ON THE STRUCTURE OF THE CORE OF SPARSE RANDOM GRAPHS

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Abstract. In this paper, we investigate the structure of the core of a sparse random graph above the critical point. We determine the asymptotic distributions of the total number of isolated cycles there as well as the joint distributions of the isolated cycles of fixed lengths. Furthermore, focusing on its giant component, we determine the asymptotic joint distributions of the cycles of fixed lengths that are contained in it and the distributions of its cyclic 2-edge-connected and 2-vertex-connected components.

1. Introduction

In this paper, we present some features of the core of sparse $\mathcal{G}_{n,m}$ random graphs. Recall that a $\mathcal{G}_{n,m}$ random graph is a uniformly random element of the set of simple graphs on $V_n = \{1,\dots,n\}$ with $m = m(n)$ edges, where $0 \leq m \leq \binom{n}{2}$. (We use the term $\mathcal{G}_{n,m}$ for both the model and the random outcome.) We say that a $\mathcal{G}_{n,m}$ random graph has a property $\mathcal{Q}_n$ (that is lies in the subset $\mathcal{Q}_n$ of the set of graphs on V_n having m edges) *asymptotically almost surely* (a.a.s.) if $\lim_{n\to\infty} \mathbb{P}[\mathcal{G}_{n,m} \in \mathcal{Q}_n] = 1$. We will be working on sparse $\mathcal{G}_{n,m}$ random graphs, i.e. $m = \lceil \theta n/2 \rceil$, where $\theta = \theta(n) = \Theta(1)$. For a graph $G = (V, E)$ and a natural number $k \geq 2$, the k-core of G is the maximal subgraph of minimum degree at least k (and it is empty if there is no such graph). We are interested in the 2-core of a $\mathcal{G}_{n,m}$ random graph, which we simply call core and denote by $\mathrm{cr}(\mathcal{G}_{n,m})$. Note that, by the definition of the core, any graph is the union of its core and a collection of trees either disjoint from the core or rooted at a vertex of it (but having no other vertex in common with it).

It is well-known that a sparse $\mathcal{G}_{n,m}$ random graph, having $\theta > 1$ fixed, typically consists of a unique "giant" (of linear order) complex component and a few "small" (of logarithmic order at most) unicyclic components, as well as some "small" tree-components (see [1] or [5], where these facts are presented in full). The typical picture of $\mathrm{cr}(\mathcal{G}_{n,m})$ is similar and was described in [12]. Namely, it consists of a giant component, which is a subgraph of the giant component of the random graph and whose order is a certain proportion of the order of the latter, along with a few isolated cycles which are the cycles of the unicyclic components of $\mathcal{G}_{n,m}$. Very precise results concerning the distribution of them as well as the distributions of the cycles that are not isolated in $\mathrm{cr}(\mathcal{G}_{n,m})$ when $\theta = \theta(n)$ is near the critical point were obtained very recently by S. Janson in [3]. The first theorem we prove achieves this (partially), but for any fixed $\theta > 1$. For such a θ, let $\lambda_2(\theta)$ be the unique root of the equation $\lambda/(1 - e^{-\lambda}) = \theta$. Also, for any real number $z > 0$ and a natural number $k \geq 2$, we define $p_k(z) = \mathbb{P}[X \geq k]$, where $X = \mathrm{Po}(z)$ is a Poisson random variable of mean z. We have:

Theorem 1.1. *For any fixed $\theta > 1$ and $m = \lceil \theta n/2 \rceil$ the following hold:*

(1) *The random graph $\mathrm{cr}(\mathcal{G}_{n,m})$ consists a.a.s. of a unique greatest component of order $np_2(\lambda_2(\theta)) + o_p(n)$ and size $n\frac{\lambda_2^2(\theta)}{2\theta} + o_p(n)$, that has more than one cycle, with the remaining components being cycles with total order $O_p(1)$.*

(2) *The number of isolated cycles of $\mathrm{cr}(\mathcal{G}_{n,m})$ converges in distribution as $n \to \infty$ to a Poisson random variable of mean*

$$-\frac{1}{2}\ln(1 - \theta\exp(-\lambda_2(\theta))) - \frac{1}{2}\theta\exp(-\lambda_2(\theta)) - \frac{1}{4}\left(\theta\exp(-\lambda_2(\theta))\right)^2.$$

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(3) *For any integer $k \geq 3$, the number of cycles having length k that are isolated in* $\mathrm{cr}(\mathcal{G}_{n,m})$ *converges in distribution as $n \to \infty$ to a Poisson random variable of mean* $\left(\theta e^{-\lambda_2(\theta)}\right)^k/2k$. *The number of cycles having length k that are not isolated there converges in distribution as $n \to \infty$ to a Poisson random variable of mean* $\theta^k(1-e^{-k\lambda_2(\theta)})/2k$. *Moreover, any finite collection of the above random variables are asymptotically independent.*

The first part of the preceding theorem was also proved using other methods by B. Pittel in [12]. In that paper, the order of the core of a $\mathcal{G}_{n,m}$ random graph beyond the critical point was given by a different formula. Numerical evidence suggests that this formula and the formula we give coincide. Though we believe that there is some identity that transforms one onto the other, we have not been able to prove this rigorously.

The second part of the present work focuses on the giant component of $\mathrm{cr}(\mathcal{G}_{n,m})$. Though there is a very precise and clear picture of the birth of the giant component of a $\mathcal{G}_{n,m}$ random graph as well as of its structure close to the critical point (see for example [4], [6], [7] - alternatively [1] or [5] for a complete description), the first attempt to give a picture of it for any fixed $\theta > 1$ was made by B.Pittel in [12]. Firstly, recall that a *2-edge-connected component* of a graph is a maximal connected subgraph having at least 3 vertices with no cutedges and a *2-vertex-connected component* of a graph is a maximal connected subgraph which has at least 3 vertices and has no cutvertices. B. Pittel proved (see Theorem 4.1 below) that for any fixed $\theta > 1$ the giant component of a $\mathcal{G}_{n,m}$ random graph typically consists of a 2-vertex-connected component of linear order, a collection of unicyclic components each sprouting from a different vertex of the 2-vertex-connected component with total order that is bounded in probability, as well as a collection of small trees each rooted at a vertex of the 2-vertex-connected component whose total order is linear. Thus, the giant component of $\mathrm{cr}(\mathcal{G}_{n,m})$ consists of a huge 2-vertex-connected component, whereas the remaining 2-vertex-connected components are small cycles and their total order is bounded in probability. Moreover, each 2-edge-connected component is joined to the giant 2-vertex-connected component by a unique path, whose internal vertices are all of degree 2 and its length is bounded in probability as well.

Our goal is to make these results more precise. For a graph G on V_n, we define $L_i(G)$ to be the i-th largest component (if there is more than one component of the same order, then we assume increasing lexicographic ordering) and let $|L_i(G)|$ denote its order. Furthermore, we define the *essential core* of G to be the largest 2-vertex-connected component of $L_1(\mathrm{cr}(G))$ (if there is more than one of them, then again we assume increasing lexicographic ordering). We denote it by $\mathrm{ess}-\mathrm{cr}(G)$. The second theorem we prove is as follows:

Theorem 1.2. *For any fixed $\theta > 1$, and $m = \lceil \theta n/2 \rceil$, we have:*

(1) *The number of 2-edge-connected components which are cycles and belong to* $L_1(\mathrm{cr}(\mathcal{G}_{n,m}))$ *converges in distribution as $n \to \infty$ to a Poisson random variable of mean*

$$\left(1-\theta e^{-\lambda_2(\theta)}\right)\ \frac{\theta e^{-\lambda_2(\theta)}\lambda_2(\theta)}{2}\left(\frac{3\left(\theta e^{-\lambda_2(\theta)}\right)^2-2\left(\theta e^{-\lambda_2(\theta)}\right)^3}{(1-\theta e^{-\lambda_2(\theta)})^2}-\frac{2\left(\theta e^{-\lambda_2(\theta)}\right)^2}{1-\theta e^{-\lambda_2(\theta)}}\right).$$

The total number of 2-vertex-connected components which are cycles and belong to the same 2-edge-connected component as $\mathrm{ess}-\mathrm{cr}(\mathcal{G}_{n,m})$ *converges in distribution as $n \to \infty$ to a Poisson random variable of mean*

$$\frac{\theta}{2}(1-e^{-\lambda_2(\theta)}-\lambda_2(\theta)e^{-\lambda_2(\theta)})\frac{\left(\theta e^{-\lambda_2(\theta)}\right)^2}{1-\theta e^{-\lambda_2(\theta)}}.$$

Furthermore, these random variables are asymptotically independent.

(2) *For any integers $k_1 \geq 3$ and $k_2 \geq 0$, the number of 2-edge-connected components of* $L_1(\mathrm{cr}(\mathcal{G}_{n,m}))$, *which are cycles of length k_1 and are joined to* $\mathrm{ess}-\mathrm{cr}(\mathcal{G}_{n,m})$ *by a path having k_2 internal vertices converges in distribution as $n \to \infty$ to a Poisson random*

variable of mean

$$\frac{\theta e^{-\lambda_2(\theta)}\lambda_2(\theta)(1-\theta e^{-\lambda_2(\theta)})}{2}\left(\theta e^{-\lambda_2(\theta)}\right)^{k_1+k_2-1}.$$

The number of 2-vertex-connected components which are cycles of length $k \geq 3$ and belong to the same 2-edge-connected component as $\mathrm{ess}-\mathrm{cr}(\mathcal{G}_{n,m})$ *converges in distribution as $n \to \infty$ to a Poisson random variable of mean*

$$\frac{\theta}{2}\left(\theta e^{-\lambda_2(\theta)}\right)^{k-1}\left(1-e^{-\lambda_2(\theta)}-e^{-\lambda_2(\theta)}\lambda_2(\theta)\right).$$

Moreover, any finite collection of the above random variables are asymptotically independent.

Both of these theorems are deduced from Theorem 4.1.1 in [2] (see Theorem 2.1 in the next section) and follow from the observation that if we condition on the degree sequence of the core of $\mathcal{G}_{n,m}$, then this is uniformly distributed over the set of graphs having this degree sequence. In particular, we define a probability space which is the product of the probability spaces of the degree sequences of $\mathrm{cr}(\mathcal{G}_{n,m})$: each element of this space is a sequence of degree sequences on n vertices, for each $n \in \mathbb{Z}^+$. Using Theorem 2.1, we determine a subspace of this space of measure 1 (see Section 2 for a precise definition), and for an arbitrary "asymptotic" degree sequence in this subspace we apply the method of moments on the space of uniform random graphs having this degree sequence in order to determine the asymptotic distributions of the graph functionals mentioned in Theorems 1.1 and 1.2 there. Finally, we use Proposition 2.5 to show that in fact these are the asymptotic distributions in the $\mathcal{G}_{n,m}$ model. Some technical but standard results of the theory of sparse random graphs are also used in the course of our proofs - see Appendix A.

For a graph G, the *kernel* $\ker(G)$ is the multigraph (possibly with loops) obtained from those components of the 2-core of G that have more than one cycles by replacing each path whose internal vertices are all of degree two by a single edge. T. Łuczak in [7] (see also [5]) obtained results concerning the structure of $\ker(\mathcal{G}_{n,m})$, when θ is close to the critical point. In Section 5, we obtain from Theorems 1.1 and 1.2 some structural properties of $\ker(\mathcal{G}_{n,m})$, for any constant $\theta > 1$. Namely, we give precise estimates for the degree sequence of the kernel, its order and its size as well as its number of loops.

2. Some preliminary results and definitions

We begin this section presenting a key-result concerning the degree sequence of the k-core of a $\mathcal{G}_{n,m}$ random graph, with $m = \lceil \theta n/2 \rceil$. For $k \geq 2$, let $k\text{–}\mathrm{cr}(\mathcal{G}_{n,m})$ denote the k-core of a $\mathcal{G}_{n,m}$ random graph and, therefore, $\mathrm{cr}(\mathcal{G}_{n,m}) = 2\text{–}\mathrm{cr}(\mathcal{G}_{n,m})$.

