1$ for all $j \in J$. By Proposition 4, $E_g$ is monotone, superadditive, and satisfies the boundary conditions; see Section 4. Hence, $E_g(\cup_{j \in J'} K_j, \cap_{j \in J'} B_j) = 1$, by superadditivity, and then $\cap_{j \in J'} B_j \neq \emptyset$, by boundary condition. Thus, EFFs $E^M$ (and $E$) are weakly superadditive.

Conversely, let $E^M$ be weakly superadditive. Let us define an EFF $E'$ by setting $E'(K, B) = 1$ if and only if $B = A$, or $K = I$ and $B \neq \emptyset$, or there is a non-empty subset $J' \subseteq J$ such that $B \supseteq \cap_{j \in J'} B_j$, $K \supseteq \cup_{j \in J'} K_j$, and the corresponding coalitions, $\{K_j \mid j \in J'\}$ are pairwise disjoint. By this definition, $E^M \leq E'$. Furthermore, it is not difficult to verify that the obtained EFF $E'$ is monotone, superadditive, and satisfies the boundary conditions. Hence, by Theorem 1, $E' = E_g$ for a game form $g$. Thus, $E^M$ and $E$ are formal-minor.

## 7 Graphs and their effectivity functions

Given a graph $G = (J, E)$, let us assign a player (outcome) to every its inclusion-maximal clique (independent set) and denote the obtained two sets by $I_G$ and $A_G$. Then, for every vertex $j \in J$ let us consider the coalition $K_j$ (block $B_j$) corresponding to all maximal cliques (independent sets) that contain vertex $j$. The obtained list $\{(K_j, B_j) \mid j \in J\}$ defines an EFF $E_G$. The next claim is instrumental in the proof of the Berge-Duchet conjecture in [4]; see also [5, 6].

**Lemma 1** *For every graph $G$ the corresponding EFF $E_G$ is formal-minor.*

**Proof**. By Theorem 2, it is enough to show that $E_G$ is weakly supperadditive. Let $J' \subseteq J$ be a set of vertices in $G$ such that the coalitions $\{K_j : j \in J'\}$ are pairwise disjoint. Then, obviously, $J'$ is an independent set of $G$. Indeed, $K_{j'} \cap K_{j''} \neq \emptyset$ if and only if $\{j', j''\}$ is an edge of $G$. Let $J''$ be a maximal independent set that contains $J'$ and $a \in A_G$ be the corresponding outcome. Then, obviously, $a \in \cap_{j \in J'} B_j \neq \emptyset$ □

Thus, there is a game form $g : \prod_{i \in I_G} X_i \to A_G$ such that $E_G \leq E_g$.

Although both sets $I_G$ and $A_G$ might be exponential in $|J|$, yet, by the construction of Theorem 2, it follows that one can choose a game form $g$ of a "pretty modest" size, namely, $|X_i| \leq |J|$ for all $i \in I_G$.

# 8 Tight game forms and self-dual EFFs

## 8.1 Dual and self-dual effectivity functions

To each EFF $E$ let us assign a dual EFF $E^d$ defined by formula:

$E^d(K, B) + E(I \setminus K, A \setminus B) = 1 \;\; \forall \, K \subseteq I, \; B \subseteq A.$

In other words, $E^d(K, B) = 1$ if and only if $E(I \setminus K, A \setminus B) = 0$.

It is not difficult to verify that two EFFs are dual if and only if the corresponding two Boolean functions are dual. (Let us also recall that an EFF is monotone if and only if the corresponding Boolean function is monotone.) Thus, our terminology for EFFs is in agreement with the standard Boolean language.

Respectively, an EFF $E$ is called *self-dual* (or *maximal*) if

$$E(K, B) + E(I \setminus K, A \setminus B) = 1, \;\; \forall \, K \subseteq I, \; B \subseteq A,$$

that is, $K$ is effective for $B$ if and only if $I \setminus K$ is not effective for $A \setminus B$.

It is easy to see that the inequality

$$E(K, B) + E(I \setminus K, A \setminus B) \leq 1, \;\; \forall \, K \subseteq I, \; B \subseteq A,$$

holds for every weakly superadditive EFF. Indeed, otherwise

$$E(K, B) = E(I \setminus K, A \setminus B) = 1,$$

in contradiction with weak superadditivity. Hence, $E(K, B) = 0$ whenever $E(I \setminus K, A \setminus B) = 1$.

An EFF $E$ is self-dual if and only if the inverse implication holds. In other words, the equalities $E(K, B) = E(I \setminus K, A \setminus B) = 0$ might hold for some $K \subseteq I, \; B \subseteq A$ of an EFF $E$; they cannot hold if and only if EFF $E$ is self-dual.

**Remark 7** *In particular, the self-dual EFFs are maximal, with respect to the partial order "$\leq$", among the weakly superadditive (as well as among superadditive, or formal, or formal-minor) EFFs. For this reason, term "maximal", rather than "self-dual", is frequent in the literature; see, for example, [22, 25, 3]. However, in this paper, we will follow Boolean terminology.*

**Remark 8** *Let us also recall that, by the boundary conditions,* $E(I, \emptyset) = 0$ *and* $E(\emptyset, A) = 1$*, in agreement with self-duality.*

## 8.2 Tight game forms; $T$-formal and $T$-formal-minor EFFs

A game form $g$ is called *tight* if its EFF $E_g$ is self-dual.

Let us recall that EFF $E$ is T-formal (T-formal-minor) if and only if $E = E_g$ (respectively, $E \leq E_g$) for a *tight* game form $g$. It is not difficult to show that the families of the formal-minor and T-formal-minor EFFs just coincide.

**Proposition 5** *An EFF is T-formal-minor if and only if it is formal-minor.*

**Proof**. Indeed, it is shown in [17] that every game form $g$ can be extended to a tight one; in other words, for each $g$ there is a tight game form $g'$ such that $g$ is a subform of $g'$ and $E_g \leq E_{g'}$. □

Furthermore, just by definition, an EFF is T-formal if and only if it is formal and self-dual. Moreover, the following statement holds.

**Theorem 4** *An EFF $E$ is T-formal if and only if it is monotone, superadditive, and self-dual. The next four properties of a self-dual EFF $E$ are equivalent:*

*(a) $E$ is T-formal;   (b) $E$ is monotone and superadditive;*

*(c) $E$ is T-formal-minor;   (d) $E$ is weakly superadditive.*

**Proof**. The first claim immediately follows from Theorem 1 and the definition of tightness and results in equivalence of (a) and (b). Furthermore, obviously, (a) implies (c). To show the inverse let us assume indirectly that the strict inequality $E < E_g$ holds for a self-dual EFF $E$ and a tight game form $g$.

Yet, let us also recall that the inequality $E_g(K, B) + E_g(I \setminus K, A \setminus B) \leq 1$ holds for a game form $g$ and the identity $E_g(K, B) + E_g(I \setminus K, A \setminus B) \equiv 1$ holds whenever $g$ is tight. Since $E < E_g$, there is a pair $K \subseteq I, B \subseteq A$ such that $E(K, B) = E(I \setminus K, A \setminus B) = 0$. Then, by duality, $E^d(K, B) = E^d(I \setminus K, A \setminus B) = 1$ and we get a contradiction, since EFF $E$ is self-dual, $E = E^d$.

The same arguments, in slightly different terms, appear already in [22].

Finally, Theorem 2 and Proposition 5 imply that (d) is equivalent to (c). □

## 8.3 On tightness and Nash-solvability

Given sets of players (voters) $I$ and outcomes (candidates) $A$, the *utility (payoff, preference) function* is introduced by a mapping $u : I \times A \to \mathbb{R}$, where $u(i, a)$ is interpreted as a profit of player $i \in I$ in case outcome $a \in A$ is realized.

