A Quantum Computational Approach to Correspondence Problems on Point Sets

max planck institut informatik

Saarland Informatics Campus SIC

Vladislav Golyanik   Christian Theobalt

http://gvv.mpi-inf.mpg.de/projects/QA/

Motivation and Contributions

+ Quantum computers are already used to solve difficult combinatorial optimisation problems, and they can be useful in computer vision
+ We show that the classical problem of *finding optimal transformation and correspondences between two point sets* can be efficiently solved on a quantum computer. The quantum annealing time is constant and does not depend on the size of the inputs in a given dimension.
+ We show how to formulate point set alignment as a *quadratic binary unconstrained optimisation problem* (QUBOP) of the form

$$\arg\min_{\mathbf{q}\in\mathbf{B}^n} \mathbf{q}^\top\mathbf{P}\mathbf{q},$$

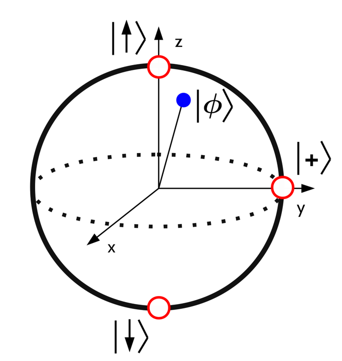
and overcome the difficulty of rotation parametrisation.

What is a Quantum Computer?

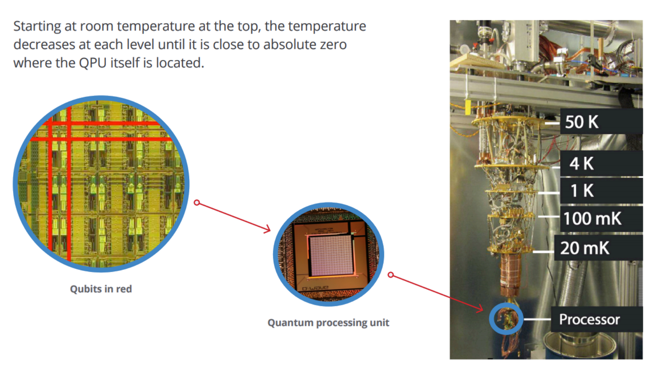
+ Quantum computers take advantage of quantum mechanical effects, *i.e.*, quantum superposition, entanglement and tunnelling [1, 2].
+ They can perform all operations which classical computers can perform, plus multiple algorithms which have lower complexity class compared to their classical counterparts (*e.g.*, prime number factorisation [3])
+ Quantum computers can be classified into two models – gates model and quantum annealers [4]
+ Quantum annealing machines relying on the *adiabatic theorem of quantum mechanics* [5] are called adiabatic quantum computers (AQC)

Adiabatic Quantum Annealers

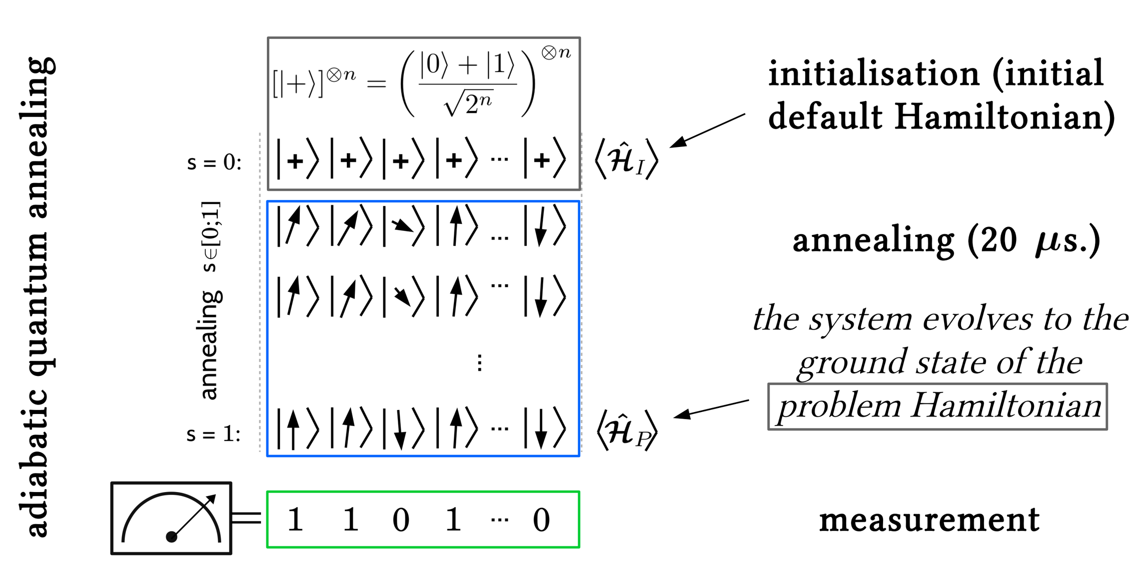
$|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$, $|\alpha|^2+|\beta|^2 = 1$