For a graph G on $V \subseteq V_n$, the sequence $(v_0(G), \ldots, v_{n-1}(G))$, where for $j = 0, \ldots, n-1$ $v_j(G)$ is the number of vertices of G of degree j, is said to be the *degree sequence* of G. For any $k \geq 2$, let

$$\gamma_k = \inf\left\{\frac{\lambda}{\pi_k(\lambda)} \ : \ \lambda > 0\right\},$$

where $\pi_k(z)$, for each real $z > 0$, is the probability of a Poisson random variable of mean z being at least $k-1$. It is easy to see that $\gamma_2 = 1$ (this is the parameter which will be used in the present work). Now, for any $k \geq 2$ and $\theta > \gamma_k$ let $\lambda_k(\theta)$ be the larger root of the equation $\theta = \lambda/\pi_k(\lambda)$, if $k \geq 3$, or the unique root, if $k = 2$ (as we have already defined it). Also, let $\rho_j(z) = \mathbb{P}[X = j]$, where $X = \mathrm{Po}(z)$. In [2] we prove the following the following:

Theorem 2.1 (Theorem 4.4.1 [2])**.** *For $k \geq 3$ ($k = 2$, respectively), let $\theta \geq \gamma_k + n^{-\delta}$, where $\delta \in (0, 1/2)$ ($\theta > 1$ - bounded away from 1). For every fixed $l \geq k$, there exist constants $c > 0$ and $\gamma, \tau \in (0, 1)$ such that $v_l(k\text{–}\mathrm{cr}(\mathcal{G}_{n,m})) = n\rho_l(\lambda_k(\theta)) + O_C(n^\gamma)$, with probability at least $1 - O(\exp(-cn^\tau))$. Thus, $v_l(k\text{–}\mathrm{cr}(\mathcal{G}_{n,m})) = n\rho_l(\lambda_k(\theta))(1 + o_p(1))$, as $n \to \infty$. Moreover, with the same bound on the probability the number of edges in the k-core is $n\frac{\lambda_k^2(\theta)}{2\theta} + O_C(n^\gamma)$.*

We now describe briefly the deletion process which was introduced in [13]. Given a graph on V_n and an integer $k \geq 2$, at each step, we choose a vertex uniformly at random amongst the non-isolated vertices of degree less than k and delete all the edges incident to it, thus making it isolated. This step is repeated so long as there are edges to be deleted and the current set of vertices of degree at least k, say H, is non-empty. At the end, either $H \neq \emptyset$ and so H is the vertex set of the k-core of the initial graph, or $H = \emptyset$ and so the k-core there is empty. We denote by $\mathrm{DP}_k(G)$ the graph on V_n which is the output of the deletion process with parameter k taking as input a graph G on V_n. Here, we apply the deletion process to a $\mathcal{G}_{n,m}$ random graph. Note that for any integer $k \geq 2$, the deletion process finds the k-core of the input graph if and only if this is non-empty. We consider S_n to be the state space consisting of n-tuples of positive integers, where the deletion process induces a Markov chain on it - see [13]. Namely, it encodes the degree sequence of the underlying graph in the course of the deletion process. Let $\mathbf{w} = (v_0, \ldots, v_{n-1}) \in S_n$. If $\mathcal{H}(\mathbf{w})$ denotes the set of graphs on V_n whose degree sequence is $\mathbf{w}$ and $h(\mathbf{w})$ is its cardinality, then for $G \in \mathcal{H}(\mathbf{w})$ Proposition 2.1 (b) in [13] yields

$$\mathbb{P}[\mathrm{DP}_k(\mathcal{G}_{n,m}) = G \mid \mathbf{w}(T) = \mathbf{w}] = \frac{1}{h(\mathbf{w})},$$

where T is the stopping time of the deletion process, that is $\mathbf{w}(T)$ is the degree sequence of $\mathrm{DP}_k(\mathcal{G}_{n,m})$. Now, let $\mathbf{d} = (d_1, \ldots, d_n)$ be a sequence of non-negative integers, such that $\sum_{i \geq 1} d_i$ is even and for every $1 \leq i < n$ we have $d_i \leq d_{i+1}$. Let $\hat{\mathcal{H}}(\mathbf{d})$ be the set of those graphs on V_n, where the vertex i ($i \in V_n$) has degree equal to d_i and let $\hat{h}(\mathbf{d})$ be the cardinality of this set. It is immediate that for each sequence $\mathbf{w} = (v_0, \ldots, v_{n-1})$ such that $\sum_{i=1}^{n-1} i v_i$ is even, we can construct such a sequence $\mathbf{d} = \mathbf{d}(\mathbf{w})$, and vice versa. More specifically, for a graph G on V_n if $\mathbf{w} = (v_0, \ldots, v_{n-1}) = (v_0(G), \ldots, v_{n-1}(G))$, then we call the corresponding $\mathbf{d}(\mathbf{w})$ the *labelled degree sequence* of G and we denote it by $\mathbf{d}(G)$. For a vector $\mathbf{d}_n = (d_1, d_2, \ldots, d_n)$ (or $(d_1, d_2, \ldots, d_n, 0, \ldots)$ - we will be using both notations interchangeably) such that its (n first) entries are in non-decreasing order and they have even sum, we let $G(\mathbf{d}_n)$ be a random graph on V_n uniformly distributed over $\hat{\mathcal{H}}(\mathbf{d}_n)$. Note that if $\mathcal{I}$ is an isomorphism class in $\hat{\mathcal{H}}(\mathbf{d}(\mathbf{w}))$ and $\mathcal{I}'$ is the corresponding class in $\mathcal{H}(\mathbf{w})$, then $|\mathcal{I}'|/|\mathcal{I}| = n!/\prod_{i=0}^{n-1} v_i!$. Thus, the probability of a property closed under automorphisms is the same in the uniform probability spaces $\mathcal{H}(\mathbf{w})$ and $\hat{\mathcal{H}}(\mathbf{d}(\mathbf{w}))$. In what follows, we assume that $\theta > 1$ is fixed and $m = \lceil \theta n/2 \rceil$.

For each $n \in \mathbb{Z}^+$, let $\mathbf{D}_n$ be the space of all infinite integer vectors of the form $(d_1, \ldots, d_n, 0, \ldots)$ for which there exists a graph G on V_n having m edges so that $(d_1, \ldots, d_n) = \mathbf{d}(\mathrm{DP}_2(G))$, endowed with the natural probability measure inherited from the $\mathcal{G}_{n,m}$ space, which we denote by μ_n. It is convenient to put these probability spaces for $n = 1, 2, \ldots$ together to form one probability space. In particular, let $\mathbf{D} = \prod_{n=1}^{\infty} \mathbf{D}_n$ be the product of these spaces and μ be the product probability measure on the product σ-algebra. If an event E that belongs to this algebra is such that $\mu(E) = 1$, then we say that E occurs *almost surely* (a.s.). An element of the space $\mathbf{D}$ is denoted by $(\mathbf{d}_n)$ and note that this is a sequence of infinite vectors indexed by the set of positive integers. For a given $(\mathbf{d}_n)$, if $\mathbf{d}_n = (d_1, \ldots, d_n, 0, \ldots)$ for $n \in \mathbb{Z}^+$, we set $D_i = D_i(n) = |\{j \in V_n \ : \ d_j = i\}|$, for $i \in \mathbb{N}$ and $\Delta = \Delta(n) = \max_{1 \leq i \leq n}\{d_i\} = d_n$. Also, we let $M = M(n) = \frac{1}{2} \sum_{i=1}^{n} d_i$. For any $n \in \mathbb{Z}^+$, we denote by π_n the projection of $\Omega = \prod_{i=1}^{\infty} \Omega_i$ onto Ω_n. We state and prove the following propositions which will be used in the sequel:

Proposition 2.2. *Let $\{(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)\}_{n \in \mathbb{Z}^+}$ be a family of probability spaces and let $\Omega = \prod_{n \in \mathbb{Z}^+} \Omega_n$ be the product space endowed with the product measure, say μ. Let $\{\mathcal{E}_n \in \mathcal{F}_n\}_{n \in \mathbb{Z}^+}$ be a family of measurable sets such that $\sum_{n \in \mathbb{Z}^+} \mathbb{P}_n[\mathcal{E}_n^c] < \infty$ and let $G = \{\omega \in \Omega \ : \ \exists N \in \mathbb{N}$ s.t. $\forall n > N \ \pi_n(\omega) \in \mathcal{E}_n\}$. Then $\mu(G) = 1$.*

Proof. For each $n \in \mathbb{Z}^+$, let $\mathcal{E}_n' = \{\omega \in \Omega \ : \ \pi_n(\omega) \in \mathcal{E}_n\}$ and note that these events are independent. Then $G^c = \cap_{N \in \mathbb{Z}^+} \cup_{n \geq N} \mathcal{E}_n'^c$ and by the second Borel-Cantelli Lemma we have $\mu(G^c) = 0$, since $\sum_{n \in \mathbb{Z}^+} \mu(\mathcal{E}_n'^c) = \sum_{n \in \mathbb{Z}^+} \mathbb{P}_n[\mathcal{E}_n^c] < \infty$. ∎

Proposition 2.3. *Under the assumptions of Proposition 2.2, for each $n \in \mathbb{Z}^+$, let $X_n : \Omega_n \to \mathbb{R}$ be a random variable and suppose that $X_n = x + o_p(1)$, for some $x \in \mathbb{R}$, and for every $\varepsilon > 0$ we have $\sum_{n\in\mathbb{Z}^+} \mathbb{P}_n[|X_n - x| > \varepsilon] < \infty$. Let $L = \{\omega \in \Omega \ : \ \lim_{n\to\infty} X_n(\pi_n(\omega)) = x\}$. Then $\mu(L) = 1$.*

Proof. For each $n \in \mathbb{Z}^+$ and for every $\varepsilon > 0$, we set $\mathcal{E}_n(\varepsilon) = \{\omega_n \in \Omega_n \ : \ |X_n(\omega_n) - x| < \varepsilon\}$ and let $L(\varepsilon) = \{\omega \in \Omega \ : \ \exists N \in \mathbb{N} \text{ s.t. } \forall n > N \ \pi_n(\omega) \in \mathcal{E}_n(\varepsilon)\}$. By the previous proposition, we have $\mu(L(\varepsilon)) = 1$, for every $\varepsilon > 0$. It is immediate that $\{L(1/t)\}_{t\in\mathbb{Z}^+}$ is a decreasing family of sets and $L = \cap_{t\in\mathbb{Z}^+} L(1/t)$. Therefore $\mu(L) = \lim_{t\to\infty} \mu(L(1/t)) = 1$. □

For some $c > 0$ let $\mathcal{D} = \mathcal{D}(c)$ be the set of $(\mathbf{d}_n) \in \mathbf{D}$ with the property that for every $0 < \varepsilon < 1$, there exist $k_0, N \in \mathbb{N}$ such that

$$Y(\mathbf{d}_n) = \sum_{i=2}^{n-1} i^4 D_i \le cn, \ X_{n,k_0}(\mathbf{d}_n) = \sum_{i=k_0}^{\Delta} \binom{i}{2} \frac{D_i}{n} < \varepsilon \text{ and } \Delta \le \lceil \ln n \rceil,$$

for any $n > N$.

Claim 2.4. *There exists $c > 0$ for which $\mu(\mathcal{D}) = 1$.*

Proof. For every real $\varepsilon > 0$ and $k \in \mathbb{N}$, we let

$$L(\varepsilon, k) = \{(\mathbf{d}_n) \in \mathbf{D} \ : \ \exists N \in \mathbb{N} \text{ s.t. } \forall n > N \ \mathbf{d}_n \in \mathcal{E}_n(\varepsilon, k)\}$$

where $\mathcal{E}_n(\varepsilon, k) = \{\mathbf{d}_n \in \mathbf{D}_n \ : \ Y(\mathbf{d}_n) \le cn, \ X_{n,k}(\mathbf{d}_n) < \varepsilon, \ \Delta \le \lceil \ln n \rceil\}$. If c is as in Lemma A.1 in Appendix A, then this implies that for every $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that $\sum_{n\in\mathbb{Z}^+} \mu_n(\mathcal{E}_n^c(\varepsilon, k)) < \infty$. Therefore, by Proposition 2.2 we deduce that $\mu(L(\varepsilon, k)) = 1$. Thus, if we let $L(\varepsilon) = \cup_{k\in\mathbb{N}} L(\varepsilon, k)$, we have $\mu(L(\varepsilon)) = 1$ as well. Now notice that $\mathcal{D} = \cap_{0<\varepsilon<1} L(\varepsilon) = \cap_{t=1}^{\infty} L(1/t)$ and that $\{L(1/t)\}_{t\in\mathbb{Z}^+}$ forms a decreasing family of sets as t increases. Therefore, $\mu(\mathcal{D}) = \lim_{t\to\infty} \mu(L(1/t)) = 1$. □

Note that Theorem 2.1 in the previous chapter along with Proposition 2.3 imply that for every $i \ge 2$, we have $\lim_{n\to\infty} D_i/n = \rho_i(\lambda_2(\theta))$ and $\lim_{n\to\infty} M/n = \lim_{n\to\infty} \sum_{i=2}^{\Delta} iD_i/n = \lambda_2^2(\theta)/2\theta$ a.s.. For c as in the previous claim, let G ("good") be the set of those $(\mathbf{d}_n) \in \mathcal{D}$ for which the preceding statement is true. We have $\mu(G) = 1$. Now, it is easy to see the following:

Proposition 2.5. *For a fixed $k \in \mathbb{Z}^+$, let X_n be a function on the set of graphs on V_n taking values in $\mathbb{R}^k$, which is invariant under any automorphism of the core, and let $x \in \mathbb{R}^k$. If there exists $p \in \mathbb{R}^+$ such that for every $(\mathbf{d}_n) \in G$, we have $\lim_{n\to\infty} \mathbb{P}[X_n(G(\mathbf{d}_n)) \le x] = p$, then $\lim_{n\to\infty} \mathbb{P}[X_n(\mathcal{G}_{n,m}) \le x] = p$.*