Given also a game form $g : X \to A$, the pair $(g, u)$ is a *game in normal form.*

A strategy profile $x = (x_i : i \in I) \in \prod_{i \in I} X_i = X$ is called a *Nash equilibrium* in game $(g, u)$ if $u(i, x) \geq u(i, x')$ for each player $i \in I$ and each strategy profile $x'$ obtained from $x$ by substituting a strategy $x'_i$ for $x_i$. In other words, $x$ is a Nash equilibrium if a player can make no profit in $x$ by choosing another strategy provided all other players keep their old strategies.

A game form $g$ is called Nash-solvable if for each utility function $u$ the obtained game $(g, u)$ has a Nash equilibrium.

**Theorem 5** *A two-person game form is Nash-solvable if and only if it is tight.*

This result was obtained in 1975 [12]; see also [14] and [8] Appendix 1, where it is also shown that in case of more than two players tightness is no longer necessary or sufficient for Nash-solvability.

In contrast, for *two-person zero-sum* games tightness remains necessary. More precisely, let $I = \{1, 2\}$, a utility function $u : I \times A \to \mathbb{R}$ is called *zero-sum* if $u(1, a) + u(2, a) = 0$ for each outcome $a \in A$. A game form $g$ is called *zero-sum-solvable* ($\pm 1$-solvable) if for every zero-sum (and taking only $\pm 1$-values) utility function $u$ the obtained zero-sum game $(g, u)$ has a saddle point.

**Theorem 6** *The following properties of a two-person game form are equivalent: (i) Nash-solvability, (ii) zero-sum-solvability, (iii) $\pm 1$-solvability, (iv) tightness.*

Equivalence of (ii), (iii), and (iv) was demonstrated in 1970 by Edmonds and Fulkerson [10] and then, independently, in [11].

To make the paper self-contained we will prove Theorem 6 in Section 10.

## 8.4 Tightness and Boolean duality

Self-duality of monotone EFFs (and hence, tightness of the corresponding game forms) can be conveniently reformulated in Boolean terms as follows. Given a monotone EFF $E : 2^I \times 2^A \to \{0, 1\}$, let us assign a Boolean variable $a$ to every outcome $a \in A$. (For simplicity we denote the outcome and the corresponding variable by the same symbol $a$.) Then, for each coalition $K \subseteq I$ let us introduce a monotone Boolean function defined by the following positive (that is, negation-free) disjunctive normal form (DNF)

$$F_K = \bigvee_{B \mid E(K,B)=1} \bigwedge_{a \in B} a.$$

By this definition, EFF $E$ is self-dual if and only if Boolean functions $F_K$ and $F_{I \setminus K}$ are dual, $F_K^d = F_{I \setminus K}$ for all $K \subseteq I$. More details about DNFs and duality can be found in any Boolean textbook; see, for example, [9].

For $K = \emptyset$ and $K = I$ duality follows from the boundary conditions. Hence, self-duality of a two-person EFF $E$ is reduced to simply $F_1^d = F_2$.

Also by definition, for a formal EFF $E = E_g$ its DNFs are given by formula:

$$F_K(g) = \bigvee_{x_K = (x_i : i \in K)} \bigwedge_{x_{I \setminus K} = (x_i : i \notin K)} g(x_K, x_{I \setminus K}) \quad \forall \; K \subseteq I. \tag{3}$$

Thus, game form $g$ is tight if and only if $F_K^d(g) = F_{I \setminus K}(g)$ for all $K \subseteq I$.

## 9 On totally tight game forms and TT-formal effectivity functions

### 9.1 Two-person case

Let us start with the case $n = 2$. A two-person game form $g$ is called *totally tight* (TT) if every $2 \times 2$ subform of $g$ is tight.

Up to an isomorphism, there are only seven $2 \times 2$ game forms:

$$\begin{matrix} a_1a_1 & a_1a_1 & a_1a_1 & a_1a_1 & & a_1a_2 & a_1a_2 & a_1a_2 \\ a_1a_1 & a_1a_2 & a_2a_2 & a_2a_3 & & a_2a_1 & a_2a_3 & a_3a_4 \end{matrix}$$

The first four are tight, while the last three are not. Thus, a $2 \times 2$ game form is tight if and only if it has a constant line, row or column.

Let $g$ be a game form with a constant line and let $g'$ be the subform of $g$ obtained by eliminating this line. Obviously, $g$ is TT if and only if $g'$ is TT.

Let us also remark that $g$ might be tight, while $g'$ is not; see [7] for the corresponding examples. However, $g$ is tight whenever $g'$ is tight.

A TT game form with a constant line is called *reducible.*

Somewhat surprisingly, all irreducible TT game forms have the same EFF.

**Theorem 7** *([7]) Let $g : X_1 \times X_2 \to A$ be an irreducible TT two-person game form. Then there are three outcomes $a_1, a_2, a_3 \in A$ such that*
$E_g(i, \{a_1, a_2\}) = E_g(i, \{a_2, a_3\}) = E_g(i, \{a_3, a_1\}) = 1$, *while* $E_g(i, \{a_j\}) = 0$, *for* $i \in I = \{1, 2\}$, $j \in J = \{1, 2, 3\}$.

It is easily seen that EFF $E_g$ is uniquely defined by the above equalities and boundary conditions. It is also uniquely defined by the corresponding (self-dual) Boolean functions: $F_1(g) = F_2(g) = a_1a_2 \vee a_2a_3 \vee a_3a_1$.

Obviously, the $1 \times 1$ game form $g_0$ is TT, too. Yet, formally, it is reducible. For the corresponding EFF $E_{g_0}$ we have $E_{g_0}(1, \{a\}) = E_{g_0}(2, \{a\}) = 1$, where $a$ is the unique outcome of $g_0$; respectively, in Boolean terms $F_1(g_0) = F_2(g_0) = a$.

We will call this EFF *trivial*, while the EFF of Theorem 7 will be called $\binom{3}{2}$-EFF. Obviously, both EFFs are self-dual and, hence, the corresponding game forms are tight. Since the addition of a constant line to a game form respects its tightness, the next statement follows.

**Corollary 2** *A totally tight game form is tight.* □

The above proof was based on Theorem 7. There is an alternative very short proof based on Theorems 5, 6, and Shapley's condition for solvability of matrix games. If $g$ is TT then every its $2 \times 2$ subform $g'$ is tight. Then, obviously, $g'$ is Nash-solvable. (This follows, for example, from Theorems 5 and 6; although "these two guns are certainly too big for a fly that small".) Yet, in 1964, Shapley [28] proved that a matrix has a saddle point whenever every its $2 \times 2$ submatrix has one. By Shapley's theorem, game $(g, u)$ has a saddle point for each zero-sum payoff $u$. Thus, $g$ is tight, by Theorem 6. $\square$

A game form is called *totally reducible* if it can be reduced to the empty one by successive elimination of constant lines, rows and columns.

By definition, every TT game form is obtained from an empty or irreducible one by recursively adding constant lines. By this operation, the corresponding EFFs are changed in an obvious way, which we will call an extension by adding constant lines or ACL-extension, for short.

Thus, we obtain a recursive characterization for the EFFs of the TT two-person game forms, or in other words, for the TT-formal two-person EFFs.

**Theorem 8** *A two-person EFF $E$ is TT-formal if and only if it is an ACL-extension of the trivial or $\binom{3}{2}$-EFF.* $\square$

A recursive characterization of the two-person TT game forms themselves is obtained in [7]. It is based on Theorem 7, yet, somewhat surprisingly, is *much* more complicated than the latter.

To make the paper self-contained we will prove Theorem 7 in Section 11.

### 9.2 $n$-person case

Now, let $g : X \to A$ be an $n$-person game form, where $X = \prod_{i \in I} X_i$ and $I = \{1, \ldots, n\}$. Each coalition $K \subseteq I$ such that $K \neq \emptyset$ and $K \neq I$ defines a two-person game form $g_K : X_K \times X_{I \setminus K} \to A$, where

$X_K = \{x_K = (x_i : \ i \in K)\}$ and $X_{I \setminus K} = \{x_{I \setminus K} = (x_i : \ i \notin K)\}$

are the sets of strategies of two complementary coalitions $K$ and $I \setminus K$.

Game form $g$ is called *totally tight* (TT) if $g_K$ is TT for all $K$.