model of a qubit (Bloch sphere)



D-WAVE 2000Q with ~2k qubits [6]



initialisation (initial default Hamiltonian)

annealing (20 μs.)

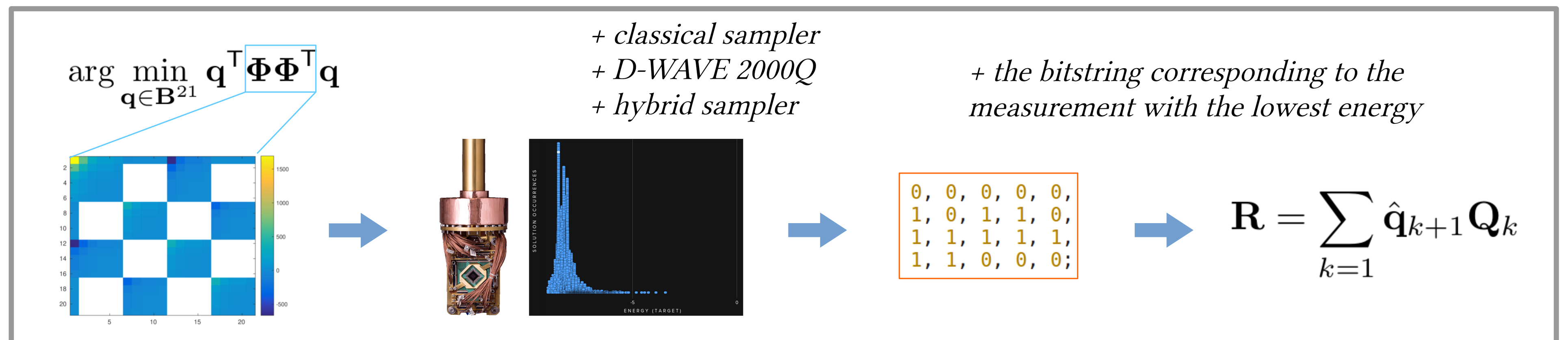
the system evolves to the ground state of the $\boxed{\textit{problem Hamiltonian}}$

measurement

classical bitstring (ready for unembedding)

qubits collapse to one of the basis states

Quantum Approach (QA)



state preparation   annealing + measurement   choosing one solution   unembedding

State Preparation:: Transformation Estimation

Reference point set: $[\mathbf{x}_n] \in \mathbf{X} \in \mathbb{R}^{D\times N}$

Template point set: $[\mathbf{y}_n] \in \mathbf{Y} \in \mathbb{R}^{D\times N}$

Power series of $\mathbf{R}$, $\mathbf{R}^{-1} = \mathbf{R}^\top$:

$$\mathbf{S} = \theta\,\mathbf{M},\ \mathbf{M} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \text{(2D)},\quad \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} \text{(3D)}$$

Cayley-Hamilton Theorem:

$$\mathbf{S}^2 + \theta^2\mathbf{I} = 0 \ \text{(2D)} \qquad -\mathbf{S}^3 - \theta^2\mathbf{S} = 0 \ \text{(3D)}$$

Exponential map for $\mathbf{R}$ with power series:

$$\mathbf{R} = \exp(\mathbf{S}) = \begin{cases} \cos(\theta)\,\mathbf{I} + \left(\frac{\sin(\theta)}{\theta}\right)\mathbf{S} \\ \mathbf{I} + \left(\frac{\sin\theta}{\theta}\right)\mathbf{S} + \left(\frac{1-\cos\theta}{\theta^2}\right)\mathbf{S}^2 \end{cases} = \begin{cases} \cos(\theta)\,\mathbf{I} + \sin(\theta)\,\mathbf{M} & \text{(2D)} \\ \mathbf{I} + \sin\theta\,\mathbf{M} + (1-\cos\theta)\,\mathbf{M}^2 & \text{(3D)} \end{cases}$$