Proof. The proof is almost straightforward:

$$\begin{aligned}
\mathbb{P}[X_n(\mathcal{G}_{n,m}) \le x] &= \sum_{\mathbf{d}_n \in \mathbf{D}_n} \mathbb{P}[X_n(\mathcal{G}_{n,m}) \le x \mid \mathbf{d}(\mathrm{DP}_2(\mathcal{G}_{n,m})) = \mathbf{d}_n] \ \mathbb{P}[\mathbf{d}(\mathrm{DP}_2(\mathcal{G}_{n,m})) = \mathbf{d}_n] \\
&= \sum_{\mathbf{d}_n \in \mathbf{D}_n} \int_{\{(\mathfrak{d}_n) \ : \ \pi_n((\mathfrak{d}_n)) = \mathbf{d}_n\}} \mathbb{P}[X_n(G(\mathbf{d}_n)) \le x] \ \mu(d(\mathfrak{d}_n)) \\
&= \int \mathbb{P}[X_n(G(\pi_n((\mathfrak{d}_n)))) \le x] \ \mu(d(\mathfrak{d}_n)).
\end{aligned}$$

Since the integrand is bounded below by 0, applying Fatou's Lemma, we obtain:

$$\begin{aligned}
\liminf_{n\to\infty} \mathbb{P}[X_n(\mathcal{G}_{n,m}) \le x] &= \liminf_{n\to\infty} \int \mathbb{P}[X_n(G(\pi_n((\mathfrak{d}_n)))) \le x] \ \mu(d(\mathfrak{d}_n)) \\
&\ge \int \liminf_{n\to\infty} \mathbb{P}[X_n(G(\pi_n((\mathfrak{d}_n)))) \le x] \ \mu(d(\mathfrak{d}_n)) \\
&= \int_G \liminf_{n\to\infty} \mathbb{P}[X_n(G(\pi_n((\mathfrak{d}_n)))) \le x] \ \mu(d(\mathfrak{d}_n)) \\
&= p \int_G \mu(d(\mathfrak{d}_n)) = p.
\end{aligned}$$

Now, applying the Reverse Fatou's Lemma (since the integrand is bounded above by 1), we have

$$\begin{aligned}\limsup_{n\to\infty} \mathbb{P}[X_n(\mathcal{G}_{n,m}) \le x] &= \limsup_{n\to\infty} \int \mathbb{P}[X_n(G(\pi_n((\mathfrak{d}_n)))) \le x]\ \mu(d(\mathfrak{d}_n)) \\ &\le \int \limsup_{n\to\infty} \mathbb{P}[X_n(G(\pi_n((\mathfrak{d}_n)))) \le x]\ \mu(d(\mathfrak{d}_n)) \\ &= \int_G \limsup_{n\to\infty} \mathbb{P}[X_n(G(\pi_n((\mathfrak{d}_n)))) \le x]\ \mu(d(\mathfrak{d}_n)) \\ &= p\int_G \mu(d(\mathfrak{d}_n)) = p,\end{aligned}$$

which concludes the proof of the proposition. □

We state without proof the following lemma, which follows from the main theorem in [11]:

Lemma 2.6. *If* $(\mathbf{d}_n) \in G$*, then*

$$\left|\hat{\mathcal{H}}(\mathbf{d}_n)\right| = \left(1 + O\left(\frac{\ln^3 n}{n}\right)\right) e^{-\lambda-\lambda^2} \frac{(2M)!}{M!2^M \prod_{i=1}^n d_i!},$$

where $\lambda = \sum_{i=1}^n \binom{d_i}{2}/(2M)$.

Now, we are ready to proceed with the proofs of Theorems 1.1 and 1.2.

3. The distribution of cycles on the core of a $\mathcal{G}_{n,m}$ random graph

In this section, we mainly investigate the asymptotic distribution of the total number of isolated cycles as well as of the number of cycles of fixed length which are either isolated or not in $\mathrm{cr}(\mathcal{G}_{n,m})$, thus proving Theorem 1.1.

Firstly, we prove parts (i) and (ii) of Theorem 1.1. We state without proof a part of a theorem which was proved by T. Łuczak in [9] (Theorem 12.2(ii) there), slightly adapted to our context:

Theorem 3.1. *Let* $(\mathbf{d}_n = (d_1,\dots,d_n))_{n\in\mathbb{Z}^+}$ *be such that for* $i = 1,\dots,D_0$ *we have* $d_i = 0$ *and* $2 \le \min_{i>D_0}\{d_i\}$ *as well as* $\max_i\{d_i\} \le n^{0.01}$*, where* $D_0 = D_0(n)$ *is the number of zeros in* $\mathbf{d}_n$ *with* $n - D_0 \to \infty$ *as* $n \to \infty$*. Also let* $D_2 = D_2(n)$ *be the number of twos in* $\mathbf{d}_n$*,* $2M = 2M(n) = \sum_{i=1}^n d_i$*,* $L_i(G(\mathbf{d}_n))$ *denote the i-th largest component of the random graph* $G(\mathbf{d}_n)$*, and* $\omega = \omega(n)$ *be a function that tends to infinity with n. If* $\lim_{n\to\infty} D_2/M = b < 1$*, then* $\lim_{n\to\infty}\mathbb{P}[|L_1(G(\mathbf{d}_n))| \ge N - \omega] = 1$*, where* $N = n - D_0$*, all of the smaller* $(i \ge 2)$ *non-trivial components are cycles with probability tending to 1 as* $n \to \infty$*, and for* $t = 0, 1, \dots,$

$$\lim_{n\to\infty} \mathbb{P}[G(\mathbf{d}_n) \text{ consists of } t+1 \text{ non-trivial components}] = e^{-\mu}\frac{\mu^t}{t!},$$

where $\mu = -\frac{1}{2}\ln(1-b) - \frac{1}{2}b - \frac{1}{4}b^2$.

Suppose that $(\mathbf{d}_n) \in G$. Firstly, note that the second part of Lemma B.1 in Appendix B implies that $\lim_{n\to\infty}\sum_{i=2}^{\Delta} D_i/n = p_2(\lambda_2(\theta))$. Now, by the definition of G we obtain $\lim_{n\to\infty} D_2/n = \rho_2(\lambda_2(\theta))$, $\lim_{n\to\infty} M/n = \frac{\lambda_2^2(\theta)}{2\theta}$ and, therefore, $\lim_{n\to\infty} D_2/M = \theta e^{-\lambda_2(\theta)} < 1$. Moreover, we have $\Delta < n^{0.01}$, for n sufficiently large. Also notice that $\frac{\lambda_2^2(\theta)}{2\theta} > p_2(\lambda_2(\theta))$, which along with the above observations and Theorem 3.1 imply that $|L_1(G(\mathbf{d}_n))|/n = p_2(\lambda_2(\theta)) + o_p(1)$ and $L_1(G(\mathbf{d}_n))$ has more than one cycle with probability tending to 1 as $n \to \infty$. On the other hand, the number of vertices that belong to smaller non-trivial components of $G(\mathbf{d}_n)$ is $O_p(1)$, with the latter being cycles with probability tending to 1 as $n \to \infty$ and finally the distribution of their total number is given by the above theorem, with $b = \theta e^{-\lambda_2(\theta)}$. Therefore, applying Proposition 2.5, we deduce Theorem 1.1(i) and (ii).

We proceed with the proof of the final part of Theorem 1.1. For any natural number $k \ge 3$ and a graph G on V_n, we let $Z_{nk}(G)$, $Z'_{nk}(G)$ be the number of isolated cycles of length k and the number of cycles of the same length that are not isolated in G, respectively. We first prove the following lemma:

Lemma 3.2. *For any integers $t, s \geq 0$, let $k_1, \ldots, k_{s+t} \geq 3$ and $l_1, \ldots, l_{s+t} \geq 1$ be natural numbers. Let $(\mathbf{d}_n) \in G$ and for any integer $k \geq 3$, we set $Z_{nk} = Z_{nk}(G(\mathbf{d}_n))$ and $Z'_{nk} = Z'_{nk}(G(\mathbf{d}_n))$. Then*

$$\lim_{n\to\infty} \mathbb{E}[(Z'_{nk_1})_{l_1} \cdots (Z'_{nk_t})_{l_t} (Z_{nk_{t+1}})_{l_{t+1}} \cdots (Z_{nk_{t+s}})_{l_{t+s}}] = \prod_{j=1}^{t} \left(\frac{\theta^{k_j}(1 - e^{-\lambda_2(\theta)k_j})}{2k_j} \right)^{l_j} \prod_{j=1}^{s} \left(\frac{(\theta e^{-\lambda_2(\theta)})^{k_{t+j}}}{2k_{t+j}} \right)^{l_{t+j}}.$$

(Here, the empty product is assumed to be equal to 1.)

Proof. We shall prove the above statement only for $t = s = 1$, as the proof of the general case is tedious but follows the same lines - see [2]. Is this particular case, the left hand side of the above equation expresses the expectation of the total number of pairs of ordered collections of cycles, with their first element consisting of ordered l_1-tuples of cycles of length k_1 which are not isolated and the second of ordered l_2-tuples of isolated cycles of length k_2. Moreover, every such pair induces a subgraph of order at most $l_1 k_1 + l_2 k_2$. By Lemma A.2 in Appendix A, the expected number of such pairs containing at least two overlapping non-isolated cycles tends to 0 as $n \to \infty$, since they induce subgraphs of bounded order which have more edges than vertices and $(\mathbf{d}_n) \in G$. Therefore, the above expectation is asymptotically equal to the expected number of those pairs consisting of pairwise disjoint cycles. Let X_n be the total number of the latter.

Given positive integers k and i, we use the following notation

$$\sum_{k}^{(i)} \equiv \sum_{p_i=1}^{k} \sum_{\substack{\text{ordered} \\ p_i-\text{partitions of } k \\ \left(k_1^{(i)}, \ldots, k_{p_i}^{(i)}\right)}} \sum_{\substack{p_i-\text{tuples} \\ 2 \leq i_1^{(i)} < \cdots < i_{p_i}^{(i)} \leq \Delta, \\ \text{such that if } p_i = 1, \text{ then } i_1^{(i)} \neq 2}}.$$

Therefore, by Lemma 2.6, we have

$$\begin{aligned}
\mathbb{E}[X_n(G(\mathbf{d}_n))] &= \left(\frac{(k_1-1)!}{2}\right)^{l_1} \left(\frac{(k_2-1)!}{2}\right)^{l_2} \sum_{k_1}^{(1)} \prod_{j=1}^{p_1} \binom{D^{(1)}_{i_j^{(1)}}}{k_j^{(1)}} \Bigg(\cdots \Bigg(\sum_{k_1}^{(l_1)} \prod_{j=1}^{p_{l_1}} \binom{D^{(l_1)}_{i_j^{(l_1)}}}{k_j^{(l_1)}} \\
&\quad \binom{D_2^{(l_1+1)}}{k_2} \cdots \binom{D_2^{(l_1+1)} - (l_2-1)k_2}{k_2} \\
&\quad \frac{(2M - 2(l_1k_1 + l_2k_2))!}{(M - (l_1k_1 + l_2k_2))! 2^{M-(l_1k_1+l_2k_2)} \prod_{i=1}^n d'_i!} \frac{M! 2^M \prod_{i=1}^n d_i!}{(2M)!} \times \\
&\quad \exp\left(-\lambda' + \lambda - (\lambda')^2 + \lambda^2\right) \Bigg) \cdots \Bigg) \left(1 + O\left(\frac{\ln^3 n}{n}\right)\right) \\
&= \left(\frac{k_1!}{2k_1}\right)^{l_1} \left(\frac{k_2!}{2k_2}\right)^{l_2} \sum_{k_1}^{(1)} \prod_{j=1}^{p_1} \frac{D^{k_j^{(1)}}_{i_j^{(1)}}}{k_j^{(1)}!} \Bigg(\cdots \Bigg(\sum_{k_1}^{(l_1)} \prod_{j=1}^{p_{l_1}} \frac{D^{k_j^{(l_1)}}_{i_j^{(l_1)}}}{k_j^{(l_1)}!} \left(\frac{D_2^{k_2}}{k_2!}\right)^{l_2} \\
&\quad \frac{M^{l_1k_1+l_2k_2}}{(2M)^{2(l_1k_1+l_2k_2)}} 2^{l_1k_1} 2^{2l_2k_2} \prod_{i=1}^{l_1} \prod_{j=1}^{p_i} \binom{i_j^{(i)}}{2}^{k_j^{(i)}} 2^{k_j^{(i)}} \Bigg) \cdots \Bigg) \left(1 + O\left(\frac{\ln^3 n}{n}\right)\right). \\
&= \left(\frac{1}{2k_1} \left(\left(\sum_{j=2}^{\Delta} \binom{j}{2} \frac{D_j}{M} \right)^{k_1} - \left(\frac{D_2}{M}\right)^{k_1} \right) \right)^{l_1} \left(\frac{1}{2k_2} \left(\frac{D_2}{M}\right)^{k_2} \right)^{l_2} \left(1 + O\left(\frac{\ln^3 n}{n}\right)\right),
\end{aligned}$$

where $D_j^{(i)}$ denotes the number of those vertices amongst the D_j vertices of degree j in the labelled degree sequence $(d_1, \ldots, d_n)$ that are left if we remove the edges of the first $i-1$ cycles of a given pair and $\mathbf{d}'_n = (d'_1, \ldots, d'_n)$ is the labelled degree sequence that remains from $(d_1, \ldots, d_n)$ after the removal of all the edges of these cycles. The following observations will clarify the intermediate steps of the above calculation. Firstly, note that $D_j^{(i)} = D_j(1 + O(1/n))$. On the other hand, if vertex i participates in a cycle, then $\mathbf{d}'_n$ has one more vertex of degree $d_i - 2$ and one less vertex of degree d_i. Let $C \subset V_n$ be the set those vertices that belong to the cycles of a given pair. Notice that for any integer $d \geq 2$, we have

$$-\binom{d-2}{2} + \binom{d}{2} = 2d - 3 \text{ and } -\binom{d-1}{2} + \binom{d}{2} = d - 1.$$