An EFF $E$ is called TT-formal (respectively, TT-formal-minor) if $E = E_g$ (respectively, $E \leq E_g$) for a TT game form $g$. By this definition, every TT-formal (TT-formal-minor) EFF $E$ is T-formal (formal-minor) and we obtain obvious necessary conditions. In particular, $E$ is (i) monotone, (ii) superadditive, and (iii) self-dual (respectively, $E$ and $E^M$ are weakly superadditive).

Furthermore, given an $n$-person EFF $E : 2^{I \cup A} \to \{0, 1\}$ and a coalition $K \subseteq I$, let us define a two-person EFF $E_K$ which is the restriction of $E$ to $K$ and $I \setminus K$. More precisely, $E_K(K', B) = 1$ if and only if $E(K', B) = 1$ and $K' = K$. Obviously, for each $K \subseteq I$ EFF $E_K$ is TT-formal (respectively, TT-formal-minor) whenever $E$ is. Thus, we obtain more necessary conditions.

Indeed, a recursive characterization of the two-person TT-formal EFFs was just obtained in the previous section. Yet, it remains open, whether the obtained necessary conditions are also sufficient for an EFF to be TT-formal. In general, characterizing TT-formal and TT-formal-minor EFFs remains an open problem.

# 10 Proof of Theorem 6 and its limits

## 10.1 Tight two-person game forms

Let $g : X_1 \times X_2 \to A$ be a two-person game form. By definition, $g$ is tight if its EFF $E_g$ is self-dual; or in Boolean terms, if $F_1^d(g) = F_2(g)$.

The next two tables represent tight and not tight game forms, respectively.

| | |
|---|---|
| $a_1$ | $a_1$ |
| $a_2$ | $a_3$ |

| | | | |
|---|---|---|---|
| $a_1$ | $a_1$ | $a_3$ | $a_3$ |
| $a_2$ | $a_4$ | $a_2$ | $a_4$ |

| | | |
|---|---|---|
| $a_0$ | $a_0$ | $a_0$ |
| $a_0$ | $a_1$ | $a_2$ |
| $a_0$ | $a_3$ | $a_4$ |

| | | |
|---|---|---|
| $a_1$ | $a_1$ | $a_3$ |
| $a_1$ | $a_2$ | $a_2$ |
| $a_3$ | $a_2$ | $a_3$ |

| | | |
|---|---|---|
| $a_1$ | $a_1$ | $a_2$ |
| $a_1$ | $a_1$ | $a_3$ |
| $a_4$ | $a_3$ | $a_3$ |

| | | | |
|---|---|---|---|
| $a_1$ | $a_2$ | $a_1$ | $a_2$ |
| $a_3$ | $a_4$ | $a_4$ | $a_3$ |
| $a_1$ | $a_4$ | $a_1$ | $a_5$ |
| $a_3$ | $a_2$ | $a_6$ | $a_2$ |

Table 4: Tight two-person game forms.

| | |
|---|---|
| $a_1$ | $a_2$ |
| $a_2$ | $a_1$ |

| | | |
|---|---|---|
| $a_0$ | $a_1$ | $a_2$ |
| $a_0$ | $a_2$ | $a_1$ |

| | | |
|---|---|---|
| $a_1$ | $a_1$ | $a_2$ |
| $a_4$ | $a_0$ | $a_2$ |
| $a_4$ | $a_3$ | $a_3$ |

Table 5: Not tight two-person game forms.

We will need more equivalent reformulations of tightness. Let us consider an arbitrary *reply mapping* $\phi_1 : X_2 \to X_1$ that assigns a strategy of player 1 (a row) to each strategy of player 2 (a column). In the special case, when this function takes a unique value $x_1 \in X_1$, we will use the notation $\phi_1^0 : X_2 \to \{x_1\}$. Let $gr(\phi_1) \subseteq X = X_1 \times X_2$ be the graph of $\phi_1$ in $X$ and $[\phi_1] = g(gr(\phi_1)) \subseteq A$ be the corresponding set of outcomes. Similarly we define $[\phi_1^0]$, $[\phi_2]$, and $[\phi_2^0]$.

**Proposition 6** *The following five properties of a game form are equivalent.*

*(j) For each $\phi_1$ there exists a $\phi_1^0$ such that $[\phi_1^0] \subseteq [\phi_1]$;*

*(jj) For each $\phi_2$ there exists a $\phi_2^0$ such that $[\phi_2^0] \subseteq [\phi_2]$;*

*(jjj) For each $\phi_1$ and $\phi_2$ we have $[\phi_1] \cap [\phi_2] \neq \emptyset$;*

*(jv) $E_g(1, B) + E_g(2, A \setminus B) \equiv 1$ for all $B \subseteq A$;*

*(v) game form $g$ is tight.*

**Proof**. (j) $\Rightarrow$ (jjj). Assume indirectly that (j) holds and (jjj) does not. The latter means that there exist $\phi_1$ and $\phi_2$ such that $[\phi_1] \cap [\phi_2] = \emptyset$, while by (j), there exists a $\phi_1^0$ such that $[\phi_1^0] \subseteq [\phi_1]$. Hence, $[\phi_1^0] \cap [\phi_2] = \emptyset$. However, this is impossible, since clearly, $gr(\phi_1^0) \cap gr(\phi_2) \neq \emptyset$ for every $\phi_1^0$ and $\phi_2$.

(jjj) ⇒ (j). Suppose that (j) does not hold, that is, there is a $\phi_1$ such that $[\phi_1^0] \subseteq [\phi_1]$ for no $\phi_1^0$. Choosing an outcome from $[\phi_1^0] \setminus [\phi_1]$ for each $\phi_1^0$ we get a mapping $\phi_2$ such that $[\phi_1] \cap [\phi_2] \neq \emptyset$. Hence, (jjj) does not hold either.

Thus, (j) and (jjj) are equivalent. Similarly, (jj) and (jjj) are equivalent. To come to this conclusion it is enough to rename players 1 and 2.

Furthermore, (jjj), (jv) and (v) are also equivalent, by definition. Indeed, $E_g(1, B) = E_g(2, A \setminus B) = 1$ hold for no $g$, since every row and column intersect. Yet, $E_g(1, B) = E_g(2, A \setminus B) = 0$ can hold. Obviously, this exactly means that $g$ is not tight and in this (and only in this) case (jjj) does not hold. □

It is a useful exercise to verify that all five properties of Proposition 6 hold for the first six and do not hold for the last three game forms of Section 10.1.

As we showed in Section 8.4, (jv) means Boolean duality $E_1^d(g) = E_2(g)$ which holds for the first six game forms and does not hold for the last three: $(a_1 \vee a_2a_3)^d = a_1a_2 \vee a_1a_3$, $(a_1a_3 \vee a_2a_4)^d = a_1a_2 \vee a_2a_3 \vee a_3a_4 \vee a_4a_1$, $a_0^d = a_0$, $(a_1a_2 \vee a_2a_3 \vee a_3a_1)^d = a_1a_2 \vee a_2a_3 \vee a_3a_1$, $(a_2a_1 \vee a_1a_3 \vee a_3a_4)^d = a_4a_1 \vee a_1a_3 \vee a_3a_2$, $(a_1a_2 \vee a_3a_4 \vee a_1a_4a_5 \vee a_2a_3a_6)^d = a_1a_3 \vee a_2a_4 \vee a_1a_4a_6 \vee a_2a_3a_5$;

$(a_1a_2)^d = a_1 \vee a_2 \neq a_1a_2$, $(a_0a_1a_2)^d = a_0 \vee a_1 \vee a_2 \neq a_0 \vee a_1a_2$,
$(a_1a_2 \vee a_2a_0a_4 \vee a_4a_3)^d = a_4a_1 \vee a_1a_0a_3 \vee a_3a_2 \vee a_2a_4 \neq a_4a_1 \vee a_1a_0a_3 \vee a_3a_2$.