It leads to our basis for $\mathbf{R}$:

$$\{\mathbf{Q}_k = \omega\,\mathbf{C} \in \mathbb{R}^{2\times 2}, \forall\omega \in \{0.5, 0.2, 0.1, 0.1, 0.05\}, \forall\mathbf{C} \in \{\mathbf{I}, \mathbf{M}, -\mathbf{I}, -\mathbf{M}\}\}. \quad \text{(2D)}$$

$$\{\mathbf{Q}_k^{3D} = \omega\,\mathbf{C}^{3D} \in \mathbb{R}^{3\times 3}, \forall\omega \in \{0.5, 0.2, 0.1, 0.1, 0.05\}, \forall\mathbf{C}^{3D} \in \{\mathbf{I}, -\mathbf{I}, \mathbf{M}_a, -\mathbf{M}_a, \mathbf{M}_b, -\mathbf{M}_b, \mathbf{M}_c, -\mathbf{M}_c, \mathbf{M}_d, -\mathbf{M}_d, \mathbf{M}_e, -\mathbf{M}_e, \mathbf{M}_f, -\mathbf{M}_f\}\}. \quad \text{(3D)}$$

$$\mathbf{M} = a\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\mathbf{M}_a} + b\underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}}_{\mathbf{M}_b} + c\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}}_{\mathbf{M}_c} \quad \text{(basis for } \mathbf{M}\text{)}$$

$$\mathbf{M}_d = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \mathbf{M}_e = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \mathbf{M}_f = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{(basis for } \mathbf{M}^2\text{)}$$

The final QUBOP and $\mathbf{P}$:

$$\arg\min_{\mathbf{q}\in\mathbf{B}^{21}} \mathbf{q}^\top\mathbf{\Phi}\mathbf{\Phi}^\top\mathbf{q} \qquad \mathbf{P} = \mathbf{\Phi}\mathbf{\Phi}^\top \ \text{(2D)}$$

$$\mathbf{\Phi} = \begin{bmatrix} \mathbf{x}_1^\top & \mathbf{x}_2^\top & \dots & \mathbf{x}_N^\top \\ -[\mathbf{Q}_1\mathbf{y}_1]^\top & -[\mathbf{Q}_1\mathbf{y}_2]^\top & \dots & -[\mathbf{Q}_1\mathbf{y}_N]^\top \\ -[\mathbf{Q}_2\mathbf{y}_1]^\top & -[\mathbf{Q}_2\mathbf{y}_2]^\top & \dots & -[\mathbf{Q}_2\mathbf{y}_N]^\top \\ \vdots & \vdots & \ddots & \vdots \\ -[\mathbf{Q}_K\mathbf{y}_1]^\top & -[\mathbf{Q}_K\mathbf{y}_2]^\top & \dots & -[\mathbf{Q}_K\mathbf{y}_N]^\top \end{bmatrix}$$

State Preparation:: Point Set Alignment

Minimise *gravitational potential energy* [7] of the system of particles (reference + template) with local point linking:

$$\mathbf{E}(\mathbf{R},\mathbf{t}) = \sum_m\sum_n \mu_{\mathbf{y}_m}\mu_{\mathbf{x}_n}\|\mathbf{R}\,\mathbf{y}_m + \mathbf{t} - \mathbf{x}_n\|_2$$

$$\arg\min_{\mathbf{q}\in\mathbf{B}^{21}} \mathbf{q}^\top\mathbf{\Phi}\mathbf{\Phi}^\top\mathbf{q} \qquad \mathbf{\Phi} = [\mathbf{\Phi}_1\mathbf{\Phi}_2\dots\mathbf{\Phi}_N] \ \text{(2D)}$$

For each template point:

$$\mathbf{\Phi}_n = \begin{bmatrix} \mathbf{x}_n^\top & \mathbf{x}_n^\top & \dots & \mathbf{x}_n^\top \\ -[\mathbf{Q}_1\mathbf{y}_1^n]^\top & -[\mathbf{Q}_1\mathbf{y}_2^n]^\top & \dots & -[\mathbf{Q}_1\mathbf{y}_{L(n)}^n]^\top \\ -[\mathbf{Q}_2\mathbf{y}_1^n]^\top & -[\mathbf{Q}_2\mathbf{y}_2^n]^\top & \dots & -[\mathbf{Q}_2\mathbf{y}_{L(n)}^n]^\top \\ \vdots & \vdots & \ddots & \vdots \\ -[\mathbf{Q}_K\mathbf{y}_1^n]^\top & -[\mathbf{Q}_K\mathbf{y}_2^n]^\top & \dots & -[\mathbf{Q}_K\mathbf{y}_{L(n)}^n]^\top \end{bmatrix}$$