(The second identity will be used in the next section.) For the two subsequent inequalities, we shall assume that $(d'_1, \ldots, d'_n)$ is the sequence that is obtained from $\mathbf{d}$, where for each $i \in V_n$ d'_i is the degree of vertex i after the removal of the edges of C. Clearly, this is a permutation of $\mathbf{d}'$ as it was defined above. Therefore, for n sufficiently large,

$$\begin{aligned} -\lambda' + \lambda &= -\frac{\sum_{i=1}^n \binom{d'_i}{2}}{2(M - (l_1 k_1 + l_2 k_2))} + \frac{\sum_{i=1}^n \binom{d_i}{2}}{2M} \\ &< \frac{1}{2M} \sum_{i \in C} \left(-\binom{d_i - 2}{2} + \binom{d_i}{2} \right) = \frac{1}{2M} \sum_{i \in C} (2d_i - 3) = O\left(\frac{\Delta}{M}\right), \end{aligned}$$

since the order of the union of the cycles (i.e. $|C|$) is bounded. Similarly (see [2] for the details),

$$-(\lambda')^2 + \lambda^2 = -\frac{\left(\sum_{i=1}^n \binom{d'_i}{2}\right)^2}{4(M - (l_1 k_1 + l_2 k_2))^2} + \frac{\left(\sum_{i=1}^n \binom{d_i}{2}\right)^2}{4M^2} = O\left(\frac{n\Delta^3}{M^2}\right),$$

since $|C|$ is bounded. Now, using the two previous relations, we obtain

$$\exp\left(-\lambda' + \lambda - (\lambda')^2 + \lambda^2\right) = \exp\left(O\left(\frac{\Delta}{M} + \frac{n\Delta^3}{M^2}\right)\right) = 1 + O\left(\frac{\ln^3 n}{n}\right),$$

since $(\mathbf{d}_n) \in G$. The first part of Lemma B.1 in Appendix B concludes the proof. □

We are ready to proceed with the proof of the final part of Theorem 1.1. We still assume that $(\mathbf{d}_n) \in G$. If we set $t = 0$ and $s = 1$ in the above lemma, then for any integers $k \geq 3$ and $l \geq 1$, we have

$$\lim_{n\to\infty} (Z_{nk})_l = \left(\frac{\left(\theta e^{-\lambda_2(\theta)}\right)^k}{2k} \right)^l,$$

and this implies that $Z_{nk} \xrightarrow{d} \text{Po}\left((\theta e^{-\lambda_2(\theta)})^k / 2k\right)$ as $n \to \infty$ (see Corollary 6.8 in [5]). Similarly, setting $t = 1$ and $s = 0$ in the above lemma, for any integers $k \geq 3$ and $l \geq 1$, we have

$$\lim_{n\to\infty} \left(Z'_{nk}\right)_l = \left(\frac{\theta^k \left(1 - e^{-k\lambda_2(\theta)}\right)}{2k} \right)^l,$$

which implies that $Z'_{nk} \xrightarrow{d} \text{Po}\left((\theta^k (1 - e^{-k\lambda_2(\theta)})/2k\right)$, as $n \to \infty$. Now, the above lemma along with Theorem 6.10 in [5] imply that the joint distribution of any finite collection of pairwise distinct random variables $(Z_{nk_1}, \ldots Z_{nk_t}, Z'_{nk_{t+1}}, \ldots, Z'_{nk_{t+s}})$ converges to that of $(Z_{k_1}, \ldots Z_{k_t}, Z'_{k_{t+1}}, \ldots, Z'_{k_{t+s}})$, where, for $i = 1, \ldots, t$, $Z_{k_i} = \text{Po}\left((\theta e^{-\lambda_2(\theta)})^{k_i} / 2k_i\right)$, and for $i = t+1, \ldots, t+s$ $Z'_{k_i} = \text{Po}\left(\theta^{k_i} (1 - e^{-k_i \lambda_2(\theta)}) / 2k_i\right)$ are independent. Finally, Proposition 2.5 concludes the proof of Theorem 1.1(iii).

4. The asymptotic distribution of the cyclic 2-vertex(-edge)- connected components of $\mathrm{cr}(\mathcal{G}_{n,m})$

We begin with a general result concerning the structure of $L_1(\mathcal{G}_{n,m})$. We state without proof the following theorem from [12] (Theorem 3(a) there):

Theorem 4.1. *For any fixed $\theta > 1$, $L_1(\mathcal{G}_{n,m})$ consists a.a.s. of a giant 2-vertex-connected subgraph of order $\Theta_p(n)$, a collection of trees and a collection of unicyclic components, each of them sprouting from a different vertex of the 2-vertex-connected component, whose total number and order is $O_p(1)$.*

The above theorem implies that $L_1(\mathrm{cr}(\mathcal{G}_{n,m}))$ consists of an essential core of linear order while each of the remaining non-trivial blocks contains precisely one cutvertex of $L_1(\mathrm{cr}(\mathcal{G}_{n,m}))$ and is a cycle. Moreover, the total number and order of the blocks of $L_1(\mathrm{cr}(\mathcal{G}_{n,m}))$ apart from the essential core is $O_p(1)$.

Let $n \in \mathbb{Z}^+$. For a graph G on V_n with no vertices of degree 1 and integers $k, k_1 \geq 3$ and $k_2 \geq 0$, let $X_k^v(G)$ be the number of the 2-vertex-connected components of G which are k-cycles, contain exactly one cutvertex of G and are not 2-edge-connected components and let $X^e_{k_1,k_2}(G)$ be the number of the 2-edge-connected components of G which are k_1-cycles and there is a unique path with k_2 internal vertices of degree 2 in G having exactly one endvertex in the cycle and the other endvertex has degree at least 3 in G (uniqueness is meant with respect to the existence of a different path which has possibly different length). We call these paths the *attaching paths* of the 2-edge-connected components. Moreover, let $X_k^e(G)$ be the number of the 2-edge-connected components of G which are k-cycles and contain precisely one cutvertex of G. For a function $\omega : \mathbb{Z}^+ \to \mathbb{Z}^+$, which will be specified later, such that $\omega(n) \to \infty$ as $n \to \infty$, and $(\mathbf{d}_n) \in \mathbf{D}$ we let $X^e(G(\mathbf{d}_n)) = \sum_{k=3}^{\omega(n)\vee 3} X_k^e(G(\mathbf{d}_n))$ and $X^v(G(\mathbf{d}_n)) = \sum_{k=3}^{\omega(n)\vee 3} X_k^v(G(\mathbf{d}_n))$. Similarly, we define $X^e(\mathcal{G}_{n,m}) = \sum_{k=3}^{\omega(n)\vee 3} X_k^e(\mathrm{cr}(\mathcal{G}_{n,m}))$ and $X^v(\mathcal{G}_{n,m}) = \sum_{k=3}^{\omega(n)\vee 3} X_k^v(\mathrm{cr}(\mathcal{G}_{n,m}))$. In this section, we investigate the asymptotic distributions of the above random variables, aiming towards the proof of Theorem 1.2. Firstly, we present the following calculations which will be used in the present section. Let G be a graph on V_n with no vertices of degree 1 and $\mathbf{d} = (d_1, \ldots, d_n)$ be its labelled degree sequence. For some $r \in \mathbb{Z}^+$, let C be a collection of subgraphs of G consisting of r vertex disjoint cyclic 2-vertex-connected components of G which are not 2-edge-connected components and cyclic 2-edge-connected components of G including their attaching paths, with each cycle containing precisely one cutvertex of G. Let t, s be the total number of vertices of degree 2 and 3 respectively of G that totally participate in these subgraphs. Also, let S_1 be the multiset of degrees of those cutvertices in G that belong to the 2-vertex-connected components of C which are not 2-edge-connected components and S_2 be the multiset of the degrees in G of those endvertices of the attaching paths of the 2-edge-connected components in C that do not belong to their cycles. Let $\mathbf{d}'$ be the labelled degree sequence of G with the edges that belong to the elements of C removed and note that this is a permutation of the sequence $(d'_1, \ldots, d'_n)$ that is obtained from $\mathbf{d}$, where for each $i \in V_n$ d'_i is the degree of vertex i after the removal of the edges of C. With λ as it was defined in Lemma 2.6 and λ' being the corresponding quantity for $\mathbf{d}'$, we have

$$\begin{aligned} -\lambda' + \lambda &< \frac{1}{2M}\left(-\sum_{j=1}^{n}\binom{d'_j}{2} + \sum_{j=1}^{n}\binom{d_j}{2}\right) \\ &= \frac{1}{2M}\left(t + 3s + \sum_{i\in S_1}\left(\binom{i}{2} - \binom{i-2}{2}\right) + \sum_{i\in S_2}\left(\binom{i}{2} - \binom{i-1}{2}\right)\right) \\ &= \frac{1}{2M}\left(t + 3s + \sum_{i\in S_1}(2i-3) + \sum_{i\in S_2}(i-1)\right). \qquad (1) \end{aligned}$$

On the other hand (see [2] for the elementary but tedious calculation),

$$
\begin{aligned}
-\lambda'^2+\lambda^2 &< -\frac{\left(\sum_{j=1}^n \binom{d_j'}{2}\right)^2}{4M^2}+\frac{\left(\sum_{j=1}^n \binom{d_j}{2}\right)^2}{4M^2} \\
&< \frac{1}{4M^2}\left(2\left(t+3s+\sum_{i\in S_1}(2i-3)+\sum_{i\in S_2}(i-1)\right)\left(\sum_{j\notin V(C)}\binom{d_j}{2}\right)\right. \\
&\quad \left. +\left(t+3s+\sum_{i\in S_1\cup S_2}\binom{i}{2}\right)^2-\left(\sum_{i\in S_1}\binom{i-2}{2}+\sum_{i\in S_2}\binom{i-1}{2}\right)^2\right). \qquad (2)
\end{aligned}
$$