## 10.2 Tightness and zero-sum-solvability

Let us recall that, by definition, a game form $g$ is zero-sum-solvable if for each utility function $u : A \to \mathbb{R}$ the obtained normal form game $(g, u)$ is solvable, that is, has a saddle point (in pure strategies). It is well-known that the latter property holds if and only if maxmin and minmax are equal, that is, if

$$v_1 = \max_{x_1 \in X_1} \min_{x_2 \in X_2} u(g(x_1, x_2)) = \min_{x_2 \in X_2} \max_{x_1 \in X_1} u(g(x_1, x_2)) = v_2.$$

**Proposition 7** *([10], see also [11]). (a) If game form $g$ is tight then it is zero-sum-solvable; (b) if $g$ is not tight then it is not $\pm 1$-solvable.*

**Proof**. . Suppose that $g$ is not tight. Then, by (jjj), there exist $\phi_1$ and $\phi_2$ such that $[\phi_1] \cap [\phi_2] = \emptyset$. Let us set $u(a) = 1$ for $a \in [\phi_1]$, $u(a) = -1$ for $a \in [\phi_2]$, and $u(a) = 1$ or $u(a) = -1$, arbitrarily, for all remaining $a \in A$. Obviously, for this $u$ we obtain $-1 = v_1 < v_2 = 1$ and hence, there is no saddle point in game $(g, u)$. Thus, game form $g$ is not $\pm 1$-solvable.

Suppose that $g$ is not zero-sum-solvable; i.e., there is a payoff $u : A \to \mathbb{R}$ such that game $(g, u)$ is not solvable, i.e., $v_1 < v_2$. Furthermore, for every $x_1 \in X_1$ there is an $x_2 \in X_2$ such that $u(g(x_1, x_2)) \leq v_1$ and for every $x_2 \in X_2$ there is an $x_1 \in X_1$ such that $u(g(x_1, x_2)) \geq v_2$. In particular, this implies that there exist $\phi_1$ and $\phi_2$ such that $[\phi_1] \cap [\phi_2] = \emptyset$. Hence, $g$ is not tight, by (jjj). □

## 10.3 Tightness implies Nash-solvability

Still, we have to prove that $g$ is Nash-solvable (not only zero-sum-solvable) whenever $g$ is tight. We will partition the set of outcomes $A$ into three pairwise disjoint subsets $A = B \cup B_1 \cup B_2$ such that

$(p1)$ $u(1,b) \geq u(1,b_1)$ for every $b \in B, b_1 \in B_1$ and

$(p2)$ $u(2,b) \geq u(2,b_2)$ for every $b \in B, b_2 \in B_2$.

Condition $(p1)$ (respectively, $(p2)$) means that any outcome of $B_1$ for player 1 (respectively, of $B_2$ for player 2) is not better than any outcome of $B$. We also assume that that the following two conditions hold for $A = B \cup B_1 \cup B_2$ too:

$(q1)$ $E_g(1, B_2) = 0$ and $(q2)$ $E_g(2, B_1) = 0$.

In other words, player 1 (resp., 2) cannot "punish" the opponent by forcing $B_2$ (resp., $B_1$). If $g$ is tight, these two conditions can be rewritten as follows:

$(q1')$ $E_g(2, B \cup B_1) = 1$ and $(q2')$ $E_g(1, B \cup B_2) = 1$.

Our proof is "dynamic". We will start with $B = A$ and reduce $B$ by sending its outcomes to $B_1$ and $B_2$ in such a way that all four above conditions hold.

Let us note that we cannot get $B = \emptyset$, since in this case $(q1')$ and $(q2')$ would imply that $E_g(2, B_1) = E_g(1, B_2) = 1$, in contradiction with $B_1 \cap B_2 = \emptyset$. (Let us remark that here is the only place where we make use of the tightness of $g$.)

Thus, there is a partition $A = B \cup B_1 \cup B_2$ such that $B$ cannot be reduced any longer. Let us fix such a partition and let $a$ be the worst outcome for player 1 in $B$, that is, $u(1,a) \leq u(1,b)$ for every $b \in B$. We know that we cannot send $a$ from $B$ to $B_1$, although this operation would be OK with $(p1)$. Clearly, it can contradict only $(q2)$ and this happens indeed if $E_g(2, (B_1 \cup \{a\})) = 1$.

Furthermore, let $B_2^a$ denote the set of all outcomes of $B$ that are not better than $a$ for player 2, that is, $u(2,b) \leq u(2,a)$ for every $b \in B_2^a$; in particular, for $a \in B_2^a$. We know that $B_2^a$ cannot be sent from $B$ to $B_2$, although this operation would be OK with $(p2)$. In fact, it can contradict only $(q1)$ and this happens indeed if $E_g(1, (B_2 \cup B_2^a)) = 1$.

Thus, we obtain $E_g(2, (B_1 \cup \{a\})) = E_g(1, (B_2 \cup B_2^a)) = 1$. By the definition of $E_g$, there are strategies $x_1^0 \in X_1$ and $x_2^0 \in X_2$ such that $g(x_1^0, x_2) \in (B_2 \cup B_2^a)$ for each $x_2 \in X_2$ and $g(x_1, x_2^0) \in (B_1 \cup \{a\})$ for each $x_1 \in X_1$. Let us note that $(B_1 \cup \{a\}) \cap (B_2 \cup B_2^a) = \{a\}$. Hence, $g(x_1^0, x_2^0) = a$ and $(x_1^0, x_2^0) \in X$ is a Nash equilibrium in the game $(g, u)$, by the definitions of $a$ and $B_2^a$. □

Now we will show that Theorem 6 does not generalize to the case $n = 3$. The concept of tightness is naturally extended to this case. Yet, for 3-person game forms tightness is no longer necessary [14] nor sufficient [12, 14] for Nash-solvability. We reproduce the corresponding two examples here.

## 10.4 Nash-solvable but not tight 3-person game form

Given three players ($|I| = 3$, $I = \{1,2,3\}$) each of which has two strategies, $X_i = \{0,1\}$ for each $i \in I$, and two outcomes ($|A| = 2$, $A = \{a_1, a_2\}$), let us define a $2 \times 2 \times 2$ game form $g : \prod_{i \in I} X_i \to A$ by formula

$g(x_1, x_2, x_3) = a_1$ if $x_1 = x_2 = x_3$ and $g(x_1, x_2, x_3) = a_2$ otherwise.

It is easy to see that every two players, say, $1, 2$, are effective for the outcome $a_2$. To enforce it they can just choose $x_1 = 0$ and $x_2 = 1$. Yet, they are not effective for $a_1$. It is also clear that a single player is effective only for the whole set $A = \{a_1, a_2\}$. Hence, game form $g$ is not tight, since, for example, $E_g(\{1, 2\}, \{a_1\}) = E_g(\{3\}, \{a_2\}) = 0$. Boolean duality, $F_K^d = F_{I \setminus K}$, fails too:

$F[1] = F_1(g) = F_2(g) = F_3(g) = a_1 a_2$,

$F[2] = F_{\{2,3\}}(g) = F_{\{3,1\}}(g) = F_{\{1,2\}}(g) = a_2$, and $F[1]^d = a_1 \vee a_2 \neq a_2 = F[2]$.

Let us show that $g$ is Nash-solvable. If all three players prefer $a_1$ to $a_2$ then, clearly, $(x \in X \mid x_1 = x_2 = x_3 = 0)$ and $(x \in X \mid x_1 = x_2 = x_3 = 1)$ are both Nash equilibria. If a player, say 1, prefers $a_2$ to $a_1$ then $(x \in X | x_1 = 1, x_2 = x_3 = 0)$ is a Nash equilibrium. Indeed, in this case $g(x) = a_2$ and no player, neither 2 nor 3, can switch it to $a_1$. Although player 1 could do this (just substituting $x_1 = 0$ for $x_1 = 1$), yet, (s)he is not interested, since (s)he prefers $a_2$ to $a_1$.