Unembedding

Unembedding is the decoding of the solution to QUBOP to the solution of the original alignment problem:

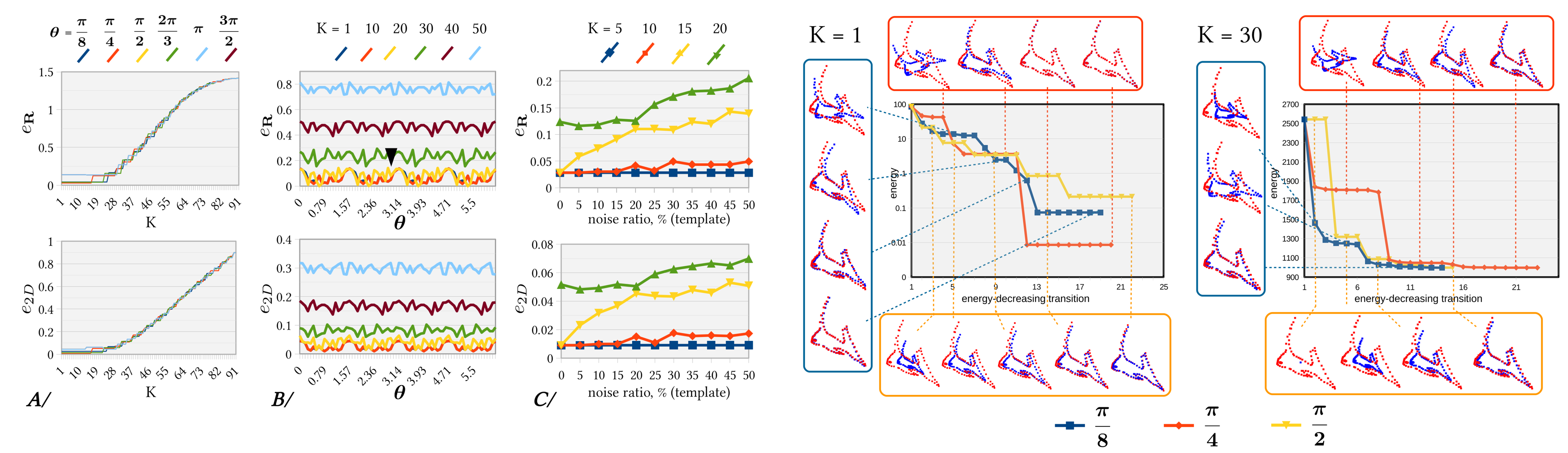
$$\mathbf{R} = \sum_{k=1}^{K} \hat{\mathbf{q}}_{k+1}\mathbf{Q}_k,$$

where a classical bitstring $\hat{\mathbf{q}}$ is the measurement of $\mathbf{q}$.

If required, the obtained approximate solution can be projected to the rotation group:

$$\min \|\mathbf{R}_\mathrm{r} - \mathbf{R}\|^2_{\mathcal{HS}}, \quad \text{s. t. } \mathbf{R}_\mathrm{r}^{-1} = \mathbf{R}_\mathrm{r}^\top \text{ and } \det(\mathbf{R}_\mathrm{r}) = 1.$$