We begin with the calculation of $\mathbb{E}[(X^e(G(\mathbf{d}_n)))_{l_1}(X^v(G(\mathbf{d}_n)))_{l_2}]$, for some positive integers l_1,l_2 and some $(\mathbf{d}_n)\in G$, where we set $\omega(n)=\lfloor \ln\ln n/\ln\ln\ln n\rfloor$ (for any $n\geq 16$ - otherwise we set $\omega(n)=1$) in the definitions of $X^e(G(\mathbf{d}_n))$ and $X^v(G(\mathbf{d}_n))$. Note that the above random variable counts the number of pairs of ordered l_1-tuples of "small" 2-edge-connected cyclic components and ordered l_2-tuples of "small" 2-vertex-connected cyclic components that are not 2-edge-connected components in $G(\mathbf{d}_n)$, each of the above containing exactly one cutvertex of $G(\mathbf{d}_n)$. Since $(\mathbf{d}_n)\in G$, by Lemma A.2 in Appendix A it is sufficient to consider in the calculation of the above expectation only those pairs that consist of graphs that are vertex disjoint. For two functions $f,g:\mathbb{N}\to\mathbb{N}$, we write $f(n)\approx g(n)$ if and only if $|f(n)-g(n)|=o(1)$. We have:

$$
\begin{aligned}
&\mathbb{E}[(X^e(G(\mathbf{d}_n)))_{l_1}(X^v(G(\mathbf{d}_n)))_{l_2}]\approx \\
&\sum_{k_1^{(1)}=3,k_1^{(2)}=0}^{\omega(n)}\binom{D_2}{k_1^{(1)}-1}\frac{(k_1^{(1)}-1)!}{2}\binom{D_2-(k_1^{(1)}-1)}{k_1^{(2)}}k_1^{(2)}!D_3^{(1)}\sum_{j_1=3}^{\Delta}(D_{j_1}^{(1)}-\delta_{j_1,3})\Bigg(\cdots \\
&\sum_{k_{l_1}^{(1)}=3,k_{l_1}^{(2)}=0}^{\omega(n)}\binom{D_2-\sum_{i=1}^{l_1-1}(k_i^{(1)}+k_i^{(2)}-1)}{k_{l_1}^{(1)}-1}\frac{(k_{l_1}^{(1)}-1)!}{2}\times \\
&\binom{D_2-\sum_{i=1}^{l_1-1}(k_i^{(1)}+k_i^{(2)}-1)-(k_{l_1}^{(1)}-1)}{k_{l_1}^{(2)}}k_{l_1}^{(2)}!D_3^{(l_1)}\sum_{j_{l_1}=3}^{\Delta}(D_{j_{l_1}}^{(l_1)}-\delta_{j_{l_1},3})\Bigg(\\
&\sum_{k_1=3}^{\omega(n)}\binom{D_2-\sum_{i=1}^{l_1}(k_i^{(1)}+k_i^{(2)}-1)}{k_1-1}\frac{(k_1-1)!}{2}\sum_{i_1=4}^{\Delta}D_{i_1}^{(l_1+1)}\Bigg(\cdots \\
&\sum_{k_{l_2}=3}^{\omega(n)}\binom{D_2-\sum_{i=1}^{l_1}(k_i^{(1)}+k_i^{(2)}-1)-\sum_{i=1}^{l_2-1}(k_i-1)}{k_{l_2}-1}\frac{(k_{l_2}-1)!}{2}\sum_{i_{l_2}=4}^{\Delta}D_{i_{l_2}}^{(l_1+l_2)}\times \\
&\frac{(2M-2(\sum_{i=1}^{l_1}(k_i^{(1)}+k_i^{(2)}+1)+\sum_{i=1}^{l_2}k_i))!}{(M-(\sum_{i=1}^{l_1}(k_i^{(1)}+k_i^{(2)}+1)+\sum_{i=1}^{l_2}k_i))!2^{M-(\sum_{i=1}^{l_1}(k_i^{(1)}+k_i^{(2)}+1)+\sum_{i=1}^{l_2}k_i)}\prod_{j=1}^n d_j'!}\times \\
&\frac{M!2^M\prod_{j=1}^n d_j!}{(2M)!}\exp\left(-\lambda'+\lambda-(\lambda')^2+\lambda^2\right)\Bigg)\cdots\Bigg)\left(1+O\left(\frac{\ln^3 n}{n}\right)\right)
\end{aligned}
$$

$$= \sum_{k_1^{(1)}=3,k_1^{(2)}=0}^{\omega(n)} \frac{D_2^{k_1^{(1)}+k_1^{(2)}-1}}{(k_1^{(1)}-1)!k_1^{(2)}!} \frac{(k_1^{(1)}-1)!k_1^{(2)}!}{2} D_3 \sum_{j_1=3}^{\Delta} D_{j_1} \Bigg(\cdots$$

$$\sum_{k_{l_1}^{(1)}=3,k_{l_1}^{(2)}=0}^{\omega(n)} \frac{D_2^{k_{l_1}^{(1)}+k_{l_1}^{(2)}-1}}{(k_{l_1}^{(1)}-1)!k_{l_1}^{(2)}!} \frac{(k_{l_1}^{(1)}-1)!k_{l_1}^{(2)}!}{2} D_3 \sum_{j_{l_1}=3}^{\Delta} D_{j_{l_1}} \Bigg($$

$$\sum_{k_1=3}^{\omega(n)} \frac{D_2^{k_1-1}}{(k_1-1)!} \frac{(k_1-1)!}{2} \sum_{i_1=4}^{\Delta} D_{i_1} \Bigg(\cdots \sum_{k_{l_2}=3}^{\omega(n)} \frac{D_2^{k_{l_2}-1}}{(k_{l_2}-1)!} \frac{(k_{l_2}-1)!}{2} \sum_{i_{l_2}=4}^{\Delta} D_{i_{l_2}} \times$$

$$\frac{(M)_{\sum_{i=1}^{l_1}(k_i^{(1)}+k_i^{(2)}+1)+\sum_{i=1}^{l_2} k_i} 2^{\sum_{i=1}^{l_1}(k_i^{(1)}+k_i^{(2)}+1)+\sum_{i=1}^{l_2} k_i}}{(2M)_{2(\sum_{i=1}^{l_1}(k_i^{(1)}+k_i^{(2)}+1)+\sum_{i=1}^{l_2} k_i)}}$$

$$\frac{\prod_{j=1}^n d_j!}{\prod_{j=1}^n d'_j!} \exp\left(-\lambda' + \lambda - (\lambda')^2 + \lambda^2\right) \Bigg) \cdots \Bigg) \left(1 + O\left(\frac{\omega(n)^2}{n} + \frac{\ln^3 n}{n}\right)\right)$$

$$= (M - D_2)^{l_1} \sum_{k_1^{(1)}=3,k_1^{(2)}=0}^{\omega(n)} \frac{3D_2^{k_1^{(1)}+k_1^{(2)}-1}}{2} D_3 \Bigg(\cdots \sum_{k_{l_1}^{(1)}=3,k_{l_1}^{(2)}=0}^{\omega(n)} \frac{3D_2^{k_{l_1}^{(1)}+k_{l_1}^{(2)}-1}}{2} D_3 \Bigg($$

$$\sum_{k_1=3}^{\omega(n)} \frac{D_2^{k_1-1}}{2} \sum_{i_1=4}^{\Delta} \binom{i_1}{2} D_{i_1} \Bigg(\cdots \sum_{k_{l_2}=3}^{\omega(n)} \frac{D_2^{k_{l_2}-1}}{2} \sum_{i_{l_2}=4}^{\Delta} \binom{i_{l_2}}{2} D_{i_{l_2}} \times$$

$$\frac{M^{\sum_{i=1}^{l_1}(k_i^{(1)}+k_i^{(2)}+1)+\sum_{i=1}^{l_2} k_i} 2^{2(\sum_{i=1}^{l_1}(k_i^{(1)}+k_i^{(2)}+1)+\sum_{i=1}^{l_2} k_i)}}{(2M)^{2(\sum_{i=1}^{l_1}(k_i^{(1)}+k_i^{(2)}+1)+\sum_{i=1}^{l_2} k_i)}} \times$$

$$\exp\left(-\lambda' + \lambda - (\lambda')^2 + \lambda^2\right) \Bigg) \cdots \Bigg) \left(1 + O\left(\frac{\omega(n)^2}{n} + \frac{\ln^3 n}{n}\right)\right)$$

$$= (M - D_2)^{l_1} \sum_{k_1^{(1)}=3,k_1^{(2)}=0}^{\omega(n)} \frac{3D_2^{k_1^{(1)}+k_1^{(2)}-1}}{2} D_3 \Bigg(\cdots \sum_{k_{l_1}^{(1)}=3,k_{l_1}^{(2)}=0}^{\omega(n)} \frac{3D_2^{k_{l_1}^{(1)}+k_{l_1}^{(2)}-1}}{2} D_3 \Bigg($$

$$\sum_{k_1=3}^{\omega(n)} \frac{D_2^{k_1-1}}{2} \sum_{i_1=4}^{\Delta} \binom{i_1}{2} D_{i_1} \Bigg(\cdots \sum_{k_{l_2}=3}^{\omega(n)} \frac{D_2^{k_{l_2}-1}}{2} \sum_{i_{l_2}=4}^{\Delta} \binom{i_{l_2}}{2} D_{i_{l_2}} \times$$

$$\frac{1}{M^{\sum_{i=1}^{l_1}(k_i^{(1)}+k_i^{(2)}+1)+\sum_{i=1}^{l_2} k_i}} \Bigg) \cdots \Bigg) \left(1 + O\left(\frac{\ln^3 n}{n}\right)\right)$$

$$= \left(\frac{M-D_2}{M} \frac{3}{2} \sum_{k=3}^{\omega(n)} (k-2) \left(\frac{D_2}{M}\right)^{k-1} \frac{D_3}{M}\right)^{l_1} \left(\frac{1}{2} \sum_{k=3}^{\omega(n)} \left(\frac{D_2}{M}\right)^{k-1} \sum_{i=4}^{\Delta} \binom{i}{2} \frac{D_i}{M}\right)^{l_2} \left(1 + O\left(\frac{\ln^3 n}{n}\right)\right),$$

where $D_j^{(i)}$ denotes the number of those vertices amongst the D_j vertices of degree j in the labelled degree sequence $(d_1, \ldots, d_n)$ that are left after the first $i-1$ subgraphs have been chosen and their edges removed and $\mathbf{d}'_n = (d'_1, \ldots, d'_n)$ be the labelled degree sequence after the removal of all the edges of the subgraphs. The indices j_l, i_l indicate the degree of that endvertex of the attaching path of the l-th cyclic 2-edge-connected component which does not belong to its cycle and the degree of the cutvertex of the l-th cyclic 2-vertex-connected component, for $l = 1, \ldots, l_1$ and $l = 1, \ldots, l_2$ respectively. The following observations will clarify the intermediate steps of the above calculation. Firstly, note that $D_j^{(i)} = D_j(1+O(\omega(n)/n))$, since $(\mathbf{d}_n) \in G$ and each subgraph of a given pair is of order $\omega(n)$ at most. Secondly, if vertex i totally participates in one of the cycles or in one of the attaching paths, then $\mathbf{d}'_n$ has one more vertex of degree 0 and one less vertex of degree d_i. Otherwise, in the case of 2-edge-connected cyclic components, if the vertex i is the endvertex of the attaching path of such a component which does not belong to its cycle, then $\mathbf{d}'_n$ has one more vertex of degree $d_i - 1$ and one less vertex of degree d_i. In the case of 2-vertex-connected cyclic components which are not 2-edge-connected components, if vertex i is the unique vertex that joins the cycle with the rest of the graph, then $\mathbf{d}'_n$ has one more vertex of degree $d_i - 2$ and one less vertex of degree d_i. Finally, using (1) and (2), we obtain the final error term of the above relation.

Note that

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{2}\sum_{k=3}^{\omega(n)} \left(\frac{D_2}{M}\right)^{k-1} \sum_{i=4}^{\Delta} \binom{i}{2}\frac{D_i}{M} &= \frac{\theta}{2}(1-e^{-\lambda_2(\theta)} - \lambda_2(\theta)e^{-\lambda_2(\theta)})\sum_{k=3}^{\infty}\left(\theta e^{-\lambda_2(\theta)}\right)^{k-1} \\ &= \frac{\theta}{2}(1-e^{-\lambda_2(\theta)} - \lambda_2(\theta)e^{-\lambda_2(\theta)})\frac{\left(\theta e^{-\lambda_2(\theta)}\right)^2}{1-\theta e^{-\lambda_2(\theta)}},\end{aligned}$$

and

$$\begin{aligned}&\lim_{n\to\infty} \frac{M-D_2}{M}\,\frac{3}{2}\sum_{k=3}^{\omega(n)}(k-2)\left(\frac{D_2}{M}\right)^{k-1}\frac{D_3}{M} = \\ &\quad\left(1-\theta e^{-\lambda_2(\theta)}\right)\frac{\theta e^{-\lambda_2(\theta)}\lambda_2(\theta)}{2}\sum_{k=3}^{\infty}(k-2)\left(\theta e^{-\lambda_2(\theta)}\right)^{k-1} \\ &= \left(1-\theta e^{-\lambda_2(\theta)}\right)\frac{\theta e^{-\lambda_2(\theta)}\lambda_2(\theta)}{2}\left(\frac{3\left(\theta e^{-\lambda_2(\theta)}\right)^2 - 2\left(\theta e^{-\lambda_2(\theta)}\right)^3}{(1-\theta e^{-\lambda_2(\theta)})^2} - \frac{2\left(\theta e^{-\lambda_2(\theta)}\right)^2}{1-\theta e^{-\lambda_2(\theta)}}\right),\end{aligned}$$

by the first and the third part of Lemma B.1 in Appendix B. From Corollary 6.8, Theorem 6.10 in [5] and Proposition 2.5, we deduce that $X^v(\mathcal{G}_{n,m}) \xrightarrow{d} X^v$, $X^e(\mathcal{G}_{n,m}) \xrightarrow{d} X^e$ and $(X^v(\mathcal{G}_{n,m}), X^e(\mathcal{G}_{n,m})) \xrightarrow{d} (X^v, X^e)$ as $n \to \infty$, where

$$X^v = \mathrm{Po}\left(\frac{\theta}{2}(1-e^{-\lambda_2(\theta)} - \lambda_2(\theta)e^{-\lambda_2(\theta)})\frac{\left(\theta e^{-\lambda_2(\theta)}\right)^2}{1-\theta e^{-\lambda_2(\theta)}}\right)$$

and

$$X^e = \mathrm{Po}\left(\left(1-\theta e^{-\lambda_2(\theta)}\right)\frac{\theta e^{-\lambda_2(\theta)}\lambda_2(\theta)}{2}\left(\frac{3\left(\theta e^{-\lambda_2(\theta)}\right)^2 - 2\left(\theta e^{-\lambda_2(\theta)}\right)^3}{(1-\theta e^{-\lambda_2(\theta)})^2} - \frac{2\left(\theta e^{-\lambda_2(\theta)}\right)^2}{1-\theta e^{-\lambda_2(\theta)}}\right)\right)$$

are independent. This yields that

$$\begin{aligned}X^v(\mathcal{G}_{n,m}) + X^e(\mathcal{G}_{n,m}) \xrightarrow{d} \mathrm{Po}\Bigg(&\frac{\theta}{2}(1-e^{-\lambda_2(\theta)} - \lambda_2(\theta)e^{-\lambda_2(\theta)})\frac{\left(\theta e^{-\lambda_2(\theta)}\right)^2}{1-\theta e^{-\lambda_2(\theta)}} \\ &+ \left(1-\theta e^{-\lambda_2(\theta)}\right)\frac{\theta e^{-\lambda_2(\theta)}\lambda_2(\theta)}{2}\left(\frac{3\left(\theta e^{-\lambda_2(\theta)}\right)^2 - 2\left(\theta e^{-\lambda_2(\theta)}\right)^3}{(1-\theta e^{-\lambda_2(\theta)})^2} - \frac{2\left(\theta e^{-\lambda_2(\theta)}\right)^2}{1-\theta e^{-\lambda_2(\theta)}}\right)\Bigg).\end{aligned}$$

These along with the remark at the beginning of this section conclude the proof of Theorem 1.2 (a).