### 10.5 Tight but not Nash-solvable 3-person game form

Given three players ($|I| = 3$, $I = \{1, 2, 3\}$) each of which has six strategies,

$$X_i = \{x_i = (x_i', x_i'') \mid x_i' \in \{0, 1\}, x_i'' \in \{0, 1, 2\}\}; i \in I,$$

and three outcomes ($|A| = 3$, $A = \{a_1, a_2, a_3\}$), let us define a $6 \times 6 \times 6$ game form $g : \prod_{i \in I} X_i \to A$ as follows:

$$g(x) = g(x_1, x_2, x_3) = g(x_1', x_1'', x_2', x_2'', x_3', x_3'') = a_j, \text{ where}$$

$$j - 1 = \begin{cases} (x_1'' + x_2'' + x_3'') \mod 3 \text{ if } x_1' = x_2' = x_3', \\ (x_1'' + x_2'') \mod 3 \text{ if } 1 = x_1' > x_2' = 0, \\ (x_2'' + x_3'') \mod 3 \text{ if } 1 = x_2' > x_3' = 0, \\ (x_3'' + x_1'') \mod 3 \text{ if } 1 = x_3' > x_1' = 0. \end{cases}$$

First let us notice that $g$ is well defined, since the above four conditions, $x_1' > x_2'$, $x_2' > x_3'$, $x_3' > x_1'$, and $x_1' = x_2' = x_3'$, form a partition of $X$. Indeed, no two of the first three inequalities can hold simultaneously, since $x_i' \in \{0, 1\}$ takes only two values for each $i \in \{1, 2, 3\}$. In fact, these four conditions partition the $6 \times 6 \times 6$ box $X$ into three $3 \times 3 \times 6$ boxes corresponding to the three inequalities and two $3 \times 3 \times 3$ boxes corresponding to the equalities.

Now, let us show that $g$ is tight. Indeed, any two players, say, $1, 2 \in I$, are effective for every outcome $a_j \in A$. To guarantee it, they just choose $1 = x_1' > x_2' = 0$ to take the control and then force $a_j$ choosing ($x_1''$ and $x_2''$) such that $x_1'' + x_2'' = j - 1 (\mod 3)$. On the other hand, a single player is effective only for the whole set $A$. It is easy to verify that $E_g(K, B) + E_g(I \setminus K, A \setminus B) \equiv 1$ and, hence, game form $g$ is tight. Equivalently, in Boolean terms we obtain:

$F[1] = F_1(g) = F_2(g) = F_3(g) = a_1 a_2 a_3$,

$F[2] = F_{2,3}(g) = F_{3,1}(g) = F_{1,2}(g) = a_1 \vee a_2 \vee a_3$, and $(F[1])^d = F[2]$.

Moreover, for each player $i \in I$ and for each strategy $x_i \in X_i$ the obtained restricted game form $g[x_i]$ of the remaining two players is tight too.

Indeed, due to symmetry, without loss of generality, we can choose any strategy. For example, let us fix $x_1 = (x_1', x_1'') = (1, 2)$. Then in the obtained game form $g[x_1]$ player 2 can enforce any outcome $a_j \in A$. To do so (s)he should just choose $x_2' = 0$ to get $1 = x_1' > x_2' = 0$ and take the control. Then (s)he should choose $x_2'' = j \mod 3$, since in this case $(x_1'' + x_2'') \mod 3 = (2 + x_2'') \mod 3 = j - 1$ which results in $a_j$. Respectively, player 3 is effective only for the whole set $A$. It is easy to verify that game form $g[x_i]$ is tight:

$$F_2 = F_2(g[x_i]) = a_1 \vee a_2 \vee a_3,\ F_3 = F_3(g[x_i]) = a_1 a_2 a_3, \text{ and } F_2^d = F_3.$$

However, game form $g$ is not Nash-solvable. To show this let us choose a utility function $u$ that realizes so-called "Condorcet" preference profile

$u(1, a_1) > u(1, a_2) > u(1, a_3)$,
$u(2, a_2) > u(2, a_3) > u(2, a_1)$,
$u(3, a_3) > u(3, a_1) > u(3, a_2)$,

and show that the obtained normal form game $(g, u)$ has no Nash equilibrium.

Let $x = (x_1, x_2, x_3) = (x_1', x_1'', x_2', x_2'', x_3', x_3'')$ be an arbitrary strategy profile.

Case 1: $x_1' = x_2' = x_3'$. In this case, by definition, $g(x) = a_j$, where $j = 1 + ((x_1'' + x_2'' + x_3'') \mod 3)$, and it is clear that each player, by changing the strategy, can get each outcome of $A$. Hence, $x$ is not a Nash equilibrium.

Case 2: equalities $x_1' = x_2' = x_3'$ do not hold. In this case, without loss of generality, we can assume that $1 = x_1' > x_2' = 0$. Then, by definition, $g(x) = a_j$, where $j = 1 + ((x_1'' + x_2'') \mod 3)$. In this situation the strategy of player 3 is irrelevant and (s)he cannot change the outcome by choosing another strategy. However, each player 2 or 3 can obtain any given outcome of $A$. Let us note that the present outcome $a_j = g(x)$ may be the best for one of these two players but not for both. Hence, $x$ is not a Nash equilibrium, since this latter player can change the strategy and get a better outcome.

A smaller, $2 \times 2 \times 4$, example was given in [12]. However, the above $6 \times 6 \times 6$ example from [14] is simpler, due to its symmetry.

## 10.6 Nash-solvability of a 3-person game form is not uniquely defined by its effectivity function

By Theorem 6, a 2-person game form $g$ is Nash-solvable if and only if it is tight, that is, the corresponding EFF $E_g$ is self-dual. In Sections 10.4 and 10.5 we demonstrated that Theorem 6 does not extend to the case of 3-person game forms, when tightness is no longer necessary (Section 10.4) nor sufficient (Section 10.5) for Nash-solvability. Of course, this is also true for $n$-person game forms with any $n \geq 3$, since we can simply introduce $n - 3$ "dummy players".

Here we extend these negative results and show that, in principle, Nash-solvability of a 3-person game form $g$ is not uniquely defined by its EFF $E_g$.

To do so, we construct two 3-person game forms $g$ and $g'$ such that $g$ is Nash-solvable, while $g'$ is not, although $E_g = E_{g'}$.

Let us take $g'$ from Section 10.4 and define $g$ by the 3-dimensional table

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $a_2$ | $a_1$ | $a_2$ | | $a_2$ | $a_2$ | $a_1$ | | $a_2$ | $a_1$ | $a_2$ |
| $a_1$ | $a_2$ | $a_1$ | | $a_2$ | $a_1$ | $a_2$ | | $a_1$ | $a_2$ | $a_2$ |
| $a_2$ | $a_2$ | $a_2$ | | $a_1$ | $a_2$ | $a_1$ | | $a_2$ | $a_1$ | $a_2$ |

Thus, $g$ and $g'$ have the same 3 players and 2 outcomes. Yet, in $g$ each player $i \in I$ has 3 (rather than 2) strategies, $X_i = \{0, 1, 2\}$; furthermore, $g(x) = g(x_1, x_2, x_3) = a_2$ whenever $x_1 + x_2 + x_3$ is even and also in three "odd cases" $x \in \{(1, 2, 0), (0, 0, 1), (2, 1, 2)\}$; otherwise $g(x) = a_1$. It is easy to verify that $g$ and $g'$ have the same EFF given in Section 10.4. Indeed, each two players are effective for $a_2$, while one player can only trivially guarantee $A = \{a_1, a_2\}$.

It is also easy to verify that if $g(x) = a_1$ then each player can switch to $a_2$ by choosing another strategy and if $g(x) = a_2$ then at least two of three players can switch to $a_1$. This observation implies that, unlike $g'$, game form $g$ is not Nash-solvable. Indeed, let us consider a utility function $u$ such that two players prefer $a_1$ to $a_2$ and one has the opposite preference. It is clear that $x$ cannot be a Nash equilibrium in both cases, $g(x) = a_1$ or $g(x) = a_2$.

