Exercise Sheet 6

Due date: Thursday, January, 26th, 2006

Problem 6-1 (20 points)

In the lecture we have sketched as an application of the semi-random method a theorem stating that a regular, triangle-free graph $G(V, E)$ with uniform degree Δ has chromatic number $\chi(G) \leq (1 - \frac{1}{2e^6})$. In the first step of the proof we colored each vertex uniformly at random with a color in $\{1, \ldots, \Delta/2\}$ and uncolored all vertices which have the same color as one of their neighbors. We then proved that with positive probability the resulting partial coloring (some vertices are uncolored, but the remaining colors do not violate the condition that an edge never joins two vertices of the same color) has the property that $\forall v : N_v$ contains at least $\frac{\Delta}{2e^6}$ repeated colors, i.e. colors that appear at least twice in v's neighborhood N_v.

Show that this property implies that any regular triangle-free graph G with uniform degree Δ has a coloring with at most $\chi(G) \leq (1 - \frac{1}{2e^6})$ colors.

Hint: Consider a simple greedy procedure to color the remaining uncolored vertices.

Problem 6-2 (30 points) On the last exercise sheet we have shown that any CNF formula $\mathcal{F} = C_1 \cap C_2 \cap C_3 \cap \ldots \cap C_m$ is satisfiable if $\sum_{i=1}^{m} 2^{-|C_i|} < 1$. Derandomize the probabilistic argument to arrive at that result to obtain a deterministic algorithm that actually produces a satisfying assignment for such a formula (use the same technique as for derandomizing the algorithm for finding a large cut in a graph).

Problem 6-3 (20 points) Consider the undirected complete graph on n nodes $G(V, E) = K_n$ with symmetric transition probabilities $1/(n-1)$ for all edges (i.e. being at vertex/state v the probability that one moves in the next step to vertex w is $1/(n-1)$ for all $w \in V$).

We define as h_{vw} the expected number of time steps to reach w starting from node v in a random walk according to the above transition probabilities within G. Equally we denote by c_v the expected number of time steps when all vertices in G have been visited at least once.

1. Prove a good upper bound for h_{vw}
2. Prove a good upper bound for c_v