Now, focusing on the asymptotic distributions of such components according to the size of their cycles and their attaching paths, we present and prove the analogue of Lemma 3.2:

Lemma 4.2. *Assume that* $(\mathbf{d}_n) \in G$. *Let* $s, t \in \mathbb{N}$ *and* $k_1^{(1)}, \ldots, k_s^{(1)}, k_{s+1}, \ldots, k_{s+t} \geq 3$, $k_1^{(2)} \ldots, k_s^{(2)} \geq 0$, $l_1, \ldots, l_{s+t} \geq 1$ *be also natural numbers. For any* $k, k_1 \geq 3$ *and* $k_2 \geq 0$, *we set* $X^e_{k_1k_2n} = X^e_{k_1k_2}(G(\mathbf{d}_n))$ *and* $X^v_{kn} = X^v_k(G(\mathbf{d}_n))$. *Then*

$$\lim_{n\to\infty} \mathbb{E}[(X^e_{k_1^{(1)}k_1^{(2)}n})_{l_1}\cdots(X^e_{k_s^{(1)}k_s^{(2)}n})_{l_s}(X^v_{k_{s+1}n})_{l_{s+1}}\cdots(X^v_{k_{s+t}n})_{l_{s+t}}] =$$
$$\prod_{j=1}^{s}\left(\frac{\theta e^{-\lambda_2(\theta)}\lambda_2(\theta)(1-\theta e^{-\lambda_2(\theta)})}{2}\left(\theta e^{-\lambda_2(\theta)}\right)^{k_j^{(1)}+k_j^{(2)}-1}\right)^{l_j}\cdot$$
$$\prod_{j=1}^{t}\left(\frac{\theta}{2}\left(\theta e^{-\lambda_2(\theta)}\right)^{k_{s+j}-1}\left(1-e^{-\lambda_2(\theta)}-e^{-\lambda_2(\theta)}\lambda_2(\theta)\right)\right)^{l_{s+j}}.$$

(Here, the empty product is assumed to be equal to 1.)

Proof. We shall prove the above lemma for $s = t = 1$ - the general case is similar but much more tedious (see [2] for the complete proof). In this case, note that the left hand side of the above equation is the expectation of the total number of pairs consisting of ordered l_1-tuples of 2-edge-connected components, which are $k_1^{(1)}$ cycles and have a unique attaching path with $k_1^{(2)}$ internal vertices, and l_2-tuples of 2-vertex-connected components which are cycles of length k_2 and are not 2-edge-connected components, each containing exactly one cutvertex of $G(\mathbf{d}_n)$. Moreover, every such family induces a subgraph of order at most $l_1(k_1^{(1)}+k_1^{(2)}+1)+l_2k_2$. So,by Lemma A.2 in Appendix A, the expected number of those families with at least two overlapping such subgraphs (whose overlap of course is a vertex of degree greater than 2) tends to 0, since they induce subgraphs of bounded order which have more edges than vertices and $(\mathbf{d}_n) \in G$. Therefore, the above expectation is asymptotically equal to the expected number of those pairs consisting of disjoint subgraphs.

Let X_n be the total number of the latter. Lemma 2.6 implies:

$$\begin{aligned}
\mathbb{E}[X_n] &= \binom{D_2^{(1)}}{k_1^{(1)}-1}\frac{(k_1^{(1)}-1)!}{2}\binom{D_2^{(1)}-(k_1^{(1)}-1)}{k_1^{(2)}}k_1^{(2)}!D_3^{(1)}\sum_{j_1=3}^{\Delta}(D_{j_1}^{(1)}-\delta_{j_1,3})\Bigg(\cdots\binom{D_2^{(l_1)}}{k_1^{(1)}-1}\\
&\frac{(k_1^{(1)}-1)!}{2}\binom{D_2^{(l_1)}-(k_1^{(1)}-1)}{k_1^{(2)}}k_1^{(2)}!D_3^{(l_1)}\sum_{j_{l_1}=3}^{\Delta}(D_{j_{l_1}}^{(l_1)}-\delta_{j_{l_1},3})\Bigg(\\
&\binom{D_2^{(l_1+1)}}{k_2-1}\frac{(k_2-1)!}{2}\sum_{i_1=4}^{\Delta}D_{i_1}^{(l_1)}\Bigg(\cdots\binom{D_2^{(l_1+l_2)}}{k_2-1}\frac{(k_2-1)!}{2}\sum_{i_{l_2}=4}^{\Delta}D_{i_{l_2}}^{(l_1+l_2)}\Bigg(\\
&\frac{(2M-2(l_1(k_1^{(1)}+k_1^{(2)}+1)+l_2k_2))!}{(M-(l_1(k_1^{(1)}+k_1^{(2)}+1)+l_2k_2))!2^{M-(l_1(k_1^{(1)}+k_1^{(2)}+1)+l_2k_2)}\prod_{i=1}^n d_i'!}\times\\
&\frac{M!2^M\prod_{i=1}^n d_i!}{(2M)!}\exp\left(-\lambda'+\lambda-(\lambda')^2+\lambda^2\right)\Bigg)\cdots\Bigg)\left(1+O\left(\frac{\ln^3 n}{n}\right)\right)\\
&= \left(\frac{D_2^{k_1^{(1)}+k_1^{(2)}-1}}{(k_1^{(1)}-1)!k_1^{(2)}!}\frac{(k_1^{(1)}-1)!k_1^{(2)}!}{2}D_3\right)^{l_1}\sum_{j_1=3}^{n}D_{j_1}\Bigg(\cdots\sum_{j_{l_1}=3}^{n}D_{j_{l_1}}\Bigg(\\
&\frac{D_2^{k_2-1}}{(k_2-1)!}\frac{(k_2-1)!}{2}\sum_{i_1=4}^{\Delta}D_{i_1}\Bigg(\cdots\frac{D_2^{k_2-1}}{(k_2-1)!}\frac{(k_2-1)!}{2}\sum_{i_{l_2}=4}^{\Delta}D_{i_{l_2}}\Bigg(\\
&\frac{(M)_{l_1(k_1^{(1)}+k_1^{(2)}+1)+l_2k_2}2^{l_1(k_1^{(1)}+k_1^{(2)}+1)+l_2k_2}}{(2M)_{2(l_1(k_1^{(1)}+k_1^{(2)}+1)+l_2k_2)}}
\end{aligned}$$

$$\frac{\prod_{i=1}^n d_i!}{\prod_{i=1}^n d_i'!} \exp\left(-\lambda' + \lambda - (\lambda')^2 + \lambda^2\right)\Bigg) \cdots \Bigg) \left(1 + O\left(\frac{\omega^2(n)}{n} + \frac{\ln^3 n}{n}\right)\right)$$

$$= \left(2(M - D_2)\frac{3! D_2^{k_1^{(1)}+k_1^{(2)}-1}}{2} D_3 \frac{2^{k_1^{(1)}+k_1^{(2)}}}{2}\right)^{l_1}$$

$$\frac{D_2^{k_2-1}}{2}\sum_{i_1=4}^{\Delta} 2^{k_2}\binom{i_1}{2} D_{i_1}\Bigg(\cdots \frac{D_2^{k_2-1}}{2}\sum_{i_{l_2}=4}^{\Delta} 2^{k_2}\binom{i_{l_2}}{2} D_{i_{l_2}}\Bigg($$

$$\frac{M^{l_1(k_1^{(1)}+k_1^{(2)}+1)+l_2k_2}2^{l_1(k_1^{(1)}+k_1^{(2)}+1)+l_2k_2}}{(2M)^{2(l_1(k_1^{(1)}+k_1^{(2)}+1)+l_2k_2)}}\left(1 + O\left(\frac{\ln^3 n}{n}\right)\right)\Bigg) \cdots \Bigg)$$

$$= \left(\frac{M - D_2}{M}\,\frac{3}{2}\left(\frac{D_2}{M}\right)^{k_1^{(1)}+k_1^{(2)}-1}\frac{D_3}{M}\right)^{l_1} \cdots \left(\frac{1}{2}\left(\frac{D_2}{M}\right)^{k_2-1}\sum_{i=4}^{\Delta}\binom{i}{2}\frac{D_i}{M}\right)^{l_2}\left(1 + O\left(\frac{\ln^3 n}{n}\right)\right),$$

where, as in the proof of Lemma 3.2, $D_j^{(i)}$ denotes the number of those vertices amongst the D_j vertices of degree j that are left in the labelled degree sequence $(d_1, \ldots, d_n)$ after the first $i-1$ subgraphs have been chosen and their edges removed and $\mathbf{d}_n' = (d_1', \ldots, d_n')$ be the labelled degree sequence after the removal of the edges of the subgraphs. The indices j_l, i_l indicate the degree of that endvertex of the attaching path of the l-th cyclic 2-edge-connected component which does not belong to its cycle and the degree of the cutvertex of the l-th cyclic 2-vertex-connected component, for $l = 1, \ldots, l_1$ and $l = 1, \ldots, l_2$ respectively. The following observations will clarify the intermediate steps of the above calculation. Firstly, note that $D_j^{(i)} = D_j(1 + O(1/n))$, as $(\mathbf{d}_n) \in G$ and all the subgraphs contained in a given pair have bounded size. Secondly, if vertex i totally participates in one of the subgraphs, then $\mathbf{d}_n'$ has an extra vertex of degree 0 and one less vertex of degree d_i. Otherwise, in the case of 2-edge-connected cyclic components, if the vertex i is the endvertex of the attaching path of a 2-edge-connected cyclic component of a given pair that does not belong to its cycle, then $\mathbf{d}_n'$ has an extra vertex of degree $d_i - 1$, whereas it has one less vertex of degree d_i. In the case of 2-vertex-connected cyclic components that are not 2-edge-connected components, for each vertex i which is the unique vertex that joins such a component with the rest of the graph, $\mathbf{d}_n'$ has an extra vertex of degree $d_i - 2$, whereas it has one less vertex of degree d_i. Finally, using (1) and (2), we obtain the error term of the above relation. The first part of Lemma B.1 in Appendix B implies the lemma. □