To obtain another similar example, let us take $g'$ from Section 10.5 and define $g$ by the following 3-dimensional table

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $a_1$ | $a_1$ | $a_1$ | | $a_1$ | $a_2$ | $a_3$ | | $a_1$ | $a_x$ | $a_x$ |
| $a_2$ | $a_2$ | $a_2$ | | $a_1$ | $a_2$ | $a_3$ | | $a_x$ | $a_2$ | $a_x$ |
| $a_3$ | $a_3$ | $a_3$ | | $a_1$ | $a_2$ | $a_3$ | | $a_x$ | $a_x$ | $a_3$ |

Thus, $g$ and $g'$ have the same 3 players and 3 outcomes. We assume that the outcomes labeled by $a_x$ can take arbitrary (perhaps, different) values in $A = \{a_1, a_2, a_3\}$. Yet, in $g$ each player $i \in I$ has 3 (instead of 6) strategies.

It is easy to verify that $g$ and $g'$ have the same EFF given in Section 10.5.

Indeed, each of two players is effective for every outcome, while one player can only trivially guarantee the whole set $A = \{a_1, a_2, a_3\}$.

It is also easy to verify that $g$ is Nash-solvable. Without loss of generality, let us assume that $u(1, a_1) \geq u(1, a_2) \geq u(1, a_3)$. Then "the upper left" strategy profile $x$ is a Nash equilibrium. Indeed, $g(x) = a_1$ and it is easy to see that outcome $a_1$ remains whenever player 2 or 3 chooses another strategy. Unlike them, player 1 can get both $a_2$ or $a_3$, by changing the strategy. Yet, (s)he is not interested, since $a_1$ is the best outcome for 1.

The above two examples show that among two game forms with the same EFF one may be Nash-solvable, while the other one not. Let us also note that the EFF is self-dual in the second example, while in the first one it is not.

**Remark 9** *It is an interesting general question which properties of game forms (and other structures) are uniquely defined by the corresponding EFFs.*

*For example, the core of a cooperative game $C(E,u)$, by definition, depends only on the EFF $E$ and utility function $u$; see e.g. [23, 22, 25].*

*By Theorem 6, a 2-person game form $g$ is Nash-solvable if and only if its EFF $E_g$ is self-dual. Yet, this result does not generalize to the case of 3-person game forms. In [18], the class of veto voting schemes is considered for which the result of elections is uniquely defined by the corresponding effectivity (equivalently, veto) function. Somewhat surprisingly, not only game structures but also quite different objects may have properties uniquely defined by some EFFs. For example, in [4, 6], an EFF $E_G$ is assigned to each graph $G$ and it is shown that such properties of $G$ as perfectness or kernel-solvability depend only on $E_G$.*

# 11 Proof of Theorem 7

Let $g$ be a totally tight game form. By Corollary 2, $g$ is tight, that is, the corresponding two monotone Boolean functions $F_1(g)$ and $F_2(g)$ are dual. Yet, Theorem 7 claims much more, namely, all TT game forms generate the same self-dual pair: $F_1(g) = F_2(g) = a_1a_2 \vee a_2a_3 \vee a_3a_1$.

## 11.1 Game correspondences and associated game forms

A *game correspondence* is defined as a mapping $G : X_1 \times X_2 \to 2^A$. In other words, to each $(x_1, x_2) \in X_1 \times X_2$ we assign a *set* of outcomes $G(x_1, x_2) \subseteq A$.

If $|G(x_1, x_2)| = 1$ for all $(x_1, x_2) \in X_1 \times X_2$, we obtain a game form.

In general, with a game correspondence $G$ we associate $k = \prod_{x_1 \in X_1, x_2 \in X_2} |G(x_1, x_2)|$ game forms $g \in G$, by choosing for each strategy profile $(x_1, x_2)$ an outcome $g(x_1, x_2) \in G(x_1, x_2)$. Let us notice that $k = 0$ whenever $G(x_1, x_2) = \emptyset$ for at least one strategy profile.

We will say that $g \in G$ is *associated with* $G$ and call $G$ *(totally) tight* if $k > 0$ and at least one $g \in G$ is (totally) tight.

## 11.2 Game correspondences associated with pairs of dual monotone DNFs

First, let us recall the following two well-known properties of dual monotone Boolean functions that will be instrumental for our analysis.

**Lemma 2** *(see, for example, [9], Part I, Chapter 4).*

*(i) Every two dual (prime) implicants $\alpha$ of $F$ and $\beta$ of $F^d$ have at least one variable in common.*

*(ii) Given a prime implicant $\alpha$ of $F$ and a variable $x$ of $\alpha$, there is a (prime) implicant $\beta$ of $F^d$ such that $x$ is the only common variable of $\alpha$ and $\beta$.* $\square$

Given arbitrary monotone (that is, negation-free) DNFs $D_1 = \bigvee_{x_1 \in X_1} B_{x_1}$ and $D_2 = \bigvee_{x_2 \in X_2} B_{x_2}$ over the set of variables $A$, let us define a game correspondence $G = G^{D_1,D_2} : X_1 \times X_2 \to 2^A$ by setting $G(x_1, x_2) = B_{x_1} \cap B_{x_2}$

| | | |
|---|---|---|
| $a_1/a_3$ | $a_1$ | $a_3$ |
| $a_1$ | $a_2/a_1$ | $a_2$ |
| $a_3$ | $a_2$ | $a_3/a_2$ |

Table 6: $\binom{3}{2}$-majority voting game correspondence;
only 2 from 8 game forms associated with this game correspondence are TT.

for each $(x_1, x_2) \in X_1 \times X_2$; see, for example, $G^{D_1,D_2}$ in Table 6, where $D_1 = D_2 = a_1a_2 \vee a_2a_3 \vee a_3a_1$.

**Lemma 3** *([14], see also [27]). If $D_1$ and $D_2$ are dual then $G(D_1, D_2)$ is tight. In particular, in this case $G(x_1, x_2) \neq \emptyset$ for all $(x_1, x_2) \in X_1 \times X_2$; moreover, all associated game forms $g \in G$ have the same Boolean functions $F_1(g)$ and $F_2(g)$ defined by DNFs $D_1$ and $D_2$, respectively. Conversely, if at least one game form $g \in G^{D_1,D_2}$ is tight then DNFs $D_1$ and $D_2$ are dual.*

**Proof**. It follows immediately from Lemma 2 (i) and (ii). □

Let us recall that, by definition, $G$ is TT if at least one $g \in G$ is TT. However, in contrast with tightness, this does not mean that *all* $g \in G$ are TT. Let us consider, for example, game correspondence $G$ in Table 6. It is not difficult to verify that only two of eight game forms associated with $G$ are TT. To get them one should choose $a_1, a_2, a_3$ or $a_3, a_1, a_2$ on the main diagonal.

Given a DNF $D$, let $D^0$ denote the corresponding irredundant DNF, that is, the disjunction of all prime (irreducible) implicants of $D$.

**Lemma 4** *Game correspondence $G^{D_1,D_2}$ is TT if and only if $G^{D_1^0,D_2^0}$ is TT.*

**Proof**. The "only if part" immediately follows, since total tightness is a hereditary property of game forms and game correspondences.

**Lemma 5** *A subcorrespondence $G'$ of $G$ is TT whenever $G$ is TT.* □

Let us prove the "if part" of Lemma 4. By assumption, there is a TT game form $g^0 \in G^0 = G^{D_1^0,D_2^0}$. Let us extend it to a TT game form $g \in G = G^{D_1,D_2}$ as follows. For $i = 1, 2$ to each strategy $x_i \in X_i$ in $G$ assign a strategy $x_i^0 \in X_i$ in $G^0$ such that $B_{x_i^0} \subseteq B_{x_i}$. Then for each strategy profile $x = (x_1, x_2)$ of $G$ choose the same outcome as for $x^0 = (x_1^0, x_2^0)$ in $g^0$. Obviously, the obtained extension $g$ of $g^0$ is totally tight, too. □

### 11.3 Totally tight Boolean functions

Thus, we can restrict ourselves by dual pairs of irredundant DNFs. In other words, keeping in mind the characterization of TT game forms, we will take as the input a monotone Boolean function $F$ rather than a game form $g$. Given $F$, we set $F_1 = F$ and $F_2 = F^d$, consider the corresponding irredundant DNFs

| | | | |
|---|---|---|---|
| $\mathbf{a_1}$ | $a_1$ | $a_3$ | $\mathbf{a_3}$ |
| $\mathbf{a_2}$ | $a_4$ | $a_2$ | $\mathbf{a_4}$ |

$g$

| | |
|---|---|
| $a_1$ | $a_3$ |
| $a_2$ | $a_4$ |

$g'$

Table 7: Game form $g$ is tight but not TT.