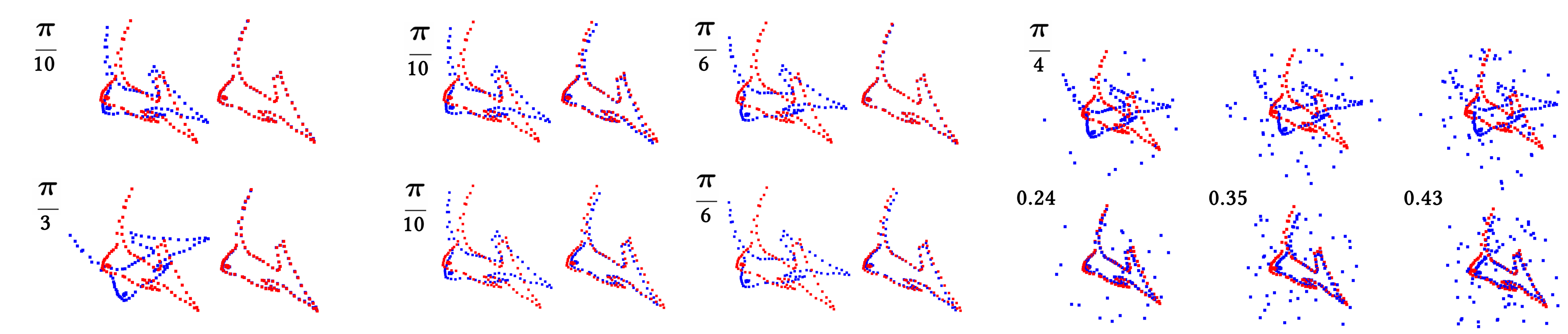
Tests on CPU Sampler and Spectral Gap Analysis (2D)



The metrics as the functions of A/: the size of the point interaction region parametrised by K; B/: the angle of initial misalignment θ; C/: the template noise ratio.

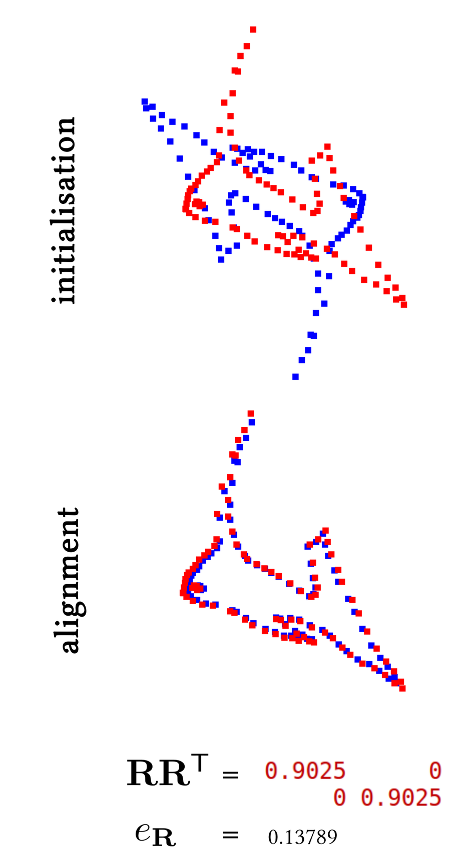
The sequences of energy-decreasing transitions and the corresponding energy values observed in our sampler.

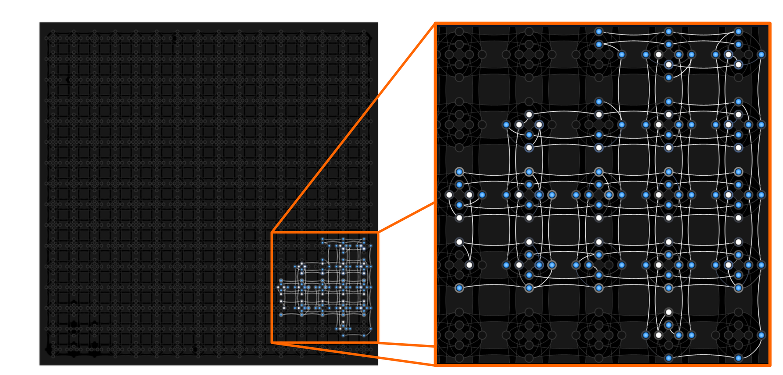
Experiments on D-WAVE 2000Q (2D)



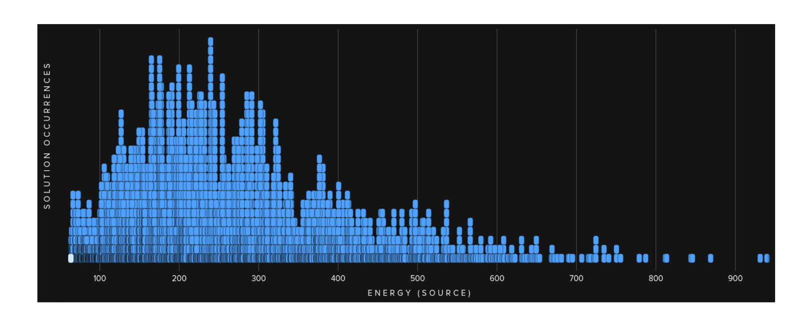
K = 1   K = 20 (top row), 30 (bottom row)   K = 30, template noise