If we set $s = 1$ and $t = 0$ in the above lemma, then for any integers $k_1 \geq 3$, $k_2 \geq 0$ and $l \geq 1$, we have

$$\lim_{n\to\infty} \mathbb{E}\left[\left(X_{k_1k_2n}^e(G(\mathbf{d}_n))\right)_l\right] = \left(\frac{\theta e^{-\lambda_2(\theta)}\lambda_2(\theta)(1 - \theta e^{-\lambda_2(\theta)})}{2}\left(\theta e^{-\lambda_2(\theta)}\right)^{k_1+k_2-1}\right)^l,$$

and this implies that $X_{k_1k_2n}^e(\mathrm{cr}(\mathcal{G}_{n,m})) \xrightarrow{d} \mathrm{Po}\left(\frac{\theta e^{-\lambda_2(\theta)}\lambda_2(\theta)(1-\theta e^{-\lambda_2(\theta)})}{2}\left(\theta e^{-\lambda_2(\theta)}\right)^{k_1+k_2-1}\right)$, as $n \to \infty$, using Corollary 6.8 in [5] and Proposition 2.5. If we set $s = 0$ and $t = 1$ in the above lemma, then for any integers $k \geq 3$ and $l \geq 1$, we have

$$\lim_{n\to\infty} \mathbb{E}\left[(X_{kn}^v(G(\mathbf{d}_n)))_l\right] = \left(\frac{\theta}{2}\left(\theta e^{-\lambda_2(\theta)}\right)^{k-1}\left(1 - e^{-\lambda_2(\theta)} - \lambda_2(\theta)e^{-\lambda_2(\theta)}\right)\right)^l,$$

and, as above, we deduce that $X_{kn}^v(\mathrm{cr}(\mathcal{G}_{n,m})) \xrightarrow{d} \mathrm{Po}\left(\frac{\theta}{2}\left(\theta e^{-\lambda_2(\theta)}\right)^{k-1}\left(1 - e^{-\lambda_2(\theta)} - \lambda_2(\theta)e^{-\lambda_2(\theta)}\right)\right)$, as $n \to \infty$. Finally, Lemma 4.2 along with Theorem 6.10 in [5] and Proposition 2.5 imply that the joint distribution of any finite collection of random variables

$$(X_{k_1^{(1)}k_1^{(2)}n}^e(\mathrm{cr}(\mathcal{G}_{n,m})), \ldots, X_{k_s^{(1)}k_s^{(2)}n}^e(\mathrm{cr}(\mathcal{G}_{n,m})), X_{k_{s+1}n}^v(\mathrm{cr}(\mathcal{G}_{n,m})), \ldots, X_{k_{s+t}n}^v(\mathrm{cr}(\mathcal{G}_{n,m})))$$

converges to the joint distribution of $(X^e_{k_1^{(1)}k_1^{(2)}}, \ldots, X^e_{k_s^{(1)}k_s^{(2)}}, X^v_{k_{s+1}}, \ldots, X^v_{k_{s+t}})$, where, for $i = 1, \ldots, s$, we have $X^e_{k_i^{(1)}k_i^{(2)}} = \mathrm{Po}\left(\frac{\theta e^{-\lambda_2(\theta)}\lambda_2(\theta)(1-\theta e^{-\lambda_2(\theta)})}{2}\left(\theta e^{-\lambda_2(\theta)}\right)^{k_i^{(1)}+k_i^{(2)}-1}\right)$, and for $i = s+1, \ldots, s+t$

$$X^v_{k_i} = \mathrm{Po}\left(\frac{\theta}{2}\left(\theta e^{-\lambda_2(\theta)}\right)^{k_i-1}\left(1 - e^{-\lambda_2(\theta)} - \lambda_2(\theta)e^{-\lambda_2(\theta)}\right)\right),$$

are independent. The remark at the beginning of this section concludes the proof of Theorem 1.2.

5. On the structure of the kernel of a sparse random graph

In this final section, we investigate the implications of Theorems 1.1, 1.2 and 2.1 to the asymptotic structure of the kernel of a $\mathcal{G}_{n,m}$ random graph. The fact that the number of vertices in $\mathrm{cr}(\mathcal{G}_{n,m})$ that do not belong to its giant component is $O_p(1)$ and all of them are of degree 2 a.a.s. implies that the order of $\ker(\mathcal{G}_{n,m})$ is equal to $n(p_2(\lambda_2(\theta))-\rho_2(\lambda_2(\theta)))+o_p(n) = np_3(\lambda_2(\theta))+o_p(n)$, by Theorems 1.1(a) and 2.1. Furthermore, since the difference between the number of edges and the number of vertices in the kernel is equal to this difference in the core, we deduce that the number of edges in $\ker(\mathcal{G}_{n,m})$ is equal to $n(\frac{\lambda_2^2(\theta)}{2\theta} - \rho_2(\lambda_2(\theta))) + o_p(n)$. The degree sequence of $\ker(\mathcal{G}_{n,m})$ is the degree sequence of $\mathrm{cr}(\mathcal{G}_{n,m})$, restricted to degrees greater than 2. Finally, the number of loops is equal to the number of cyclic 2-vertex-connected components of $L_1(\mathrm{cr}(\mathcal{G}_{n,m}))$, which follows asymptotically the Poisson distribution of mean equal to the sum of the two expressions in the first part of Theorem 1.2.

6. Conclusive remarks

This paper continues the study of the structure of the core of a $\mathcal{G}_{n,m}$ random graph, that started in [12], giving the asymptotic distributions of the 2-vertex and 2-edge-connected components of it as well as the distributions of small cycles which are either isolated or not there. Thus, it is natural to turn our attention to the essential core, since this is the dominant 2-vertex-connected component of the giant component. It seems that there is a series of natural questions concerning the essential core of a $\mathcal{G}_{n,m}$ random graph above the critical point that remain to be answered. For example, is it a.a.s. Hamiltonian? What is the exact connectivity of its kernel? Does it have a perfect matching a.a.s.? Also, there are questions concerning the k-core, for any $k \geq 3$, after its appearance. It is known that it is a.a.s. k-vertex-connected [8], but for example is it a.a.s. Hamiltonian? How does it look like in general? All these questions are open for further research.

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Appendix A. Some results from the theory of random graphs

We show the following lemma which is used mainly in Section 2.

Lemma A.1. *Let $\Delta(n) = \lceil \ln n \rceil$. Also, let $Y(\mathcal{G}_{n,m}) = \sum_{i=2}^{n-1} i^4 v_i(\mathrm{DP}_2(\mathcal{G}_{n,m}))$ and, for $k \in \mathbb{N}$, $X_{n,k} = X_{n,k}(\mathcal{G}_{n,m}) = \sum_{i=k}^{n-1} \binom{i}{2} \frac{v_i(\mathrm{DP}_2(\mathcal{G}_{n,m}))}{n}$. Then, there exists a $c > 0$ such that for every $\varepsilon > 0$, there exists a $k_0 \in \mathbb{N}$ so that*

$$\mathbb{P}[Y(\mathcal{G}_{n,m}) > cn \text{ or } X_{n,k_0} \geq \varepsilon \text{ or } \textit{max.deg.}(\mathrm{DP}_2(\mathcal{G}_{n,m})) > \Delta(n)] = o\left(\frac{1}{n^2}\right).$$

Proof. Let us fix $\varepsilon > 0$. Note that

$$Y(\mathcal{G}_{n,m}) \leq \sum_{i=1}^{n-1} i^4 v_i(\mathcal{G}_{n,m}) \text{ and } X_{n,k} \leq \sum_{i=k}^{n-1} \binom{i}{2} \frac{v_i(\mathcal{G}_{n,m})}{n} \leq \sum_{i=k}^{n-1} i^2 \frac{v_i(\mathcal{G}_{n,m})}{n}, \tag{3}$$

always for $k \geq 2$. Let $Y'(\mathcal{G}_{n,m})$, $X'_{n,k}(\mathcal{G}_{n,m})$ be the quantities on the right hand sides of the above inequalities. We shall focus on $Y'(\mathcal{G}_{n,m})$ and $X'_{n,k}(\mathcal{G}_{n,m})$ rather than $Y(\mathcal{G}_{n,m})$ and $X_{n,k}$. Also, note that max.deg.$(\mathrm{DP}_2(\mathcal{G}_{n,m})) \leq$ max.deg.$(\mathcal{G}_{n,m})$. For some $k \geq 2$ and some $c > 0$ which will be specified later we set $E_1 = \{\text{max.deg.}(\mathcal{G}_{n,m}) > \Delta(n)\}$, $E_2 = \{X'_{n,k}(\mathcal{G}_{n,m}) \geq \varepsilon\}$ and $E_3 = \{Y'(\mathcal{G}_{n,m}) > cn\}$ and $P_1 = \mathbb{P}[E_1]$, $P_2 = \mathbb{P}[E_2]$, $P_3 = \mathbb{P}[E_3]$.

A $\mathcal{G}_{n,m}$ random graph can be viewed as the multigraph process $MG(n,m)$ conditioned on the event

$$A_n = \{MG(n,m) \text{ has no loops and no multiple edges}\}.$$

(At each stage of the multigraph process we form an edge by picking uniformly at random, independently and with replacement two vertices, i and j, ignoring the previous choices; if $i = j$ then the multigraph gets a loop at i.) Using $\mathbb{P}[U \mid V] \leq P[U]/P[V]$ (this is the first conditioning), and denoting by P'_i the analogous probabilities in $MG(n,m)$, for $i = 1,2,3$, we have

$$P_i \leq \frac{P'_i}{A_n} = O(P'_i), \tag{4}$$

since

$$\lim_{n\to\infty} \mathbb{P}[A_n] = \lim_{n\to\infty} \frac{\binom{\binom{n}{2}}{m} m! 2^m}{n^{2m}} = \exp(-\theta/2 - \theta^2/4) > 0.$$

The second conditioning comes from the fact that the degree sequence of the random multigraph produced by the multigraph process can be viewed as the random sequence of occupancy numbers when we throw $2m$ distinguishable balls into n bins. These are jointly distributed as n independent copies of $Z(\theta)$ (a Poisson random variable with mean θ) conditioned on $S_n = \sum_{j=1}^n Z_j(\theta) = 2m$. Since S_n is $Z(2m)$, we have

$$\mathbb{P}[S_n = 2m] = \mathbb{P}[Z(2m) = 2m] = e^{-2m} \frac{(2m)^{2m}}{(2m)!} \geq \text{const.} \times m^{-1/2}.$$

Also, let k be such that $\mathbb{E}\left[\sum_{i=k}^{\infty} i^2 v_i\right] < \varepsilon n/2$, where $v_i = |\{j \in [n] \ : \ Z_j(\theta) = i\}|$ and note that $\mathbb{E}\left[\sum_{i=k}^{\infty} i^2 v_i\right] = \Theta(n)$. If $X''_{n,k} = \sum_{i=k}^{\infty} i^2 v_i$, then

$$\mathbb{P}\left[X''_{n,k} \geq \varepsilon n\right] \leq \mathbb{P}\left[X''_{n,k} > 2\mathbb{E}\left[X''_{n,k}\right]\right].$$

Moreover, we set $c = 2\mathbb{E}[Z^4(\theta)]$. Therefore,

$$P'_i = O(n^{1/2} P''_i), \tag{5}$$

where

$$\begin{aligned} P_1'' &= \mathbb{P}\left[\max_{1\le j\le n} Z_j(\theta) > \Delta(n)\right], \\ P_2'' &= \mathbb{P}\left[X_{n,k}'' > 2\mathbb{E}\left[X_{n,k}''\right]\right], \\ P_3'' &= \mathbb{P}\left[\sum_{j=1}^n Z_j^4(\theta) > 2\mathbb{E}[Z^4(\theta)].\right] \end{aligned}$$

We have

$$P_1'' \le n\mathbb{P}[Z(\theta) > \Delta(n)] = n\sum_{r>\Delta(n)} e^{-\theta}\frac{\theta^r}{r!} = o\left(\frac{1}{n^4}\right). \tag{6}$$

Now, for each $j = 1, \ldots, n$ we define

$$Y_j = \begin{cases} Z_j(\theta) & k \le Z_j(\theta) \le \Delta(n) \\ 0 & \text{otherwise} \end{cases},$$

and note that $\{Y_j\}_{j\in[n]}$ is independent. Then, $X_{n,k}'' = \sum_{j=1}^n Y_j^2$ if we condition on $Z_j(\theta) \le \Delta(n)$ for each $j = 1, \ldots, n$. Also, note that $X_{n,k}'' \ge \sum_{j=1}^n Y_j^2$ always. Using the bounded differences lemma (see Corollary 2.27 in [5] or [10]) we obtain:

$$\begin{aligned} &\mathbb{P}\left[X_{n,k}'' > 2\mathbb{E}[X_{n,k}''] \,\cap\, \left(\cap_{j=1}^n Z_j(\theta) \le \Delta(n)\right)\right] = \\ &\quad \mathbb{P}\left[\sum_{j=1}^n Y_j^2 > 2\mathbb{E}[X_{n,k}''] \,\cap\, \left(\cap_{j=1}^n Z_j(\theta) \le \Delta(n)\right)\right] \\ &\le \mathbb{P}\left[\sum_{j=1}^n Y_j^2 > 2\mathbb{E}\left[\sum_{j=1}^n Y_j^2\right] \,\cap\, \left(\cap_{j=1}^n Z_j(\theta) \le \Delta(n)\right)\right] \\ &\le \mathbb{P}\left[\sum_{j=1}^n Y_j^2 > 2\mathbb{E}\left[\sum_{j=1}^n Y_j^2\right]\right] \\ &\le \exp\left(-\frac{\mathbb{E}\left[\sum_{j=1}^n Y_j^2\right]^2}{2n\Delta^4(n)}\right) = \exp\left(-\Theta\left(\frac{n}{\ln^4 n}\right)\right). \end{aligned}$$

Thus, by (6) we obtain

$$\begin{aligned} P_2'' &\le \mathbb{P}\left[X_{n,k}'' > 2\mathbb{E}[X_{n,k}''] \,\cap\, \left(\cap_{j=1}^n Z_j(\theta) \le \Delta(n)\right)\right] + \mathbb{P}\left[\max_{1\le j\le n} Z_j(\theta) > \Delta(n)\right] \\ &\le \exp\left(-\Theta\left(\frac{n}{\ln^4 n}\right)\right) + o\left(\frac{1}{n^4}\right) = o\left(\frac{1}{n^4}\right). \end{aligned} \tag{7}$$