$D_1^0$, $D_2^0$, and game correspondence $G = G^F = G^{D_1^0, D_2^0}$. We will call $F$ TT if $G$ is TT, or in other words, if there is a TT $g \in G$. By construction, $F$ is TT if and only if $F^d$ is TT. Let us consider several examples.

**Example 5** *$F = F_1 = a_1a_3 \vee a_2a_4$ and $F^d = F_2 = a_1a_2 \vee a_2a_3 \vee a_3a_4 \vee a_4a_1$. It is easily seen that every two prime implicants, one of $F$ and another of $F^d$, have exactly one variable in common. In other words, game correspondence $G^F$ is, in fact, a game form, since $|G^F(x_1, x_2)| = 1$ for every $(x_1, x_2) \in X_1 \times X_2$. However, this game form is not TT, since it has a $2 \times 2$ subform $g'$ that is not tight; see Table 7.*

**Remark 10** *In general, a (monotone) Boolean function $F$ is called* read-once *if it can be expressed via its variables by a $(\vee, \wedge)$-formula in which every variable appears only once. The following four claims are equivalent:*

*(i) $F$ is* read-once*; (ii) $F^d$ is read-once; (iii) $G^F$ is a game form, $G^F = g^F$;*

*(iv) every two prime implicants, one of $F$ and another of $F^d$, have exactly one variable in common; see [13] and also [20] and [9], Part II, Chapter 10.*

*A (monotone) Boolean function $F$ is called reducible if $F = a \vee F'$ or $F = a \wedge F'$, where $F'$ does not not depend on $a$.*

*Furthermore, $F$ is called totally reducible if it can be successively reduced to $F = a$ (and then, by the next step, to $F \equiv 0$ or to $F \equiv 1$). Obviously, the following four properties of a (monotone) function $F$ are equivalent:*

*(v) $F$ is totally reducible; (vi) $F^d$ is totally reducible;*

*(vii) $F$ is read-once and all parentheses of its read-once formula are nested;*

*(viii) game correspondence $G^F$ is a totally reducible game form.*

*Let us recall that the totally reducible game forms are totally tight.*

*Yet, we are looking for the irreducible TT game forms.*

**Example 6** *As another example let us consider the dual pair*

*$F = F_1 = a_1a_2 \vee a_2a_3 \vee a_3a_4$ and $F^d = F_2 = a_1a_3 \vee a_3a_2 \vee a_2a_4$.*

*Obviously, $G^F$ is not TT, since it contains a not tight $2 \times 2$ subform $g'$; see Table 8.*

A case analysis might be needed for more difficult examples.

| | | |
|---|---|---|
| $\mathbf{a_1}$ | $a_2$ | $\mathbf{a_2}$ |
| $a_3$ | $a_2/a_3$ | $a_2$ |
| $\mathbf{a_3}$ | $a_3$ | $\mathbf{a_4}$ |

$G$

| | |
|---|---|
| $a_1$ | $a_2$ |
| $a_3$ | $a_4$ |

$g'$

Table 8: No TT game form is associated with this game correspondence.

| | 123 | 145 | 245 | 345 |
|---|---|---|---|---|
| 123 | **1**23 | 1 | 2 | **3** |
| 145 | 1 | 145 | 45 | 45 |
| 245 | **2** | 45 | 245 | **45** |
| 345 | 3 | 45 | 45 | 345 |

$G$

| | |
|---|---|
| **1** | **3** |
| **2** | **45** |

$G'$

Table 9: $\binom{5}{3}$-majority voting, a $4 \times 4$ subcorrespondence;
this subcorrespondence is not TT, since no TT game form is associated with it.

**Example 7** *Let* $F = F\binom{5}{3} := \bigvee_{\{i,j,k\} \subseteq \{1,2,3,4,5\}} a_i a_j a_k$*, where* $i, j,$ *and* $k$ *are pairwise distinct triplets; in other words,* $F = 1$ *if and only if at least* 3 *of its* 5 *variables are equal to* 1*. To show that* $G^F$ *is not TT let us consider its* $4 \times 4$ *subcorrespondence* $G$ *given in Table 9 (where, to save space, we substitute only the subscript* $j \in \{1,2,3,4,5\}$ *for* $a_j$*). Let us choose an arbitrary game form* $g \in G$*. Due to obvious symmetry, we can choose* $a_1$ *from* $\{a_1, a_2, a_3\}$*, without any loss of generality. Yet, in this case* $G$ *already contains a* $2 \times 2$ *subconfiguration* $G'$ *that is clearly not TT; see Table 9. Hence,* $g$ *cannot be TT and, by Lemma 5,* $G$ *and* $G^F$ *are not TT, either.*

The next Lemma is instrumental in characterizing TT Boolean functions.

Given $F$, let us choose two distinct prime implicants and denote by $B, B' \subseteq A$ the corresponding two set of variables. Obviously, $B \setminus B' \neq \emptyset$ and $B' \setminus B \neq \emptyset$.

**Lemma 6** *If* $F$ *is totally tight then* $|B \setminus B'| = 1$ *or* $|B' \setminus B| = 1$*.*

**Proof**. Let us assume indirectly that $|B \setminus B'| \geq 2$ and $|B' \setminus B| \geq 2$, say, $a_1, a_2 \in B \setminus B'$ and $a_3, a_4 \in B' \setminus B$, where $a_1, a_2, a_3, a_4 \in A$ are four pairwise distinct outcomes, yet, $F$ is TT.

| | | |
|---|---|---|
| $B$ | $a_1$ | $a_2$ |
| $B'$ | $a_3$ | $a_4$ |

| | | | |
|---|---|---|---|
| $B$ | $a_1$ | $a_2$ | $b$ |
| $B'$ | $b'$ | $a_3$ | $a_4$ |

| | | | | |
|---|---|---|---|---|
| $B$ | $a_1$ | $a_2$ | $b_1$ | $b_2$ |
| $B'$ | $b_3$ | $b_4$ | $a_3$ | $a_4$ |

| | | | |
|---|---|---|---|
| $a_1$ | $a_2$ | $b$ | $b$ |
| $b'$ | $b'$ | $a_3$ | $a_4$ |

| | | | |
|---|---|---|---|
| $\mathbf{a_1}$ | $a_2$ | $a_2$ | $\mathbf{a_2}$ |
| $\mathbf{a_3}$ | $a_3$ | $a_3$ | $\mathbf{a_4}$ |

Table 10: $|B \setminus B'| = 1$ or $|B' \setminus B| = 1$.

By Lemma 2 (ii), there are four prime implicants $\beta_1, \beta_2, \beta_3, \beta_4$ in $F^d$ whose sets of variables $B_1, B_2, B_3, B_4$ are such that $B_1 \cap B = \{a_1\}$, $B_2 \cap B = \{a_2\}$, $B_3 \cap B' = \{a_3\}$, $B_4 \cap B' = \{a_4\}$. Obviously, $\beta_1 \neq \beta_2$ and $\beta_3 \neq \beta_4$. Hence, among these four implicants two, three, or four are pairwise distinct.

Let us fix a game form $g \in G^F$ and consider the corresponding subform $g'$ in $g$ of size $2 \times 2$, $2 \times 3$, or $2 \times 4$. All cases are considered in Table 10, where the first row of each game form is assigned to $B$ and it contains $a_1$ and $a_2$, while the second one is assigned to $B'$ and it contains $a_3$ and $a_4$.