Minor-Embedding and Quantum Annealing

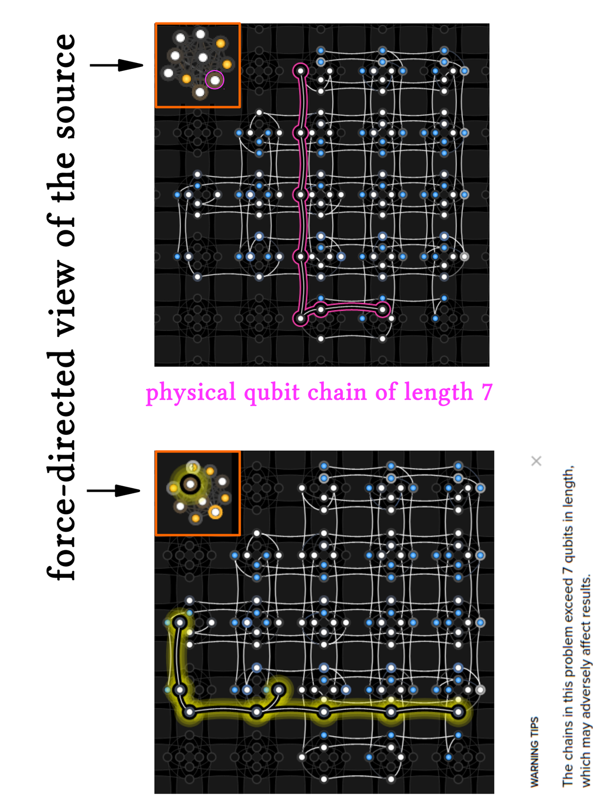




entire *Chimera* graph with [16 x 16] unit cells   the region with the embedded problem



distribution of solutions after 5k runs



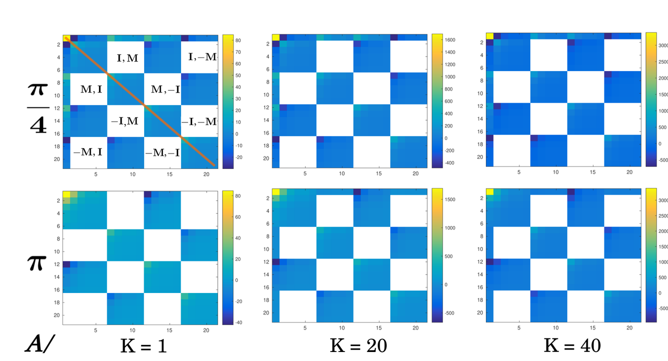
mapping of the problem

quantum notion	classical counterpart
qubit (states $\|0\rangle$ and $\|1\rangle$)	bit (states 0 and 1)
(time-dependent) Hamiltonian	energy functional
eigenstate	some energy state
ground state	globally optimal energy state
quantum system evolution	optimisation process
quantum annealing	simulated annealing

Quantum notions and their classical counterparts.

	TE	K = 10	20	30	40	50
e_{2D}	0.023	0.026	0.041	0.078	0.17	0.3
σ_{2D}	0.012	0.013	0.012	0.012	0.012	0.013
$e_\mathbf{R}$	0.058	0.062	0.083	0.22	0.47	0.764
$\sigma_\mathbf{R}$	0.041	0.044	0.041	0.036	0.031	0.03

The accuracy of QA under random initial misalignments.



Examples of $\mathbf{P} = \mathbf{\Phi}\mathbf{\Phi}^\top$ in QA.

Complexity of state preparation:

$\mathcal{O}(K^2DN)$ (transformation estimation)

$\mathcal{O}(K^2DN\bar{L})$ (point set alignment)

Error metrics:

$$e_{2D} = \frac{\|\mathbf{R}\mathbf{Y}-\mathbf{X}\|_{\mathcal{HS}}}{\|\mathbf{X}\|_{\mathcal{HS}}} \quad \text{(alignment error)}$$

$$e_\mathbf{R} = \|\mathbf{I} - \mathbf{R}\mathbf{R}^\top\|_{\mathcal{HS}} \quad \text{(transformation discrepancy)}$$

References

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