Finally, using Chernoff's bound we obtain

$$P_3'' \le \exp(-dn), \tag{8}$$

for some $d > 0$. The estimates (4)-(8) along with (3) imply the statement of the lemma. □

Now, for $n \ge 16$ we let $\omega(n) = \lfloor \ln\ln n \rfloor$ and for $1 \le n \le 15$ we let $\omega(n) = 2$. For a graph G on V_n, let $S(G)$ be the number of subgraphs of order at most $\omega(n)$ with their size exceeding their order. Then the following holds:

Lemma A.2. *For any $(\mathbf{d}_n) \in G$, we have*

$$\mathbb{E}[S(G(\mathbf{d}_n))] = o(1).$$

Proof. For n sufficiently large, if $l \in \mathbb{Z}^+$, then for every $2 \leq k \leq \omega(n)$ the expected number of subgraphs of $G(\mathbf{d}_n)$ of order k and size $k+l$ is, by Lemma 2.6, at most

$$\begin{aligned}
& \binom{n}{k} \binom{\binom{k}{2}}{k+l} \frac{(2M-2k-2l)!}{(M-k-l)!2^{M-k-l}} \frac{M!2^M}{(2M)!} \max_{\{\text{graphs on } [k]\}} \max_{[V_n]^k} \frac{\prod_{i=1}^n d_i!}{\prod_{i=1}^n d_i'!} \exp(-\lambda' - \lambda'^2 + \lambda + \lambda^2) \\
\leq & \frac{n^k}{k!} \left(\frac{k^2 e}{k+l}\right)^{k+l} \frac{2^{k+l}(M)_{k+l}}{(2M)_{2k+2l}} \max_{\{\text{graphs on } [k]\}} \max_{[V_n]^k} \frac{\prod_{i=1}^n d_i!}{\prod_{i=1}^n d_i'!} \exp(\lambda + \lambda^2) \\
= & \frac{n^k}{k!} \left(\frac{k^2 e}{k+l}\right)^{k+l} \frac{2^{k+l} M^{k+l}}{(2M)^{2k+2l}} \max_{\{\text{graphs on } [k]\}} \max_{[V_n]^k} \frac{\prod_{i=1}^n d_i!}{\prod_{i=1}^n d_i'!} \exp(\lambda + \lambda^2)\ (1+o(1)) \\
= & \frac{n^k}{k!} \left(\frac{k^2 e}{k+l}\right)^{k+l} \frac{2^{-k-l}}{M^{k+l}} \max_{\{\text{graphs on } [k]\}} \max_{[V_n]^k} \frac{\prod_{i=1}^n d_i!}{\prod_{i=1}^n d_i'!} \exp(\lambda + \lambda^2)\ (1+o(1)) \\
\leq & \Theta(1) \left(\frac{e^2}{2}\right)^k \left(\frac{ke}{2}\right)^{k^2} \frac{1}{M} \left(\frac{(\lceil \ln n \rceil)!}{(\lceil \ln n \rceil - \lceil \ln \ln n \rceil)!}\right)^k \\
\leq & \Theta(1) \left(\frac{e^2 (\ln n)^{\ln \ln n}}{2}\right)^k \left(\frac{e \ln \ln n}{2}\right)^{k \ln \ln n} \frac{1}{M},
\end{aligned}$$

since $(\mathbf{d}_n) \in G$, where λ is defined as in Lemma 2.6 and λ' is the corresponding quantity for the sequence $(d_i')_{i=1,\ldots,n}$, which is the labelled degree sequence that is left after the removal of the edges of the subgraph on the chosen k vertices. Therefore, since $M = \Theta(n)$ we obtain:

$$\begin{aligned}
\mathbb{E}[S(G(\mathbf{d}_n))] \leq & \frac{\ln^2 \ln n}{M} \sum_{k=2}^{\lfloor \ln \ln n \rfloor} \left(\frac{e^2 (\ln n)^{\ln \ln n}}{2}\right)^k \left(\frac{e \ln \ln n}{2}\right)^{k \ln \ln n} \\
= & O\left(\left(\frac{e^2 (\ln n)^{\ln \ln n}}{2}\right)^{\ln \ln n + 1} \left(\frac{e \ln \ln n}{2}\right)^{\ln^2 \ln n + \ln \ln n} \frac{\ln^2 \ln n}{M}\right) = o(1).
\end{aligned}$$

□

Appendix B. Limiting properties of the degree sequences in G

Finally, we prove the following technical statement which was used in the course of our proof:

Lemma B.1. *For any $(\mathbf{d}_n) \in G$ the following hold:*

(1) *For any integer $k \geq 2$, we have*

$$\lim_{n \to \infty} \sum_{i=k}^{\Delta} \binom{i}{2} \frac{D_i}{M} = \theta \left(1 - \exp(-\lambda_2(\theta)) \sum_{i=0}^{k-3} \lambda_2^i(\theta)\right).$$

If $k = 2$, then we assume that the sum on the right hand side is equal to 0.

(2) *We have*

$$\lim_{n \to \infty} \sum_{i=2}^{\Delta} \frac{D_i}{n} = \sum_{i=2}^{\infty} \rho_i(\lambda_2(\theta)) = p_2(\lambda_2(\theta)).$$

(3) *For every function $\omega(n)$ that tends to infinity as $n \to \infty$, we have*

$$\lim_{n \to \infty} \sum_{k=2}^{\omega(n)} \left(\frac{D_2}{M}\right)^k = \sum_{k=2}^{\infty} \left(\theta e^{-\lambda_2(\theta)}\right)^k = \frac{\left(\theta e^{-\lambda_2(\theta)}\right)^2}{1 - \theta e^{-\lambda_2(\theta)}},$$

and

$$\lim_{n \to \infty} \sum_{k=3}^{\omega(n)} (k-2) \left(\frac{D_2}{M}\right)^{k-1} = \sum_{k=3}^{\infty} (k-2) \left(\theta e^{-\lambda_2(\theta)}\right)^{k-1}.$$

Proof. We prove the first part. For every integer $k_0 \geq k$, we have

$$\sum_{i=k}^{k_0} \binom{i}{2} \frac{D_i}{M} \leq \sum_{i=k}^{\Delta} \binom{i}{2} \frac{D_i}{M}.$$

Thus,

$$\lim_{n\to\infty} \sum_{i=k}^{k_0} \binom{i}{2} \frac{D_i}{M} \leq \liminf_{n\to\infty} \sum_{i=k}^{\Delta} \binom{i}{2} \frac{D_i}{M},$$

where

$$\lim_{n\to\infty} \sum_{i=k}^{k_0} \binom{i}{2} \frac{D_i}{M} = \sum_{i=k}^{k_0} \binom{i}{2} \frac{2\theta e^{-\lambda_2(\theta)} \lambda_2^i(\theta)}{i! \lambda_2^2(\theta)} = \theta e^{-\lambda_2(\theta)} \sum_{i=k-2}^{k_0-2} \frac{\lambda_2(\theta)^i}{i!}.$$

Since this is true for any integer $k_0 \geq 2$, we deduce that

$$\theta \left(1 - \exp(-\lambda_2(\theta)) \sum_{i=0}^{k-3} \lambda_2^i(\theta)\right) = \theta e^{-\lambda_2(\theta)} \sum_{i=k-2}^{\infty} \frac{\lambda_2(\theta)^i}{i!} \leq \liminf_{n\to\infty} \sum_{i=k}^{\Delta} \binom{i}{2} \frac{D_i}{M}.$$

On the other hand, since $(\mathbf{d}_n) \in G$, for every $\varepsilon > 0$ there exist $k_0, N \in \mathbb{N}$ such that for every $n > N$, we have

$$\sum_{i=k}^{\Delta} \binom{i}{2} \frac{D_i}{M} \leq \sum_{i=k}^{k_0} \binom{i}{2} \frac{D_i}{M} + \varepsilon.$$

Therefore,

$$\begin{aligned}
\limsup_{n\to\infty} \sum_{i=k}^{\Delta} \binom{i}{2} \frac{D_i}{M} &\leq \lim_{n\to\infty} \sum_{i=k}^{k_0} \binom{i}{2} \frac{D_i}{M} + \varepsilon = \sum_{i=k}^{k_0} \binom{i}{2} \frac{2\theta e^{-\lambda_2(\theta)} \lambda_2^i(\theta)}{i! \lambda_2^2(\theta)} + \varepsilon \\
&\leq \sum_{i=k}^{\infty} \binom{i}{2} \frac{2\theta e^{-\lambda_2(\theta)} \lambda_2^i(\theta)}{i! \lambda_2^2(\theta)} + \varepsilon = \theta \left(1 - \exp(-\lambda_2(\theta)) \sum_{i=0}^{k-3} \lambda_2^i(\theta)\right) + \varepsilon.
\end{aligned}$$

Since ε is arbitrary, we deduce that

$$\begin{aligned}
\theta \left(1 - \exp(-\lambda_2(\theta)) \sum_{i=0}^{k-3} \lambda_2^i(\theta)\right) &\leq \liminf_{n\to\infty} \sum_{i=k}^{\Delta} \binom{i}{2} \frac{D_i}{M} \\
&\leq \limsup_{n\to\infty} \sum_{i=k}^{\Delta} \binom{i}{2} \frac{D_i}{M} \leq \theta \left(1 - \exp(-\lambda_2(\theta)) \sum_{i=0}^{k-3} \lambda_2^i(\theta)\right),
\end{aligned}$$

which concludes the proof of the first part of the lemma.

The proof of the second part of the lemma is nearly identical to the above and we omit it (see [2] for the details). To get the upper bound one has to observe that $\sum_{i=k}^{\Delta} D_i/n \leq \sum_{i=k}^{\Delta} \binom{i}{2} D_i/n$ and use the fact that the latter quantity can become arbitrarily small if k is chosen to be sufficiently large and for n sufficiently large, since $(\mathbf{d}_n) \in G$.

We now proceed with the third part of the lemma. We shall prove the first statement, as the proof of the second one is identical. For every integer $k_0 \geq 2$, we have

$$\sum_{k=2}^{k_0} \left(\frac{D_2}{M}\right)^k \leq \sum_{k=2}^{\omega(n)} \left(\frac{D_2}{M}\right)^k.$$

Thus,

$$\liminf_{n\to\infty} \sum_{k=2}^{\Delta} \left(\frac{D_2}{M}\right)^k \geq \lim_{n\to\infty} \sum_{k=2}^{k_0} \left(\frac{D_2}{M}\right)^k = \sum_{k=2}^{k_0} \left(\frac{2\theta e^{-\lambda_2(\theta)} \lambda_2^2(\theta)}{2\lambda_2^2(\theta)}\right)^k = \sum_{k=2}^{k_0} \left(\theta e^{-\lambda_2(\theta)}\right)^k.$$

Since this is true for any integer $k_0 \geq 2$, we deduce that

$$\sum_{k=2}^{\infty} \left(\theta e^{-\lambda_2(\theta)}\right)^k \leq \liminf_{n\to\infty} \sum_{k=2}^{\omega(n)} \left(\frac{D_2}{M}\right)^k .$$

On the other hand, there exists a real $0 < \lambda < 1$ and $N \in \mathbb{N}$, such that for every $n > N$, we have $D_2/M < \lambda$. Therefore, for any $\varepsilon > 0$, there exists an integer $k_0 \geq 2$ such that

$$\sum_{k=2}^{\omega(n)} \left(\frac{D_2}{M}\right)^k \leq \sum_{k=2}^{k_0} \left(\frac{D_2}{M}\right)^k + \sum_{k=k_0+1}^{\omega(n)} \lambda^k \leq \sum_{k=2}^{k_0} \left(\frac{D_2}{M}\right)^k + \sum_{k=k_0+1}^{\infty} \lambda^k \leq \sum_{k=2}^{k_0} \left(\frac{D_2}{M}\right)^k + \varepsilon,$$

for every $n > N$. Therefore,

$$\limsup_{n\to\infty} \sum_{k=2}^{\omega(n)} \left(\frac{D_2}{M}\right)^k \leq \lim_{n\to\infty} \sum_{k=2}^{k_0} \left(\frac{D_2}{M}\right)^k + \varepsilon \leq \sum_{k=2}^{k_0} \left(\theta e^{-\lambda_2(\theta)}\right)^k + \varepsilon \leq \sum_{k=2}^{\infty} \left(\theta e^{-\lambda_2(\theta)}\right)^k + \varepsilon.$$

Since ε is arbitrary, we deduce that

$$\limsup_{n\to\infty} \sum_{k=2}^{\omega(n)} \left(\frac{D_2}{M}\right)^k \leq \sum_{k=2}^{\infty} \left(\theta e^{-\lambda_2(\theta)}\right)^k ,$$

which concludes the proof of the first formula of the third part of the lemma. □