By assumption, Boolean function $F$ and game correspondence $G^F$ is TT. Hence, we can assume that the associated game form $g \in G^F$ is TT, too.

Case $2 \times 2$ is easy, since $g'$ itself is not tight; see Table 10.1.

In case $2 \times 3$, by definition of $B$ and $B'$, we have $b \neq a_4$ and $b' \neq a_1$. Hence, the $2 \times 2$ subform with entries $a_2, b, a_3, a_4$ (respectively $a_1, a_2, b', a_3$ is not tight unless $b = a_2$ (respectively, $b' = a_3$). Yet, if both these equalities hold then the remaining $2 \times 2$ subform, with entries $a_1, b, b', a_4$, is not tight; see Table 10.2.

In case $2 \times 4$, outcomes $b_1, b_2, b_3, b_4 \in A$ are not necessarily pairwise distinct, yet, $\{b_1, b_2\} \cap \{a_3, a_4\} = \{b_3, b_4\} \cap \{a_1, a_2\} = \emptyset$, since $b_1, b_2 \in B$ and $b_3, b_4 \in B'$; see Table 10.3. Furthermore, $b_1 = b_2$ and $b_3 = b_4$, since otherwise the first or the last two columns of $g'$ form a not tight subform. Let us set $b_1 = b_2 = b$ and $b_3 = b_4 = b'$, as in Table 10.4. Yet, $b$ (respectively, $b'$) cannot be equal to both $a_1$ and $a_2$ (respectively, $a_3$ and $a_4$), since they are distinct. Without loss of generality, let us assume that $b \neq a_1$ and $b' \neq a_4$; see Table 10.5. Then the first and last columns of $g'$ form a not tight subform, even if $b = b'$.

Thus, in each case, we obtain a contradiction. □

## 11.4 All irreducible TT Boolean functions are self-dual

In Boolean terms, reducible game forms are obviously characterized as follows.

**Lemma 7** *Game correspondence $G^F$ contains a constant row (column) whose every entry is an outcome $a \in A$ if and only if $F = a \vee F'$ (respectively, $F^d = a \vee F''$). In both cases, every associated game form $g \in G^E$ is reducible.* □

Thus, we can reformulate Theorem 7 as follows:

If $F$ is TT then either $F = a \vee F'$, or $F^d = a \vee F''$, or $F = F^d = a_1 a_2 \vee a_2 a_3 \vee a_3 a_1$.

In the first two cases $F$ is called *reducible*.

**Lemma 8** *If $F$ is TT and irreducible then every two of its prime implicants have a variable in common.*

**Proof**. Let us assume indirectly that there are two prime implicants of $F$ with disjoint set of variables $B, B' \subseteq A$. By Lemma 6, if $F$ is TT then $|B| = 1$ or $|B'| = 1$, in other words, $F$ is reducible and we get a contradiction. □

**Lemma 9** *If $F$ is TT and irreducible then it is self-dual, $F = F^d$.*

**Proof**. It is both obvious and well-known (see, for example, [9]) that $F$ is dual-minor, $F \leq F^d$, if and only if every two prime implicants of $F$ have a variable in common. Thus, by Lemma 8, if $F$ is irreducible and TT then it is dual-minor, $F \leq F^d$. Furthermore, $F$ is irreducible and TT if and only if $F^d$ is irreducible and TT. To see this, it would suffice just to rename players 1 and 2. Hence, $F$ and $F^d$ are both dual-minor: $F \leq F^d$ and $F^d \leq (F^d)^d = F$. Hence, $F = F^d$, that is, $F$ is self-dual. □

We will show that only one self-dual function is TT, all others are not.

**Example 8** *The next function is associated with the Fano projective plane:*

$F = a_1a_2a_3 \vee a_3a_4a_5 \vee a_5a_6a_1 \vee a_0a_1a_4 \vee a_0a_2a_5 \vee a_0a_3a_6 \vee a_2a_4a_6.$

*It is well-known and not difficult to verify that $F$ is self-dual, $F = F^d$.*

*Yet, by Lemma 6, $F$ is not TT. Indeed, rows $\{a_1, a_2, a_3\}$, $\{a_3, a_4, a_5\}$ and columns $\{a_0, a_1, a_4\}$, $\{a_0, a_2, a_5\}$ form a $2 \times 2$ game form that is not tight.*

As another example, let us recall that $\binom{5}{3}$-majority EFF is self-dual but not TT; see Table 9.

## 11.5 2-Wheel is a unique irreducible TT Boolean function

**Example 9** *The $k$-*wheel *is defined for all $k \geq 2$ by formula*

$F_k = a_0a_1 \vee a_0a_2 \vee \ldots \vee a_0a_k \vee a_1a_2 \ldots a_k.$

*Again, it is well-known and easy to check that $F_k$ is self-dual, $F_k = F_k^d$ for any $k \geq 2$. Game correspondences, $G^{F_k}$ are given in Table 11 for $k = 2, 3$, and in general. (Again, to save space we substitute for an outcome $a_j$ only its subscript $j$.) Let us fix an arbitrary $g \in G^{F_k}$. Due to obvious symmetry, without loss of generality, we can choose $a_k$ from $\{a_1, a_2, \ldots, a_k\}$. Yet, then a $2 \times 2$ not tight subform $g'$ appears in $g$ whenever $k \geq 3$; see Table 11.*

As we already know, 2-wheel $F_2$ is TT. There are two TT game forms associated with $G^{F_2}$; see Table 6, in which we substitute $i + 1$ for $i = 0, 1$ and 2.

Furthermore, we can strengthen Lemma 9 as follows.

**Theorem 9** *If $F$ is TT and irreducible then it is the 2-wheel.*

**Proof**. Let us fix a prime implicant of $F$ with the largest set of variables, say, $B = \{a_1, \ldots, a_k\} \subseteq A$. Since $F$ is irreducible, $k \geq 2$.

By Lemma 9, $F$ is self-dual, $F = F^d$. Then, by Lemma 2 (ii), for every $j = 1, \ldots, k$ function $F$ contains a prime implicant with the set of variables $B_j$ such that $B \cap B_j = \{a_j\}$. Furthermore, by Lemma 6, $|B \setminus B_j| = 1$ or $|B_j \setminus B| = 1$.

Let us assume that $k \geq 3$. Then $|B \setminus B_j| \geq 2$. Hence, $|B_j \setminus B| = 1$, that is, $B_j = \{a_j, b_j\}$ for each $j = 1, \ldots, k$. Moreover, by Lemma 2 (i), all $b_j$ must coincide, that is, $B_j = \{a_0, a_j\}$ for each $j = 1, \ldots, k$. In other words, $F$ is a $k$-wheel with $k \geq 3$. Yet, as we already know, in this case $F_k$ is not TT. Hence,

| | 01 | 02 | 12 |
|---|---|---|---|
| 01 | 01 | 0 | 1 |
| 02 | 0 | 02 | 2 |
| 12 | 1 | 2 | 12 |

| | 01 | 02 | 03 | 123 |
|---|---|---|---|---|
| 01 | 01 | **0** | 0 | **1** |
| 02 | 0 | 02 | 0 | 2 |
| 03 | 0 | 0 | 03 | 3 |
| 123 | 1 | **2** | 3 | 12**3** |

| | 01 | 02 | $\cdots$ | $0k$ | $12\ldots k$ |
|---|---|---|---|---|---|
| 01 | 01 | **0** | $\cdots$ | 0 | **1** |
| 02 | 0 | 02 | | 0 | 2 |
| $\vdots$ | | | $\ddots$ | | $\vdots$ |
| $0k$ | 0 | 0 | | $0k$ | $k$ |
| 12...k | 1 | **2** | $\cdots$ | k | 12...**k** |

Table 11: 2-wheel, 3-wheel, and $k$-wheel.

$k = 2$, that is, every prime implicant of $F$ has exactly two variables; in other words, $F = a_1a_2 \vee a_0a_1 \vee a_0a_2$ is the 2-wheel. □

Thus, all TT irreducible game forms have a unique EFF, the 2-wheel.

This completes the proof of Theorem 7. □

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